Feedback stabilization of discrete-time quantum systems subject to non-demolition measurements with imperfections and delays

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A B S T R A C T

We consider a controlled quantum system whose finite dimensional state is governed by a discrete-time nonlinear Markov process. In open-loop, the measurements are assumed to be quantum non-demolition (QND). The eigenstates of the measured observable are thus the open-loop stationary states: they are used to construct a closed-loop supermartingale playing the role of a strict control Lyapunov function. The parameters of this supermartingale are calculated by inverting a Metzler matrix that characterizes the impact of the control input on the Kraus operators defining the Markov process. The resulting state feedback scheme, taking into account a known constant delay, provides the almost sure convergence to the target state. This convergence is ensured even in the case where the filter equation results from imperfect measurements corrupted by random errors with conditional probabilities given as a left stochastic matrix. Closed-loop simulations corroborated by experimental data illustrate the interest of such nonlinear feedback scheme for the photon box, a cavity quantum electrodynamics system.

1. Introduction

Manipulating quantum systems allows one to accomplish tasks far beyond the reach of classical devices. Quantum information is paradigmatic in this sense: quantum computers will substantially outperform classical machines for several problems (Nielsen & Chuang, 2000). Though significant progress has been made recently, severe difficulties still remain, amongst which decoherence is certainly the most important. Large systems consisting of many qubits must be prepared in fragile quantum states, which are rapidly destroyed by their unavoidable coupling to the environment. Measurement-based feedback and coherent feedback are possible routes towards the preparation, protection and stabilization of such states. For coherent feedback strategy, the controller is also a quantum system coupled to the original one (see Gough and James (2009) and James and Gough (2010) and the references therein). This paper is devoted to the measurement-based feedback where the controller and the control input are classical objects (Wiseman & Milburn, 2009). The results presented here are directly inspired by a recent experiment (Sayrin, 2011; Sayrin et al., 2011) demonstrating that such a quantum feedback scheme achieves the on-demand preparation and stabilization of non-classical states of a microwave field.

Following Geremia (2006) and relying on continuous-time Lyapunov techniques exploited in Mirrahimi and van Handel (2007), an initial measurement-based feedback was proposed in Dotsenko et al. (2009). This feedback scheme stabilizes photon-number states (Fock states) of a microwave field (see e.g. Haroche and Raimond (2006) for a physical description of such cavity quantum electrodynamics (QED) systems). The controller consists of a quantum filter that estimates the state of the field from discrete-time measurements performed by probe atoms, and secondly a stabilizing state-feedback that relies on Lyapunov techniques. The discrete-time behavior is crucial for a possible real-time implementation of such controllers. Closed-loop simulations reported in Dotsenko et al. (2009) have been confirmed by the stability analysis performed in Amini, Mirrahimi, and Rouchon (2012). In the experimental implementation (Sayrin, 2011; Sayrin et al., 2011), the...
state-feedback has been improved by considering a strict Lyapunov function: this ensures better convergence properties by avoiding the passage by high photon numbers during the transient regime. The goal of this paper is to present, for a class of discrete-time quantum systems, the mathematical methods underlying such improved Lyapunov design. Our main result is given in Theorem 5 where closed-loop convergence is proved in the presence of delays and measurement imperfections.

The state-feedback scheme may be applied to generic discrete-time finite-dimensional quantum systems using controlled measurements in order to deterministically prepare and stabilize the system at some pre-specified target state. The dynamics of these systems may be expressed in terms of a (classical) nonlinear controlled Markov chain, whose state space consists of the set of density matrices on some Hilbert space. The jumps in this Markov chain are induced by quantum measurements and the associated jump probabilities are state-dependent. These systems are subject to a discrete-time sequence of positive operator valued measurements (POVMs) and we use these POVMs to stabilize the system at the target state. By controlled measurements, we mean that at each time step the chosen POVM is not fixed but is a function of some classical control signal $u$, similar to Wiseman and Doherty (2005). However, we assume that when the control is zero, the chosen POVM performs a quantum non-demolition (QND) measurement (Haroche & Raimond, 2006; Wiseman & Milburn, 2009) for some orthonormal basis that includes the target state. The feedback-law is based on a Lyapunov function that is a linear combination of a set of martingales corresponding to the open-loop QND measurements. This Lyapunov function determines a "distance" between the target state and the current state. The parameters of this Lyapunov function are given by inverting Metzler matrices characterizing the impact of the control input on the Kraus operators defining the Markov processes and POVMs. The (graph theoretic) properties of the Metzler matrices are used to construct families of open-loop supermartingales that become strict supermartingales in closed-loop. This fact provides directly the convergence to the target state without using the invariance principle.

A common problem that occurs in quantum feedback control is that of delays between the measurement process and the control process (Kashima & Yamamoto, 2009; Nishio, Kashima, & Imura, 2009). In this paper we demonstrate, using a predictive quantum filter, that the proposed scheme works even in the presence of delays. Convergence analysis is done for perfect and imperfect measurements. For imperfect measurements, the dynamics of the system are governed by a nonlinear Markov chain given in Somaraju, Dotenko, Sayrin, and Rouchon (2011).

In both the perfect and imperfect measurement situations we prove a robustness property of the feedback algorithm: the convergence of the closed-loop system is ensured even when the feedback law is based on the state of a quantum filter that is not initialized correctly. This robustness property, is similar in spirit to the separation principle proven in Boutsen and van Handel (2008) and Bouten, van Handel, and James (2009). We use the fact that the state space is a convex set and the target state, being a pure state, is an extreme point of this convex state space and therefore cannot be expressed as a convex combination of any other states. One then uses the linearity of the conditional expectation to prove the robustness property. Our result is only valid for target quantum states that are pure states.

The paper is organized as follows. In Section 2, we describe the finite dimensional Markov model together with the main modeling assumptions in the case of perfect measurements and study the open-loop behavior (Theorem 1) which can be seen as a non-deterministic protocol for preparing a finite number of isolated and orthogonal quantum states. In Section 3, we present the main ideas underlying the construction of these control-Lyapunov functions $W_\mu$: a Metzler matrix attached to the second derivative of the measurement operators and a technical lemma assuming this Metzler matrix is irreducible. Finally, Theorem 2 describes the stabilizing state feedback derived from $W_\mu$. The same analysis is done for the case of imperfect measurements in Section 4. For the state estimations used in the feedback scheme we propose a brief discussion on the quantum filters and prove a rather general robustness property for perfect measurements in Section 3 with Theorem 3 and for imperfect ones in Section 4 with Theorem 5. Section 5 is devoted to the experimental implementation that has been done at Laboratoire Kastler–Brossel of Ecole Normale Supérieure de Paris. Closed-loop simulations and experimental data complementary to those reported in Sayrin et al. (2011) and Sayrin (2011) are presented.

2. System model and open-loop dynamics

2.1. The nonlinear Markov model

We consider a finite dimensional quantum system (the underlying Hilbert space $\mathcal{H} = \mathbb{C}^d$ is of dimension $d > 0$) being measured through a generalized measurement procedure at discrete-time intervals. The dynamics are described by a nonlinear controlled Markov chain. Here, we suppose perfect measurements and no decoherence. The system state is described by a density operator $\rho$ belonging to $\mathcal{D}$, the set of non-negative Hermitian matrices of trace one: $\mathcal{D} = \{ \rho \in \mathbb{C}^{d \times d} \mid \rho = \rho^\dagger, \text{Tr}(\rho) = 1, \rho \geq 0 \}$. To each measurement outcome $n \in \{1, \ldots, m\}$, $m$ being the numbers of possible outcomes, is attached the Kraus operator $M^0_n = C^\dagger_n (\rho)$ depending on $\mu$ and on a scalar control input $u \in \mathbb{R}$. For each $u$, $(M^0_n)_{n \in \{1, \ldots, m\}}$ satisfy the constraint $\sum_{n=1}^m M^0_n M^0_n^\dagger = I$, the identity matrix. The Kraus map $K^0_n$ is defined by

$$ \mathcal{D} \ni \rho \mapsto K^0_n(\rho) = \sum_{\mu=1}^m M^0_\mu \rho M^0_\mu^\dagger \in \mathcal{D}. $$

The random evolution of the state $\rho_k$ $\in \mathcal{D}$ at time step $k$ is modeled through the following dynamics:

$$ \rho_{k+1} = W_{\mu_k}^{n_k}(\rho_k) := \frac{M_{n_k,\mu_k} M_{n_k,\mu_k}^\dagger}{\text{Tr}(M_{n_k,\mu_k}^\dagger M_{n_k,\mu_k})} \rho_k M_{n_k,\mu_k}^\dagger, $$

for any $\mu_k$ randomly taking values $\mu_k \in \{1, \ldots, m\}$ with probability $p_{\mu_k,n_k} = \text{Tr}(M_{n_k,\mu_k}^\dagger M_{n_k,\mu_k})$. For each $\mu_k$, $M^\mu_\rho(\rho_k)$ is defined when $p_{\mu_k,n_k} = \text{Tr}(M^\mu_\rho M^\mu_\rho^\dagger) \neq 0$.

We now state some assumptions that we will be using in the remainder of this paper.

Assumption 1. For $u = 0$, all $M^0_\mu$ are diagonal in the same orthonormal basis $\{|n\rangle \mid n \in \{1, \ldots, d\} \}$: $M^0_\mu = \sum_{n=1}^d c^\mu_n |n\rangle \langle n|$, with $c^\mu_n \in \mathbb{C}$.

Assumption 2. For all $n_1 \neq n_2$ in $\{1, \ldots, d\}$, there exists $\mu \in \{1, \ldots, m\}$ such that $|c^\mu_{n_1}|^2 \neq |c^\mu_{n_2}|^2$.

Assumption 3. All $M^\mu_\rho$ are $\mathbb{C}^d$ functions of $u$.

Assumption 1 means that when $u = 0$ the measurements are quantum non-demolition (QND) measurements over the states $\{|n\rangle \mid n \in \{1, \ldots, d\} \}$; when $u_k \equiv 0$, any $\rho = |n\rangle \langle n|$ (orthogonal projector on the basis vector $|n\rangle$) is a fixed point of (2). Since $\sum_{\mu=1}^m M^\mu_\mu M^\mu_\mu^\dagger = I$, we have $\sum_{\mu=1}^m |c^\mu_n|^2 = 1$ for all $n \in \{1, \ldots, d\}$.
according to Assumption 1. Assumption 2 means that there exists a \( \mu \) such that the statistics when \( u_k \equiv 0 \) for obtaining the measurement result \( \mu \) are different for the fixed points \( |n_1\rangle |n_1\rangle \) and \( |n_2\rangle |n_2\rangle \). This follows by noting that \( \text{Tr}(M^0_\mu |n_1\rangle |n_1\rangle M^0_\mu) = |c_{u,n}|^2 \) for \( n \in \{1, \ldots, d\} \). Assumption 3 is a technical assumption we will use in our proofs.

### 2.2. Convergence of the open-loop dynamics

When the control input vanishes \( (u \equiv 0) \), the dynamics are simply given by

\[
\rho_{k+1} = M^0_\mu \rho_k, \tag{3}
\]

where \( \mu_k \) is a random variable with values in \( \{1, \ldots, m\} \). The probability \( p^0_{\mu_{k-1},\mu_k} \) to have \( \mu_k = \mu \) depends on \( p^0_{\mu_{k-1},\mu_k} = \text{Tr}(M^0_{\mu_{k-1}} M^0_{\mu_k}) \).

**Theorem 1.** Consider a Markov process \( \rho_k \) obeying the dynamics of (3) with an initial condition \( \rho_0 \in \mathcal{D} \). Then with probability one, \( \rho_k \) converges to one of the \( d \) states \( |n\rangle \langle n| \) with \( n \in \{1, \ldots, d\} \) and the probability of convergence towards the state \( |n\rangle \langle n| \) is given by \( \langle n | \rho_0 | n \rangle \).

This Theorem is already proved in Amini et al. (2012); Amini, Rouchon, and Mirrahimi (2011) and also in Bauer and Bernard (2011) in a slightly different formulation. A direct proof is based on the following Lyapunov function:

\[
\Gamma(\rho) := -\sum_{n=1}^d \frac{\langle n | \rho | n \rangle^2}{2}. \tag{4}
\]

Each \( \langle n | \rho | n \rangle \) is a martingale and we have

\[
\Gamma(\rho_k) - \mathbb{E}(\Gamma(\rho_{k+1}) | \rho_k) = Q_{\theta}(\rho_k) := \sum_{n,m=1}^d p^0_{\mu_{k-1},\mu_k} \left( |c_{u,n}|^2 \langle n | \rho_k | n \rangle - \frac{|c_{u,n}|^2 \langle n | \rho_k | n \rangle^2}{p^0_{\mu_{k-1},\mu_k}} \right). \tag{5}
\]

### 3. Feedback stabilization with perfect measurements

In Section 3.1, we give an overview of the control method and then in Sections 3.2 and 3.3, we prove the main results. Finally in Section 3.4, we prove a robustness principle that explains how we can ensure convergence even if the initial state \( \rho_0 \) is unknown.

#### 3.1. Overview of the control method

**Theorem 1** shows that the open-loop dynamics are stable in the sense that in each realization \( \rho_k \) converges (non-deterministically) to one of the pure states \( |n\rangle \langle n| \) with probability \( \langle n | \rho_0 | n \rangle \). The control goal is to make this convergence deterministic toward a chosen \( \bar{n} \in \{1, \ldots, d\} \) playing the role of controller set point. We build on the ideas in Amini et al. (2012, 2011), Dotsenko et al. (2009) and Somaraju, Mirrahimi, and Rouchon (2013) to design a controller that is based on a strict Lyapunov function for the target state.

In this paper, we assume arbitrary controlled Kraus operators \( M^u_k \) that cannot be decomposed into QND measurement operators \( M^0_k \) followed by a unique controlled unitary operator \( D_\theta \) with \( D_\theta = I \) as assumed in Amini et al. (2012, 2011), Dotsenko et al. (2009) and Somaraju et al. (2013) where \( M^u_k = D_\theta M^0_k \). It can be argued that the control we are proposing is non-Hamiltonian control (Romano & D’Alessandro, 2006a; Romano, 2006b), as the control parameter \( u \) is not necessarily a parameter in the interaction Hamiltonian and could indeed be any parameter of an auxiliary system such as the measurement device.

To convey the main ideas involved in the control design, we begin with the case where there are no delays \( (\tau = 0) \) and we assume that the initial state is known. We wish to use the open-loop supermartingales to design a Lyapunov function for the closed-loop system. By an open-loop supermartingale we mean any function \( V : \mathcal{D} \rightarrow \mathbb{R} \) such that \( \mathbb{E}(V(\rho_{k+1}) | \rho_k = \rho, u_k = u) \leq V(\rho) \) for all \( \rho \in \mathcal{D} \). The Lyapunov function underlying **Theorem 1** demonstrates how we can construct such open-loop supermartingales. Now, at each time-step \( k \), the feedback signal \( u_k \) is chosen by minimizing this supermartingale \( V \) knowing the state \( \rho_k \):

\[
u_k = \bar{u}(\rho) := \arg\min_{u \in \{-\bar{u}, \bar{u}\}} \mathbb{E}(V(\rho_{k+1}) | \rho_k = \rho, u_k = u).\]

Here \( \bar{u} \) is some small positive number that needs to be determined. Because \( 0 \in [-\bar{u}, \bar{u}] \) and the control \( u_k \) is chosen to minimize \( V \) at each step, we directly have that \( V \) is a closed-loop supermartingale, i.e.,

\[
\bar{Q}(\rho) := \mathbb{E}(V(\rho_{k+1}) | \rho_k = \rho, u_k = \bar{u}(\rho)) - V(\rho) \leq 0
\]

for all \( \rho \in \mathcal{D} \). If this supermartingale \( V \) is bounded from below then we can directly apply the convergence **Theorem 6** in the Appendix to prove that \( \rho_k \) converges with probability one to the set \( \mathcal{I}_C := \{ \rho : \bar{Q}(\rho) = 0 \} \).

What remains to be done is to choose an appropriate open-loop supermartingale \( V \) so that the set \( \mathcal{I}_C \) is restricted to the target state \( |\bar{n}\rangle \langle \bar{n}| \) (c.f. **Fig. 2** below which shows how \( \sigma_n \) are chosen for the experimental setting).

A state \( \rho \) is in the set \( \mathcal{I}_C \) if and only if for all \( u \in [-\bar{u}, \bar{u}] \), we have

\[
\mathbb{E}(V(\rho_{k+1}) | \rho_k = \rho, u_k = u) - V(\rho) \geq 0. \tag{6}
\]

Also from the fact that \( V \) is an open-loop supermartingale, we have for all \( \rho \in \mathcal{D} \)

\[
\mathbb{E}(V(\rho_{k+1}) | \rho_k = \rho, u_k = 0) - V(\rho) \leq 0. \tag{7}
\]

We prove in **Lemma 2** below that given any \( \bar{n} \in \{1, \ldots, d\} \), we can always choose the weights \( \gamma_1, \ldots, \gamma_d \) so that \( V \) satisfies the following property: \( \forall n \in \{1, \ldots, d\} \), \( u \mapsto \mathbb{E}(V(\rho_{k+1}) | \rho_k = |n\rangle \langle n|, u_k = u) \) has a strict local minimum at \( u = 0 \) for \( n = \bar{n} \) and strict local maxima at \( u = 0 \) for \( n \neq \bar{n} \). This combined with Eq. (7) then ensures that for any \( n \neq \bar{n} \), there is some \( u \in [-\bar{u}, \bar{u}] \) such that

\[
\mathbb{E}(V(\rho_{k+1}) | \rho_k = |n\rangle \langle n|, u_k = u) - V(\rho_k) < 0 \tag{9}
\]

Therefore using Eq. (6), we know that \( |n\rangle \langle n| \) is in the limit set \( \mathcal{I}_C \) if and only if \( n = \bar{n} \).

This idea can easily be extended to the situations where the delay \( \tau \) is non-zero. Take \( \rho_{k-1}, \ldots, \rho_{k-\tau-1} \) as state at step \( k \) by denote by \( \gamma = (\rho, \beta_1, \ldots, \beta_\tau) \) this state where \( \beta_r \) stands for the control input \( u \) delayed \( r \) steps. Then the state form of the delayed dynamics (2) is governed by the following Markov chain

\[
\begin{cases}
\rho_{k+1} = M^{\beta_{k+1}}_\mu (\rho_k) \\
\beta_{r+1} = u_k + \beta_{r+1-k} \text{ for } r = 2, \ldots, \tau.
\end{cases}
\tag{8}
\]

The goal is to design a feedback law \( u_k = f(\chi_k) \) that globally stabilizes this Markov chain \( \chi_k \) towards a chosen target state \( \bar{\chi} = (\bar{n} \langle \bar{n}|, 0, \ldots, 0) \) for some \( \bar{n} \in \{1, \ldots, d\} \). In **Theorem 2**, we show...
how to design a feedback relying on the control Lyapunov function $W_\epsilon(\chi) = V_\epsilon(\chi^P(\chi^P(\cdots \chi^P(\rho \cdots )))).$ The idea is to use a predictive filter to estimate the state of the system $\tau$ time-steps later.

Finally, we address the situation where the initial state of the system is not fully known but only estimated by $\rho_0^\text{est}$. We show under some assumptions on the initial condition that, the feedback law based on the state $\rho_0^\text{est}$ of the miss-initialized filter still ensures the convergence of $\rho_0^\text{est}$ as well as the well-initialized conditional state $\rho_\epsilon$ towards $[\epsilon]/[\epsilon]$. This demonstrates how the control algorithm is robust to uncertainties in the initialization of the estimated state of the quantum system.

### 3.2. Choosing the weights $\sigma_n$

The construction of the control Lyapunov function relies on two lemmas.

**Lemma 1.** Consider the $d \times d$ matrix $R$ defined by

$$R_{n_1, n_2} = \mu \sum_{\mu} \left( 2 \left| \frac{dM_{\mu}^u}{du} \right|_{u=0}^2 \right) + 2\delta_{n_1, n_2} \beta \left( \frac{d^2M_{\mu}^u}{du^2} \left|_{u=0}^2 \right| \right).$$

When $R \neq 0$, the non-negative $P = I - R/\text{Tr}(R)$ is a right stochastic matrix.

**Proof.** For $n_1 \neq n_2$, $R_{n_1, n_2} \geq 0$. Thus $R$ is a Metzler matrix. Let us prove that the sum of each row vanishes. This results from identity

$$\sum_{n_2} (n_1 | M_{\mu}^u | n_2) = \sum_{n_2} (n_1 | M_{\mu}^u | n_2) = 1,$$

yields

$$\sum_{n_2} (n_1 | M_{\mu}^u | n_2)^2 = \sum_{n_2} \frac{d^2M_{\mu}^u}{du^2} \left|_{u=0}^2 \right| (n_1 | n_2) = 0.$$

Since for $u = 0$, $(n_2 | M_{\mu}^u | n_0) = \delta_{n_1, n_2} \beta$, the above sum corresponds to $\sum_{n_2} R_{n_1, n_2}$. Therefore, the diagonal elements of $R$ are non-positive. If $R \neq 0$, then $\text{Tr}(R) < 0$ and the matrix $P = I - R/\text{Tr}(R)$ is well defined with non-negative entries. Since the sum of each row of $P$ vanishes, the sum of each row of $P$ is equal to 1. Thus $P$ is a right stochastic matrix.

To the Metzler matrix $R$ defined in Lemma 1, we associate its directed graph denoted by $G$. This graph admits $d$ vertices labeled by $n \in \{1, \ldots, d\}$. To each strictly positive off-diagonal element of the matrix $R$, say, on the $n_1$th row and the $n_2$th column we associate an edge from vertex $n_1$ towards vertex $n_2$.

**Lemma 2.** Assume that the directed graph $G$ of the matrix $R$ defined in Lemma 1 is strongly connected, i.e., for any $n, n' \in \{1, \ldots, d\}, n \neq n'$, there exists a path of $\tau$ distinct elements $(n_i)_{i=1,...,\tau}$ of $\{1, \ldots, d\}$ such that $n_1 = n, n_\tau = n'$ and for any $j = 1, \ldots, \tau - 1, R_{n_j, n_{j+1}} \neq 0$. Take $n \in \{1, \ldots, d\}$. Then, there exist $d - 1$ strictly positive real numbers $\epsilon_n, n \in \{1, \ldots, d\} \setminus \{n\}$, such that

- for any real $\lambda_n, n \in \{1, \ldots, d\} \setminus \{n\}$, there exists a unique vector $\sigma = (\sigma_n)_{n=1,...,d}$ of $\mathbb{R}^d$ with $\sigma_n = 0$ such that $R\sigma = \lambda \sigma$ where $\lambda_n$ is the vector of $\mathbb{R}^d$ of components $\lambda_{n_\tau}$ for $n \in \{1, \ldots, d\} \setminus \{n\}$ and $\lambda_n = -\sum_{n \neq n_\tau} \epsilon_n \lambda_n$, if additionally $\lambda_n < 0$ for all $n \in \{1, \ldots, d\} \setminus \{n\}$, then $\sigma_n > 0$ for all $n \in \{1, \ldots, d\} \setminus \{n\}$.
- for any vector $\sigma \in \mathbb{R}^d$, solution of $R\sigma = \lambda \in \mathbb{R}^d$, the function $V_\epsilon(\rho) = \sum_{n=1}^d \sigma_n |\rho_n| |\rho_n|$ satisfies

$$\frac{dV_\epsilon(\mathbb{R}^d(\mathbb{R}^d(\mathbb{R}^d(\rho)|\rho_n)|\rho_n)|\rho_n)|\rho_n)}{du^2} = \lambda_n \ \forall n \in \{1, \ldots, d\}.$$

**Proof.** Since the directed graph $G$ coincides with the directed graph of the right stochastic matrix $P$ defined in Lemma 1, $P$ is irreducible. Since it is a right stochastic matrix, its spectral radius is equal to 1. By Perron–Frobenius theorem for non-negative irreducible matrices, this spectral radius, i.e., 1, is also an eigen-value of $P$ and of $P^2$, with multiplicity one and associated to vectors having strictly positive entries: the right eigen-vector $(Pw = w)$ is obviously $w = (1, \ldots, 1)^T$; the left eigen-vector $e = (e_1, \ldots, e_d)^T$ can be chosen such that $e_i = 1$. Consequently, the rank of $R = d - 1$ with ker($R$) = span($w$) and im($R$) = $e^\perp$ where $e^\perp$ is the hyper-plane orthogonal to $e$. Since $e^\perp \lambda = 0$, $\lambda \in \text{im}(R)$, exists such that $R\sigma = \lambda$. Since ker($R$) = span($w$), there is a unique $\sigma$ solution of $R\sigma = \lambda$, such that $\sigma_n = 0$. The fact that $\sigma_n > 0$ when $\lambda_n < 0$ for $n \neq n$, comes from elementary manipulations of $P\sigma = \sigma - \lambda/\text{Tr}(R)$ showing that $\min \sigma_n > 0$.

Thus, $\forall n$, set $V_\epsilon(n) = V_\epsilon(\beta n^\perp(\beta n^\perp(\beta n^\perp(\beta n^\perp(\cdots)))) = \sum_{l} \sigma_l (\beta n^\perp(\beta n^\perp(\beta n^\perp(\beta n^\perp(\cdots)))))$. Set $P^2 := (l|\beta n^\perp(\beta n^\perp(\beta n^\perp(\beta n^\perp(\cdots))))$. Then $\frac{dV_\epsilon}{du^2} = \sum_{n} V_\epsilon(\mathbb{R}^d(\mathbb{R}^d(\mathbb{R}^d(\mathbb{R}^d(\rho)|\rho_n)|\rho_n)|\rho_n)|\rho_n)|\rho_n)}{du^2}$ and

$$\frac{d\epsilon_n}{du^2} = \sum_{l} (l|\beta n^\perp(\beta n^\perp(\beta n^\perp(\beta n^\perp(\cdots)))) - \lambda_n (l|\beta n^\perp(\beta n^\perp(\beta n^\perp(\beta n^\perp(\cdots))))^2) \frac{d\epsilon_n}{du^2}.$$

Therefore $\frac{d\epsilon_n}{du^2} = R_{n,1}$ and

$$\frac{dV_\epsilon}{du^2} = \sum_{n=1}^d R_{n,1} \epsilon_n = \lambda_n.$$

### 3.3. The global stabilizing feedback

The main result of this section is expressed through the following theorem.

**Theorem 2.** Consider the Markov chain (8) with Assumptions 1–3. Take $n \in \{1, \ldots, d\}$ and assume that the directed graph $G$ associated to the Metzler matrix $R$ of Lemma 1 is strongly connected. Take $\epsilon > 0$, $\sigma \in \mathbb{R}^d$, the solution of $R\sigma = \lambda \sigma$ with $\sigma_n = 0$, $\lambda_n < 0$ for $n \in \{1, \ldots, d\} \setminus \{n\}$, $\lambda_n = -\sum_{n \neq n_\tau} \epsilon_n \lambda_n$ (see Lemma 2) and $V_\epsilon(\rho) = \sum_{n=1}^d \sigma_n |\rho_n| |\rho_n|$. So
Since $\omega$ According to Theorem 6, the dependence of $D_{\bar{\omega}}$ in the closed-loop Markov chain of state $x_\tau$ with the feedback law of Theorem 2 converges almost surely towards $(|\bar{\omega}| |\bar{\omega}|, 0, 0, \ldots, 0)$ for any initial condition $x_0 \in \mathcal{D} \times [-\bar{\omega}, \bar{\omega}]$.

Proof. For the sake of simplicity, first we demonstrate this Theorem for $\tau = 1$ and thus for $x = (\rho, \beta)$. We then explain how this proof may be extended to arbitrary $\tau > 1$. Here, $\mathbb{E}(W_{\tau}(x_{\tau+1})|x_{\tau}, u_{\tau})$ is given by $\sum_{n=1}^{\infty} P_{\rho,\beta}^n V_\tau(\mathbb{E}^\rho(\mathbb{E}^\beta(|\bar{\omega}| |\bar{\omega}|)))$ that can also be presented as $\sum_{n=1}^{\infty} P_{\rho,\beta}^n V_\tau(\mathbb{E}^\rho(\mathbb{E}^\beta(|\bar{\omega}| |\bar{\omega}|))) = \mathcal{D}$ with $Q_2(x_\tau) = 0$.

Proof of Step 3. For the sake of simplicity, first we demonstrate this Theorem for $\tau = 1$ and thus for $x = (\rho, \beta)$. We then explain how this proof may be extended to arbitrary $\tau > 1$. Here, $\mathbb{E}(W_{\tau}(x_{\tau+1})|x_{\tau}, u_{\tau})$ is given by $\sum_{n=1}^{\infty} P_{\rho,\beta}^n V_\tau(\mathbb{E}^\rho(\mathbb{E}^\beta(|\bar{\omega}| |\bar{\omega}|)))$ that can also be presented as $\sum_{n=1}^{\infty} P_{\rho,\beta}^n V_\tau(\mathbb{E}^\rho(\mathbb{E}^\beta(|\bar{\omega}| |\bar{\omega}|))) = \mathcal{D}$ with $Q_2(x_\tau) = 0$.

Proof of Step 1. For all $\alpha > 0$, there exists $u > 0$ such that $I_{\alpha} \subset \bigcup_{n=1}^{\infty} \{ (\rho, \beta) \in \mathcal{D} \times [-\bar{\omega}, \bar{\omega}] | (\bar{\omega}, \bar{\omega}) \}$.

Proof of Step 2. For all $\alpha > 0$, there exists $u > 0$ such that $I_{\alpha} \subset \bigcup_{n=1}^{\infty} \{ (\rho, \beta) \in \mathcal{D} \times [-\bar{\omega}, \bar{\omega}] | (\bar{\omega}, \bar{\omega}) \}$.

Proof of Step 3. By construction of $V_{\tau}$, we have for $\rho = |\bar{\omega}| |\bar{\omega}|$ and $\beta_1 = 0$, $\frac{d^2F_{\rho,\beta_1}}{dT^2} |_{T=0} < 0$. Therefore, such $(\rho, \beta_1)$ and $\xi = 0$ cannot minimize $F_{\rho,\beta_1}$ and we have $I_{\alpha} \cap \{ (\rho, \beta_1) \in \mathcal{D} \times [-\bar{\omega}, \bar{\omega}] | (\bar{\omega}, \bar{\omega}) \}$.

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Proof of Step 3. By construction of $V_{\tau}$, we have for $\rho = |\bar{\omega}| |\bar{\omega}|$ and $\beta_1 = 0$, $\frac{d^2F_{\rho,\beta_1}}{dT^2} |_{T=0} < 0$. Therefore, such $(\rho, \beta_1)$ and $\xi = 0$ cannot minimize $F_{\rho,\beta_1}$ and we have $I_{\alpha} \cap \{ (\rho, \beta_1) \in \mathcal{D} \times [-\bar{\omega}, \bar{\omega}] | (\bar{\omega}, \bar{\omega}) \}$.
4. Imperfect measurements

We now consider the feedback control problem in the presence of classical measurement imperfections with the possibility of detection errors. This model is a direct generalization of the ones used in Dotsenko et al. (2009), Gardiner and Zoller (2000) and Sayrin et al. (2011) (see e.g. Boutein, van Handel, and James (2007); Boutein et al. (2009) for an introduction to quantum filtering). The imperfections in the measurement process are described by a classical probabilistic model relying on a left stochastic matrix $\eta_{\mu',\mu}$, $\mu' \in \{1, \ldots, m\}$ and $\mu \in \{1, \ldots, m\}$: $\eta_{\mu,\mu'} \geq 0$ and for any $\mu$, $\sum_{\mu'} \eta_{\mu',\mu} = 1$. The integer $m'$ corresponds to the number of imperfect outcomes and $\eta_{\mu',\mu}$ is the probability of having the imperfect outcome $\mu'$ knowing the perfect one $\mu$. Set $\hat{\rho}_k = \mathbb{E}(\rho_k | \mu_0, \mu_0, \ldots, \mu_{k-1}, u_r, \ldots, u_{r-k+1})$. (12)

Since $\hat{\rho}_k$ follows (2), $\hat{\rho}_k$ is also governed by a recursive eq. (So- maraju et al., 2011):

$$\rho_{k+1}^m = L^m_{\mu',\mu}(\hat{\rho}_k),$$

where for each $\mu', \xi^m_\mu$ is the super-operator defined by $\rho \mapsto L^m_{\mu',\mu}(\rho) = \mathbb{E}(\xi^m_\mu(\rho) | \rho, u_k, \ldots, u_{r-k+1})$ and where $\mu'$ is a random variable taking values in $\mu', \ldots, m'$. Set $\xi^m_\mu(\rho) = \mathbb{E}(\eta_{\mu',\mu}\rho_{\mu'}(\rho) | \rho, u_k, \ldots, u_{r-k+1})$. The following Markov process remains a fixed point of the Markov process (13) when $\eta_{\mu,\mu'} = 0$.

We now consider the dynamics of the filter state $\hat{\rho}$ in the presence of delays in the feedback control. Similar to the case with perfect measurements, let $\hat{\rho}_k = (\hat{\rho}_k, \hat{\beta}_k, \ldots, \hat{\beta}_k, \ldots)$ be the filter state at step $k$, where $\hat{\beta}_k$ is the feedback control at time-step $k$ delayed $r$ steps. Then the delay dynamics are determined by the following Markov chain

$$\hat{\rho}_{k+1} = L^m_{\mu',\mu}(\hat{\rho}_k)$$

$$\hat{\beta}_{k+1} = u_k$$

Instead of Assumption 2 we now assume

Assumption 4. For all $n_1 \neq n_2$ in $\{1, \ldots, d\}$, there exists $\mu' \in \{1, \ldots, m\}$ such that $\sum_{\mu=1}^m \eta_{\mu',\mu} |n_1| \neq \sum_{\mu=1}^m \eta_{\mu',\mu} |n_2|^2$.

Assumption 4 means that there exists a $\mu'$ such that the statistics when $\eta_{\mu,\mu'} = 0$ for obtaining the measurement result $\mu'$ are different for the fixed points $|n_1|$ and $|n_2|$. This follows by noting that $Tr(\xi^m_\mu(\rho)|n_1|) = \sum_{\mu=1}^m \eta_{\mu',\mu} |n_1|^2$ for $n \in \{1, \ldots, d\}$. We now state the analogue of Theorem 2 in the case of imperfect measurements.

Theorem 4. Consider the Markov chain (14) with Assumptions 1, 3 and 4. Take $\bar{n} \in \{1, \ldots, d\}$. Assume that the directed graph of

5. The photon box

In this section, we give the explicit expression of the feedback controller which has been experimentally tested in Laboratoire Kastler-Brossel (LKB) at Ecole Normale Supérieure (ENS) de Paris. We briefly summarize how the control design elucidated in this paper is applied to the LKB experiment. We refer the interested reader to Sayrin (2011) and Sayrin et al. (2011) for more details. This feedback controller has been obtained by the Lyapunov design discussed in previous sections.
5.2. The controlled Markov process and quantum filter

The state to be stabilized is the cavity state. The underlying Hilbert space is $\mathcal{H} = \mathbb{C}^{n_{\text{ph}}_{\text{max}}+1}$ which is assumed to be finite-dimensional (truncations to $n_{\text{ph}}_{\text{max}}$ photons). It admits ($|0\rangle$, $|1\rangle$, $\ldots$, $|n_{\text{ph}}_{\text{max}}\rangle$) as an orthonormal basis. Each basis vector $|n_{\text{ph}}\rangle \in \mathbb{C}^{n_{\text{ph}}_{\text{max}}+1}$ corresponds to a pure state, called photon-number state (Fock state), with precisely $n_{\text{ph}}$ photons in the cavity, $n_{\text{ph}} \in \{0, \ldots, n_{\text{ph}}_{\text{max}}\}$. With notations of Section 2.1, $d = n_{\text{ph}}_{\text{max}} + 1$ and the index $n = n_{\text{ph}} + 1$. The three main processes that drive the evolution of our system are decoherence, injection, and measurement. Their action onto the system’s state can be described by three super-operators $\mathbb{T}$, $\mathbb{D}$ and $\mathbb{F}$, respectively. We refer to Dotsenko et al. (2009) for more details. All operators are expressed in the Fock-basis ($|n_{\text{ph}}\rangle|n_{\text{ph}}\rangle_{\text{max}}$ truncated to $n_{\text{ph}}_{\text{max}}$ photons).

After taking into account our full knowledge about the experiment, we finally get the following state estimate at step $k + 1$:

$$\hat{\rho}_{k+1} = \sum_{\mu} P_{\mu} (D_{\mu} \hat{\rho}_{k} \mathbb{T}_{\mu} (\hat{\rho}_{\text{est}}))) \equiv \sum_{\mu} u_{\mu} \cdot \hat{\rho}_{\text{est}}$$

(16)

with the following short descriptions for the super-operators $\mathbb{T}$, $\mathbb{D}$ and $\mathbb{F}$.

- The decoherence manifests itself through spontaneous loss or capture of a photon to or from the environment which is described as follows:

$$\rho \mapsto \mathbb{T}_{\rho} (\rho) = L_{\rho} \rho L_{\rho}^{\dagger}$$

(17)

where $L_{\rho} = 1 - \theta (1/2 + n_{\text{ph}}) \mathbb{N} - (\theta n_{\text{ph}}/2) \mathbb{L}_- + \sqrt{\theta (1 + n_{\text{ph}})} \mathbb{L}_+$. Here, $\mathbb{L}_+$ and $\mathbb{L}_-$ are the annihilation and creation operators $a a^\dagger$ and $\mathbb{N} = a^\dagger a$ is the photon number operator ($\mathbb{N}|n_{\text{ph}}\rangle = n_{\text{ph}}|n_{\text{ph}}\rangle$). Besides, $n_{\text{ph}}$ is the mean number of photons in the cavity mode at thermal equilibrium with its environment.

- The evolution of the state $\rho$ after the control injection is modeled through

$$\rho \mapsto \mathbb{D}_{\rho} (\rho) = D_{\rho} \rho D_{\rho}^{\dagger}$$

(18)

with $D_{\rho} = \exp(u a^\dagger - u a)$. In reality, the control at step $k$, $u_k$, is subject to a delay of $\tau > 0$ steps which corresponds to the number of flying atoms between the cavity C and the detector D.

- In the real experiment, the atom source is probabilistic and is characterized by a truncated Poisson probability distribution $P_{\rho}(n_{\text{ph}}) \geq 0$ to have $n_{\text{ph}} \in \{0, 1, 2\}$ atom(s) in a sample (we neglect events with more than 2 atoms). This expands the set of the possible detection outcomes to $m = 7$ values $\mu \in \{0, g, e, \ldots, g, e, g\}$, related to the following measurement operators, $L_{\rho} = \sqrt{P_{\rho}(0)} \mathbb{I}$, $L_{g} = \sqrt{P_{\rho}(1)} \cos(\phi_{\rho})$, $L_{e} = \sqrt{P_{\rho}(1)} \sin(\phi_{\rho})$, $L_{eg} = \sqrt{P_{\rho}(2)} \cos^2(\phi_{\rho})$, $L_{ge} = \sqrt{P_{\rho}(2)} \sin^2(\phi_{\rho})$ and $L_{ee} = \sqrt{P_{\rho}(2)} \cos(\phi_{\rho}) \sin(\phi_{\rho})$ where $\phi_{\rho} = \arctan(n_{\text{ph}} + 1/2)$, $\phi_{\rho}$ and $\phi_{\rho}$ are physical parameters.

The real measurement process is not perfect: the detection efficiency is limited to $\epsilon < 1$ and the state detection errors are non-zero ($0 < \eta_{\rho,\mu} < 1$). These imperfections are taken into account by considering the left stochastic matrix $\eta_{\rho,\mu}$ which is given in Somaraju et al. (2011). Consequently, the optimal state estimate after measurement outcome $\mu'$ gets the following form:

$$\mathbb{P}_{\mu'} (\rho) = \frac{\sum_{\mu=1}^{m} \eta_{\mu',\mu} P_{\mu} \rho L_{\mu}^{\dagger}}{\text{tr} \left( \sum_{\mu=1}^{m} \eta_{\mu',\mu} P_{\mu} \rho L_{\mu}^{\dagger} \right)}.$$
The measurement operators $(L_{\mu})_{1 \leq \mu \leq m}$ are diagonal in the Fock basis $|n_{ph}\rangle$, illustrating their quantum non-demolition nature with respect to the photon number operator and thus fulfilling Assumption 1. Besides, Assumption 4 can also be fulfilled by a proper choice of the experimental parameters $\phi_{r}$ and $\phi_{o}$.

5.3. Feedback controller

For $\theta = 0$ (no cavity decoherence) the Markov model of density matrix $\bar{\rho}$ associated to the filter (16) is exactly of the form (14) with $\langle |n_{ph}\rangle \langle n_{ph}| \rangle_{\bar{\rho}_{ph}} = \sum_{n>\bar{n}_{ph}} |n_{ph}\rangle \bar{\rho}_{ph} |n_{ph}\rangle$ being fixed-points in open-loop. Similarly, the underlying Markov process of the true cavity state $\rho$, which is unobservable in practice because of detection errors and delays, admits the same fixed points in open-loop. With parameters given in Section 5.4 (except $\theta = 0$), these Markov processes satisfy Assumptions 1–3 for $\rho$ and Assumptions 1–4 for $\bar{\rho}$. Moreover the Metzler matrix $R$ of Lemma 2 is irreducible. Consequently the assumptions of Theorem 4 are satisfied. The feedback law proposed in Theorem 5 and relying on the filter state $\bar{\rho}^{est}$ will stabilize globally the unobservable state $\rho$ towards the target photon-number state $|n_{ph}\rangle \langle n_{ph}|_{\bar{\rho}_{ph}}$. Numerous closed-loop simulations show that taking $\epsilon = 0$ in the feedback law does not destroy stability and does not affect the convergence rates. This explains why in the simulations and experiments, we set $\epsilon = 0$ despite the fact that Theorem 5 guarantees convergence only for arbitrary small but strictly positive $\epsilon$.

For $\theta$ positive and small, the $|n_{ph}\rangle \langle n_{ph}|$ are no more fixed-points for $\rho$ and $\bar{\rho}$ in open-loop. Let us detail how to adapt the feedback scheme of Theorem 5. At each step of an ideal experiment the control $u(t)$ minimizes the Lyapunov function $V_{\bar{\rho}}(\bar{\rho}_{ph}) = \sum_{n>\bar{n}_{ph}} \sigma_{nph} \langle n_{ph}| \bar{\rho}_{ph} |n_{ph}\rangle$ ($\epsilon$ is set to zero) calculated for state $\bar{\rho}_{ph} = \bar{D}_{u_{\rho}}(\rho_{ph})$. In our real experiment however, we also take into account decoherence and $\tau$ flying not-yet-detected samples and therefore choose $u(t)$ to minimize $V_{\bar{\rho}}(\bar{\rho}_{ph})$ for

$$\bar{\rho}_{ph} = \bar{D}_{u_{\rho}}(T_{\rho}(\bar{\rho}_{ph}) = \sum_{n>\bar{n}_{ph}} \sigma_{nph} \langle n_{ph}| \bar{\rho}_{ph} |n_{ph}\rangle$$

with $\bar{\rho}^{est}$ given by (16). Here we have introduced the Kraus map of the real experiment $\bar{\rho}_{ph} = \sum_{n>\bar{n}_{ph}} \sigma_{nph} \langle n_{ph}| \bar{\rho}_{ph} |n_{ph}\rangle$. The control $u$ minimizing $V_{\bar{\rho}}$ is approximated by argmax$\bar{\rho}_{ph} \in \{0,1/2,2\}$ with $a_{1} = \text{Tr}(|a_{1}|a_{1} \sigma_{nph})$ and $a_{2} = \text{Tr}(|a_{1}|a_{1} \sigma_{nph})$, where $\sigma_{nph}$ is the diagonal operator $\sum_{n>\bar{n}_{ph}} \sigma_{nph} |n_{ph}\rangle \langle n_{ph}|$. The coefficients $\sigma_{nph}$ are computed using Lemma 2 where, for $n_{ph} \neq \bar{n}_{ph}$, $\gamma_{nph}$ are chosen negative and with a decreasing modulus versus $n_{ph}$ in order to compensate cavity decay. For $\bar{n}_{ph} = 3$, we have compared in simulations different setting and selected the profile displayed in Fig. 2.

5.4. Simulations and experimental results

Closed-loop simulation of Fig. 3 shows a typical Monte-Carlo trajectory of the feedback loop aiming to stabilize the 3-photon state $|n_{ph} = 3\rangle$. The experimental parameters used in the simulations are the following: $n_{\text{max}} = 8$, $\phi_{0} = 0.245 \pi$, $\phi_{1} = \pi/2 - \phi_{0}(\bar{n}_{ph} + 1/2)$, $\phi_{2} = 0.6$, $\epsilon = 0.35$, $\eta_{f} = 0.13$, $\eta_{g} = 0.11$, $\theta = 0.014$, $n_{h} = 0.05$, and $\tau = 4$. For the feedback, $\sigma_{nph}$ are given in Fig. 2, $\epsilon = 0$ and $\bar{\sigma}_{1} = 1/10$. The initial states $\rho_{0}$ and $\bar{\rho}_{0}^{est}$ take the following form: $\mathcal{D}_{\bar{n}_{ph}} \rho_{0}$.

The results of the experimental implementation of the feedback scheme are presented in Fig. 4. Fig. 4(e) shows that the average fidelity of the target state is about 47%. Besides, the asymmetry between the distributions for $n_{ph} < \bar{n}_{ph}$ and $n_{ph} > \bar{n}_{ph}$ indicates the presence of quantum jumps occurring preferentially downwards ($\bar{n}_{ph} \rightarrow \bar{n}_{ph} - 1$). Contrarily to the simulations of Fig. 3.
the cavity photon number probabilities relying on $\rho$ are not accessible in the experimental data of Fig. 4 since we do not have access to the detection errors and to the cavity decoherence jumps (Brune et al., 2008; Guerlin et al., 2007; Wang et al., 2008). Nevertheless, green curves in simulations of Figs. 3(d) and (e) indicate that when $|\langle n_{\text{ph}} | 2^\text{est} | n_{\text{ph}} \rangle|^2$ exceeds $8/10$, $\rho$ coincides, with high probability, with $|\langle n_{\text{ph}} | 2_{\text{est}} | n_{\text{ph}} \rangle|^2$.

6. Conclusion

We have proposed a Lyapunov design for state-feedback stabilization of a discrete-time finite-dimensional quantum system with QND measurements. Extensions of this design are possible in different directions such as:

- replacing the continuous and one-dimensional input $u$ by a multi-dimensional one $(u_1, \ldots, u_p)$;
- assuming that $u$ belongs to a finite set of discrete values;
- taking an infinite dimensional state space as in Somaraju et al. (2013) where the truncation to finite photon numbers is removed;
- considering continuous-time systems similar to the ones investigated in Mirrahimi and van Handel (2007);
- ensuring convergence towards a sub-space (Bolognani & Ticozzi, 2010) instead of a pure-state and thus achieving a goal similar to error correction code as already proposed in Ahn, Doherty, and Landahl (2002).

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Appendix

The following theorem is just an application of Theorem 1 in Kushner (1971, chapter 8).

**Theorem 6.** Let $X_t$ be a Markov chain on the compact state space $S$. Suppose that there exists a continuous function $V(X)$ satisfying

$$E(V(X_{t+1})|X_t) - V(X_t) = -Q(X_t),$$

where $Q(X)$ is a non-negative continuous function of $X$, then the o-limit set $\Omega$ (in the sense of almost sure convergence) of $X_t$ is contained by the following set $\mathcal{B}_0 = \{ X \mid Q(X) = 0 \}$.

**Lemma 3.** Consider the function $Q_1$ defined by (5) and $\overline{u} > 0$. Then there exists $C > 0$ such that for all $\rho$, $\beta_1$ in $\mathcal{D} \times [-\overline{u}, \overline{u}]$ satisfying $Q_1(\rho, \beta_1) = 0$, there exists $n \in \{1, \ldots, d\}$ such that $\rho_{n,n} = |\langle n | \rho | n \rangle|^2 \geq 1 - C|\beta_1|^2$.

The proof is the following. For all $n$, $\mu$, $\nu$, condition $Q_1 = 0$ implies that $p_{\mu 2}^{\nu}(n|M^{\mu}_\rho p M^{\nu}_\rho)^{-1}|n\rangle = p_{\mu 2}^{\nu}(n|M^{\mu}_\rho p M^{\nu}_\rho)^{-1}|n\rangle$. Taking the sum over all $\nu$, we get $|\langle n | M^{\mu}_\rho p M^{\nu}_\rho | n \rangle|^2 = (\rho_{\mu\nu} |\langle \nu | \rho | \nu \rangle|^2) \rho_{\mu\nu} + b_{\mu\nu}(\rho, \beta_1)$ where $\rho_{\mu\nu}$ stands for $(n_{\mu} | \rho | n_{\nu})$ and the scalar functions $b_{\mu\nu}$ depend continuously on $\rho$ and $\beta_1$. Let us finish the proof by contradiction. Assume that for all $C > 0$, there exists $(\rho^0, \beta_1^0) \in \mathcal{D} \times [-\overline{u}, \overline{u}]$ satisfying $Q_1(\rho^0, \beta_1^0) = 0$, such that $\forall n \in \{1, \ldots, d\}$, $\rho_{n,n}^0 \leq 1 - C|\beta_1^0|^2$. Take $C$ tending towards $+\infty$. Since $\rho^0$ and $\beta_1^0$ remain in a compact set, we can assume, up to some extraction process, that $\rho^0_{n,n}$ and $\beta_1^0$ converge towards $\rho_n$ and $\beta_1$ in $\mathcal{D}$ and $[-\overline{u}, \overline{u}]$. Since $|\beta_1|^2 \leq (1 - \rho_{n,n})^2/C \leq 1/C$, we have $\beta_1^0 = 0$ since

$$|c_{\mu,n}|^2 \rho_{\mu,n}^0 = \rho_{\mu,n}^0 \sum_{\nu} |c_{\mu,n}|^2 \rho_{\mu,n}^\nu + \beta_1^0 b_{\mu,n}(\rho^0, \beta_1^0),$$

(A.2)

we have by continuity for $C$ tending to $+\infty$ and for all $n$ and $\mu$:

$$|c_{\mu,n}|^2 \rho_{\mu,n}^n = \left( \sum_{\nu} |c_{\mu,n}|^2 \rho_{\mu,n}^\nu \right) \rho_{\mu,n}^n.$$  

Thus there exists $n^* \in \{1, \ldots, d\}$ such that $\rho_n = |\langle n^* | |\langle n^* | |\rangle|^2$ (see the proof of Theorem 1).

Since $\rho_{\mu,n}^n = 1$, for $C$ large enough, $\rho_{\mu,n}^n < 1/2$ and thus

$$\sum_{\nu} |c_{\mu,n}|^2 \rho_{\mu,n}^\nu = |c_{\mu,n}|^2 - \rho_{\mu,n}^n b_{\mu,n}(\rho^0, \beta_1^0).$$

Replacing (A.3) in (A.2) yields:

$$|c_{\mu,n}|^2 = |c_{\mu,n}|^2 + \beta_1^0 b_{\mu,n}(\rho^0, \beta_1^0).$$

Thus, there exists $C_0 > 0$, such that for $n \neq n^*$ and $C$ large enough $\rho_{n,n}^n \leq C_0 |\beta_1|^2$. But $\rho_{\mu,n}^n = 1 - \sum_{\nu \neq n} \rho_{\mu,n}^\nu \geq 1 - C_0 (d - 1)|\beta_1|^2$. This is in contradiction with $\rho_{\mu,n}^n \leq 1 - C|\beta_1|^2$ as soon as $C > C_0 (d - 1)$.

References


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