Euler-Lagrange Models With Complex Currents of Three-Phase Electrical Machines and Observability Issues

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Abstract—A new Lagrangian formulation with complex currents is developed and yields a direct and simple method for modeling three-phase permanent-magnet and induction machines. The Lagrangian is the sum of a mechanical one and of a magnetic one. This magnetic Lagrangian is expressed in terms of rotor angle, complex stator and rotor currents. A complexification procedure widely used in quantum electrodynamics is applied here in order to derive the Euler-Lagrange equations with complex stator and rotor currents. Such complexification process avoids the usual separation into real and imaginary parts and simplifies notably the calculations. Via simple modifications of such magnetic Lagrangians we derive new dynamical models describing permanent-magnet machines with both saturation and saliency, and induction machines with both magnetic saturation and space-harmonics. For each model we also provide its Hamiltonian thus its magnetic energy. This energy is also expressed with complex currents and can be directly used in Lyapunov and/or passivity based control. Further, we briefly investigate the observability of this class of Euler-Lagrange models, in the so-called sensorless case when the measured output is the stator current and the load torque is constant but unknown. For all the dynamical models obtained via such variational principles, we prove that their linear tangent systems are unobservable around a one-dimensional family of steady-states attached to the same constant stator voltage and current.

This negative result explains why sensorless control of three-phase electrical machines around zero stator frequency remains yet a difficult control problem.

Index Terms—Induction machine, Lagrangian with complex coordinates, magnetic saturation, permanent-magnet machine, sensorless control, space-harmonics.

I. INTRODUCTION

Modeling electrical machines with magnetic-saturation and space-harmonics effects is not a straightforward task and could lead to complicated developments when a detailed physical description is included (see, e.g., [1], [3]). Even if such effects are not dominant they could play an important role for sensorless control (no rotor position or velocity sensor). For a permanent-magnet machine, the rotor position will be unobservable without saliency. For standard models of induction machines (no magnetic saturation, no space-harmonics) the rotor velocity is always unobservable at zero stator frequency [2], [6]. A global observability analysis based on such standard models is given in [7]. In this note we develop a systematic method to include into such standard control models magnetic-saturation, saliency and/or space-harmonics effects. Our initial motivation was to see whether this non-observability is destroyed by such modelling changes or not. It appears that any physically-consistent model admits the same kind of observability deficiency at zero stator frequency.

By physically consistent models we mean Lagrangian-based models. We propose here an extension of Lagrangian modeling of three-phase machines with real variables (see, e.g., [9]) to complex electrical variables. It is directly inspired from quantum electro-dynamics where Lagrangian with complex generalized positions and velocities are widely used (see, e.g., [4, page 87]). We obtain, from such Lagrangian functions, physically consistent and synthetic Euler-Lagrange models directly expressed with complex stator and rotor currents. Such modeling method by-passes the usual detailed physical descriptions that are not easily accessible to the control community. Here we propose a much more direct way: it just consists in modifying the magnetic part of the Lagrangian directly expressed with complex currents and then in deriving the dynamic model from the Euler-Lagrange equations with complex variables. We obtain automatically the dynamics of the electrical part as a set of complex differential equations. We suggest here simple Lagrangians modeling simultaneously magnetic-saturation, saliency and space-harmonic effects. The obtained dynamics extend directly the ones used in almost all control-theoretic papers and include also more elaborate ones that can be found in specialized books such as [1], [3].

For permanent-magnet three-phase machines, the general structure of any physically consistent model including magnetic-saturation, saliency and other conservative effects is given by (4) with magnetic Lagrangian $L_m$ and Hamiltonian $H_m$ related by (5). For induction three-phase machines the physically consistent models are given by (15) where the magnetic Lagrangian $L_m$ is related to the magnetic energy $H_m$ by (16). Such synthetic formulation of the dynamical equations is new and constitutes the first contribution of this note. It provides new control models including nonlinear magnetic effects and their corresponding energies. These energy functions could be used in the future to construct controlled Lyapunov functions and/or storage functions for passivity-based feedback laws.

From a control theoretic point of view we just prove here that the severe observability difficulties encountered in sensorless control and well explained in [7] for the standard model resulting from the quadratic Lagrangian (13), remain present for models (15) where the magnetic Lagrangian is any function of the rotor angle, stator...
and rotor currents. Consequently, addition of magnetic saturations, saliency and harmonics effects, do not remove observability issues at zero stator frequency in the sensorless case (see proposition 1). This observability obstruction has neither been stated for models with magnetic-saturation and space-harmonics of three-phase machines and constitutes the second contribution of this notes. Contrarily to observability, non-observability is not a generic property and could be destroyed by generic and small changes in the equations. Since proposition 1 is based on the class of models derived from (1) or (15) with arbitrary magnetic Lagrangian $L_m$, we prove here that any physically consistent model of three-phase machines where the non-conservative effects result only from voltage supply and Ohmic losses, such non-observability holds true around zero stator frequency. This means that non-observability around zero stator frequency is robust to generic and physically consistent modifications of the equations. As far as we know this negative and physically robust result is new. It indicates that sensorless control of three-phase electrical machines around zero stator frequency cannot be just addressed via refined physical models but also requires advanced and nonlinear control techniques.

In Section II we recall the simplest model of a permanent-magnet machine and its Euler-Lagrange formulation based on the two scalar components of the complex stator current. Using the complexification procedure detailed in Appendix, we show how to use complex representation of stator-current in Lagrangian formulation of the dynamics. This leads us to the general form of physically consistent models (4). Finally we obtain, just by simple modifications of the magnetic Lagrangian, physically consistent models with magnetic saturation and saliency effects (10). Section III deals with induction machines and admits the same progression as the previous one: we start with the usual ($\alpha, \beta$) model, describe its complex Lagrangian formulation, derive physically consistent models (15) and specialize them to saturation and space-harmonics effects (21). In Section IV, we prove proposition 1 that states the main observability issues of these Euler-Lagrange models at zero stator frequency. In conclusion we show how to transpose this modelling based on complex currents associated to a Lagrangian formulation to complex fluxes associated to a Hamiltonian formulation with complex generalized positions and momentums. Appendix details the complexification procedure. It explains how to derive the Euler-Lagrange equations when some generalized positions and velocities are treated as complex quantities. Throughout the technical note, we define models in ($\alpha, \beta$) frame, using the standard transformation from three phases frame (see, e.g., [8]).

II. PERMANENT-MAGNET THREE-PHASE MACHINES

A. Usual Model and its Magnetic Energy

In the ($\alpha, \beta$) frame (total power invariant transformation), the dynamic equations read (see, e.g., [3], [8]):

\[
\begin{align*}
\frac{d}{dt}(J\dot{\theta}) &= n_p \frac{3}{2} \left( \ddot{\phi} e^{i\nu_\beta} \right) \cdot \dot{s}_\alpha - \tau_L, \\
\frac{d}{dt}(\lambda s + \ddot{\phi} e^{i\nu_\beta}) &= u_s - R_s i_s,
\end{align*}
\]

where

- $\cdot$ stands for complex-conjugation, 3 means imaginary part, $J = \sqrt{-1}$ and $n_p$ is the number of pairs of poles.
- $\theta$ is the rotor mechanical angle, $J$ and $\tau_L$ are the inertia and load torque, respectively.
- $s_\alpha \in \mathbb{C}$ is the stator current, $u_s \in \mathbb{C}$ the stator voltage.
- $\lambda = (L_d + L_s)/2$ with inductances $L_d = L_s > 0$ (no saliency here).
- The stator flux is $\phi_s = \lambda s + \ddot{\phi} e^{i\nu_\beta}$ with the constant $\ddot{\phi} > 0$ representing to the rotor flux due to permanent magnets.

The Lagrangian associated to this system is the sum of the mechanical one $L_v$ and magnetic one $L_m$ defined as follows:

\[
L_v = \frac{3}{2} \ddot{\theta}^2, \quad L_m = \frac{\lambda}{2} \left| \phi_s + \ddot{\phi} e^{i\nu_\beta} \right|^2
\]

where $\ddot{\phi} > 0$ is the permanent magnetizing current.

It is well known that (1) derives from a variational principle (see, e.g., [9]) and thus can be written as Euler-Lagrange equations with source terms corresponding to energy exchange with the environment. Consider the additional complex variable $q_{\nu_\beta} \in \mathbb{C}$ defined by $\frac{d}{dt} q_{\nu_\beta} = i_s$.

Take the Lagrangian $L = L_v + L_m$ as a real function of the generalized coordinates $q = (\theta, q_{\nu_\beta}, q_{\nu_\alpha})$ and generalized velocities $\dot{q} = (\dot{\theta}, \dot{q}_{\nu_\beta}, \dot{q}_{\nu_\alpha})$.

\[
L(q, \dot{q}) = \frac{3}{2} \ddot{\theta}^2 + \frac{\lambda}{2} \left( (q_{\nu_\beta} + \ddot{\phi} \cos \alpha)^2 + (q_{\nu_\alpha} - \ddot{\phi} \sin \alpha)^2 \right).
\]

Then the mechanical equation in (1) reads

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = -\tau_L
\]

where $-\tau_L$ corresponds to the energy exchange through the mechanical load torque. Similarly, the real part of complex and electrical equation in (1) reads $\frac{d}{dt} (\Re L(q) \dot{q}_{\nu_\beta}) - \Re (\frac{\partial L}{\partial q_{\nu_\beta}}) = u_s - R_s i_s$, and its imaginary part $\frac{d}{dt} (\Im L(q) \dot{q}_{\nu_\alpha}) = \Im \frac{\partial L}{\partial q_{\nu_\alpha}} = 0$.

The energy exchanges here are due to the power supply through the voltage $u_s$, and also to dissipation and irreversible phenomena due to stator resistance represented by the Ohm law $-R_s i_s$.

B. Euler-Lagrange Equation With Complex Current

The drawback of such Lagrangian formulation is that we have to split into real and imaginary parts the generalized complex coordinates with $q_{\nu_\beta} = q_{\nu_\beta} + j q_{\nu_\alpha}, (q_{\nu_\alpha}$ and $q_{\nu_\beta}$ real) and velocities $\dot{q}_{\nu_\beta} = i_s = \ddot{\phi} + j \ddot{\phi}$ real). We do not preserve the elegant formulation of the electrical part through complex variables and equations.

Let us apply the complexification procedure detailed in Appendix to the Lagrangian $L(\theta, q_{\nu_\beta}, q_{\nu_\alpha}; \dot{\theta}, \dot{q}_{\nu_\beta}, \dot{q}_{\nu_\alpha})$ defined in (3). The complexification process only focuses on $q_{\nu_\beta}$ and $q_{\nu_\alpha}$ by considering $L$ as a function of $(\theta, q_{\nu_\beta}, q_{\nu_\alpha}, \dot{\theta}, \dot{q}_{\nu_\beta}, \dot{q}_{\nu_\alpha})$.

\[
L(\theta, \dot{\theta}, q_{\nu_\beta}, i_s^*; \dot{\theta}, \dot{q}_{\nu_\beta}, \dot{q}_{\nu_\alpha}) = \frac{3}{2} \ddot{\theta}^2 + \frac{\lambda}{2} \left( i_s + \ddot{\phi} e^{i\nu_\beta} \right) \left( i_s^* + \ddot{\phi} e^{-i\nu_\beta} \right).
\]

Following the notations in Appendix, $n_{\nu_\beta} = 1$ with $q'_{\nu_\beta} = q_{\nu_\beta}, n'_{\nu_\alpha} = 1$ with $q'_{\nu_\alpha} = \theta, S'_{\nu_\alpha} = -\tau_L$, and $S' = u_s - R_s i_s$. Then according to (25) the usual equations (1) read:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = -\tau_L, \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_{\nu_\beta}} \right) = u_s - R_s i_s,
\]

where $\frac{\partial L}{\partial q_{\nu_\beta}} = 0$ and $\frac{\partial L}{\partial q_{\nu_\alpha}} = \frac{\partial L}{\partial q_{\nu_\alpha}} = 0$.

More generally, for any magnetic Lagrangian $L_m$ that is a real value function of $\theta, i_s$ and $s_\alpha$ and that is $2\pi/n_p$ periodic versus $\theta$, we get the general model (with saliency, saturation) of three-phase permanent-magnet machine:

\[
\frac{d}{dt} (J\dot{\theta}) = \frac{d}{dt} (L_m) - \tau_L, \quad \frac{d}{dt} (\lambda \ddot{\phi} e^{i\nu_\beta}) = u_s - R_s i_s.
\]

We recover the usual equation with $\dot{q}_{\nu_\beta} \equiv 2L_m / \ddot{\phi}$ corresponding to the stator flux and to the conjugate momentum $p'_{\nu_\beta}$ of $q'_{\nu_\beta}$ as shown in Appendix. According to (28), the Hamiltonian $H$ is the sum of two
energy: $H = H_c + H_m$. The mechanical kinetic energy $H_c = J/2 \dot{\theta}^2$ and the magnetic energy

$$H_m (\theta, \dot{\theta}, \dot{r}, \dot{s}) = \frac{\partial L_m}{\partial \dot{r}} \dot{r} + \frac{\partial L_m}{\partial \dot{s}} \dot{s} - L_m.$$  

(5)

The standard model (1) derives from a magnetic Lagrangian of the form $L_m = \lambda / 2 \| \dot{r} + e^{-2 i \theta} \|^2$ with two positive parameters, $\lambda$ and $\tau$. Its corresponding magnetic energy reads $H_m = \lambda / 2 \| \dot{r} \|^2$. We recover the usual magnetic energy $\lambda / 2 \| \dot{r} \|^2$ up to the constant magnetizing energy $\lambda / 2 \tau^2$.

In (4), many other formulations of $L_m$ are possible and depend on particular modeling issues. Usually, the dominant part of $L_m$ will be of the form $\lambda / 2 \| \dot{r} + e^{-2 i \theta} \|^2 (\lambda, \tau$ positive constants) to which correction terms that are "small" scalar functions of $(\theta, \dot{\theta}, \dot{r}, \dot{s})$ are added.

C. Saliency Models

Adding to $L_m$ the correction $-\mu / 2 \Re (i \dot{r} e^{-2 i \theta} \dot{e} m \rho)$ with $|\mu| < \lambda$ ($\Re$ means real part) provides a simple way to represent saliency phenomena while the dominant part of the magnetic Lagrangian (and thus of the dynamics) remains attached to $\lambda / 2 \| \dot{r} + e^{-2 i \theta} \|^2$. With a magnetic Lagrangian of the form

$$L_m = \frac{\lambda}{2} \left( \dot{r} + e^{-2 i \theta} \rho \right) \left( \dot{r} + e^{-2 i \theta} \rho \right) - \frac{\mu}{4} \left( \left( i \dot{r} e^{-2 i \theta} \rho \right)^2 + \left( i \dot{r} e^{-2 i \theta} \rho \right)^2 \right),$$

(6)

where $\lambda = (L_d + L_s) / 2$ and $\mu = (L_s - L_d) / 2$ (inductances $L_d > 0$ and $L_s > 0$), (4) becomes $(\lambda \tau = \tilde{\tilde{\tau}})$

$$\begin{align*}
\frac{d}{dt} (\dot{\theta}) &= n_p \lambda \left( \lambda \dot{r} + \dot{e} m \rho \dot{e} m \rho \right) - \tau_l \\
\frac{d}{dt} (\lambda \dot{r} + e^{-2 i \theta} \rho \dot{e} m \rho) &= u_s - R_r \dot{r}
\end{align*}$$

(7)

and we recover the usual model with saliency effect. In this case the magnetic energy is given by

$$H_m = \lambda / 2 \left( \| \dot{r} \|^2 - \tau^2 \right) - \mu / 4 \left( \left( i \dot{r} e^{-2 i \theta} \rho \right)^2 + \left( i \dot{r} e^{-2 i \theta} \rho \right)^2 \right).$$

(8)

D. Saturation and Saliency Models

We can also take into account magnetic saturation effects, i.e., the fact that inductances depend on the currents. Let us assume that only the inductances $\lambda$ and $\mu$ in (6) depend on the modulus $\rho = \| \dot{r} + e^{-2 i \theta} \rho \|$ of the magnetic Lagrangian now reads

$$L_m = \frac{\lambda}{2} \left( \dot{r} + e^{-2 i \theta} \rho \right) \left( \dot{r} + e^{-2 i \theta} \rho \right) - \frac{\mu}{4} \left( \left( i \dot{r} e^{-2 i \theta} \rho \right)^2 + \left( i \dot{r} e^{-2 i \theta} \rho \right)^2 \right).$$

(9)

The dynamics are given by (4) with $L_m$ defined here above. Set $\lambda' = d \lambda / d \rho$. With $\partial \lambda / \partial \theta = n_p \left( \lambda \dot{r} + \dot{e} m \rho \dot{e} m \rho \right)$ and $\partial (\lambda \lambda / \partial r) = (\dot{r} + e^{-2 i \theta} \rho) / \| \dot{r} + e^{-2 i \theta} \rho \|$ we get the following dynamical model with both saliency and magnetic-saturation effects:

$$\begin{align*}
\frac{d}{dt} (\dot{\theta}) &= n_p \lambda \left( \lambda \dot{r} + \dot{e} m \rho \dot{e} m \rho - \tau_l \right) \\
\frac{d}{dt} (\lambda \dot{r} + e^{-2 i \theta} \rho \dot{e} m \rho) &= u_s - R_r \dot{r}
\end{align*}$$

(10)

with $\Lambda = \lambda + \{ |\dot{r} + e^{-2 i \theta} \rho|^2 / 2 \} \lambda'$ and $M = \mu + \{ |\dot{r} + e^{-2 i \theta} \rho|^2 / 2 \} \mu'$. It is interesting to compute the magnetic energy $H_m$ from general formula (5)

$$H_m = \lambda / 2 \| \dot{r} \|^2 - \frac{\mu}{2} \left( \frac{\| \dot{r} + e^{-2 i \theta} \rho \|^2}{4} \right) \times \cdots \left( \left( i \dot{r} e^{-2 i \theta} \rho \right)^2 + \left( i \dot{r} e^{-2 i \theta} \rho \right)^2 \right).$$

(11)

Such magnetic energy formulae are not straightforward. They are not obtained by replacing $\lambda$ and $\mu$ in the standard magnetic energy (8) by $\Lambda$ and $M$, respectively.

III. INDUCTION THREE-PHASE MACHINES

We will now proceed as for permanent-magnet machines. Let us recall first the usual dynamical equations of an induction machine with complex stator and rotor currents. They admit the following form:

$$\begin{align*}
\frac{d}{dt} (\dot{J}_d) &= n_p \lambda \left( L_m \dot{r} e^{-2 i \theta} \rho \dot{e} m \rho \right) - \tau_l \\
\frac{d}{dt} (\lambda \dot{r} + e^{-2 i \theta} \rho \dot{e} m \rho) &= - \tau_r \\
\frac{d}{dt} (\lambda \dot{r} + e^{-2 i \theta} \rho \dot{e} m \rho) &= u_s - R_r \dot{r}
\end{align*}$$

(12)

where

- $n_p$ is the number of pairs of poles, $\theta$ is the rotor mechanical angle, $J$ and $\tau_l$ are the inertia and load torque, respectively.
- $r_s \in \mathbb{C}$ is the rotor current (in the rotor frame, different from the $(d, q)$ frame), $s = \Re \{ r_s \}$ is the stator current (in the stator frame, i.e., the $(d, q)$ frame) and $u_s \in \mathbb{C}$ is the stator voltage (in the stator frame). The stator and rotor resistances are $R_s > 0$ and $R_r > 0$.
- The inductances $L_m$, $L_f$, and $L_s$ are positive parameters with $L_f, L_s \ll L_m$.
- The stator (resp. rotor) flux is $\phi_s = L_m \left( r_s + i \dot{r} e^{-2 i \theta} \rho \right) + L_f \dot{r}_s$ (resp. $\phi_r = L_m \left( r_s + i \dot{r} e^{-2 i \theta} \rho \right) + L_f \dot{r}_s$).

A. Euler-Lagrange Equation With Complex Current

With notations of Appendix, $n' = 2$ with $q' = (r_s, r_s)$, $n'' = 1$ with $q'' = (r_s, r_s)$, and $S'' = -\tau_l$. The Lagrangian associated to (12), expressed with complex currents $r_s$ and $r_s$, reads

$$\begin{align*}
\mathcal{L}(\theta, \dot{\theta}, r_s, \dot{r}_s, \dot{r}_s, \dot{r}_s) &= \frac{J}{2} \dot{r}^2 \\
&+ \frac{L_m}{2} \left( r_s + i \dot{r} e^{-2 i \theta} \rho \right)^2 + \frac{L_f}{2} \left| \dot{r}_s \right|^2 + \frac{L_f}{2} \left| \dot{r}_s \right|^2.
\end{align*}$$

(13)

The first term $J / 2 \dot{r}^2$ represents the mechanical Lagrangian and the remaining sum the magnetic Lagrangian $L_m$. The dynamics (12) read

$$\begin{align*}
\frac{d}{dt} (\dot{J}_d) - \frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= - \tau_l \\
2 \frac{d}{dt} (\dot{r}_s) - \frac{\partial \mathcal{L}}{\partial \dot{r}_s} &= - \tau_r. \end{align*}$$

Here magnetic Lagrangian $L_m$ coincides with the magnetic energy $H_m$

$$L_m = \frac{L_m}{2} \left( r_s + i \dot{r} e^{-2 i \theta} \rho \right)^2 + \frac{L_f}{2} \left| \dot{r}_s \right|^2 + \frac{L_f}{2} \left| \dot{r}_s \right|^2.$$

(13)
More generally physically consistent model should be obtained with a Lagrangian of the form

$$L_{1m} = \frac{1}{2} \dot{\theta}^2 + L_m(\theta, \tau, s, r, i)$$

(14)

where $L_m$ is the magnetic Lagrangian expressed with the rotor angle and currents. It is $2\pi/n_p$ periodic versus $\theta$. Any physically admissible model of a three-phase induction machine reads

$$\frac{d}{dt}(\Phi) = \frac{\partial L_m}{\partial \theta} - \tau_L$$

$$2 \frac{d}{dt} \left( \frac{\partial L_m}{\partial \tau} \right) = -R_s i_s$$

$$2 \frac{d}{dt} \left( \frac{\partial L_m}{\partial i_s} \right) = u_s - R_s i_s$$

(15)

where the rotor and stator flux are given by

$$\Phi = 2 \frac{\partial L_m}{\partial \tau}, \quad \Phi_s = 2 \frac{\partial L_m}{\partial i_s}.$$

In general the magnetic energy does not coincide with $L_m$. It is given by (28) that yields

$$\frac{\partial L_m}{\partial \tau} + \frac{\partial L_m}{\partial i_s} i_s + \frac{\partial L_m}{\partial \theta} i_s - L_m = \frac{\partial L_m}{\partial \theta}.$$

(16)

B. Saturation Models

A simple way to include saturation effects is to consider that the main inductances $L_m$ appearing in (13) depends on the modulus $\rho = |i_s + x_e e^{j\omega_{ps}}|$. Thus we consider the following magnetic-saturation Lagrangian:

$$L_m = \frac{L_m}{2} \left( i_s + x_e e^{j\omega_{ps}} \right) \left( i_s + x_e e^{j\omega_{ps}} \right)^* + L_{fs} i_s + L_{fs} i_s^* - L_m.$$

(17)

Since

$$\frac{d L_m}{d \rho} = \partial L_m / \partial \theta = \partial L_m / \partial i_s = \partial L_m / \partial \tau,$$

the saturation model (formula (15) with $L_m$ given by (17)) reads

$$\frac{d}{dt}(\Phi) = \frac{\partial L_m}{\partial \theta} - \tau_L$$

$$2 \frac{d}{dt} \left( \frac{\partial L_m}{\partial \tau} \right) = -R_s i_s$$

$$2 \frac{d}{dt} \left( \frac{\partial L_m}{\partial i_s} \right) = u_s - R_s i_s$$

(18)

Now the saturation model (18) is changed as follows:

$$\frac{d}{dt}(\Phi) = \frac{\partial L_m}{\partial \theta} - \tau_L$$

$$2 \frac{d}{dt} \left( \frac{\partial L_m}{\partial \tau} \right) = -R_s i_s$$

$$2 \frac{d}{dt} \left( \frac{\partial L_m}{\partial i_s} \right) = u_s - R_s i_s.$$

(21)

Following (16), the associated magnetic energy reads then:

$$\frac{\partial L_m}{\partial \tau} + \frac{\partial L_m}{\partial i_s} i_s + \frac{\partial L_m}{\partial \theta} i_s - L_m.$$

Several space-harmonics can be included in a similar way. Moreover saturation of space-harmonics can be also tackled just by choosing $L_m$ as a function of $|i_s + x_e e^{j\omega_{ps}}|$. As far as we know such explicit models including magnetic saturation and space harmonics have never been given.

IV. OBSERVABILITY ISSUES AT ZERO STATOR FREQUENCY

The sensorless control case is characterized by a load torque $\tau_L$ constant but unknown, control inputs $u_s$ and measured outputs $i_s$. Models derived from (4) for permanent-magnet machines (resp. from (15) for inductions machines) can be always written in state-space form

$$\frac{d}{dt}X = f(X, U), \quad Y = h(X)$$

(22)

where $X = (\tau_L, \theta, \dot{\theta}, \Phi, \Phi_s)$ (resp. $X = (\tau_L, \theta, \dot{\theta}, \Phi, \Phi_s, \Phi_s)$) with $U = (U_{d}, U_{q}, U_{f_d}, U_{f_q})$. $Y = (\Phi, \Phi_s, \Phi_s)$ and $d_{\Phi}/dt = 0$. A stationary regime at zero stator frequency corresponds then to a steady state $(\vec{X}, \vec{U}, \vec{Y})$ of (22) satisfying $f(\vec{X}, \vec{U}) = 0$ and $\vec{Y} = h(\vec{X})$. The tangent linear system around this steady state is then

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx$$

(23)

where $A = \partial f/\partial X(\vec{X}, \vec{U}), B = \partial f/\partial U(\vec{X}, \vec{U})$ and $C = \partial h/\partial X(\vec{X})$. If we assume that the linearized system (23) is ob-
servable, the Kalman criteria implies that the rank of the matrix
\[
\begin{bmatrix}
C \\
A
\end{bmatrix}
\]
must be equal to \(\dim(X)\). If it is the case, the mapping
\(X \mapsto (f(X, C), h(X))\) is maximum rank around \(\bar{X}\). This maximum rank condition just means that the set of algebraic equations characterizing the steady-state from the knowledge of \(C\) and \(\bar{Y}, f(X, C) = 0\) and \(h(X) = \bar{Y}\) admits around \(\bar{X}\) the maximum rank \(\dim(X)\). Such rank is not changed by any invertible manipulations of this set of equations characterizing the steady-state from the knowledge of the input and output values, \(C\) and \(\bar{Y}\). Putting the implicit Euler-Lagrange (4) and (15) into their explicit state-space forms (22) involves such invertible manipulations.

For permanent-magnet machines described by (4), this set of equations yields the following mapping:
\[
(t_L, \theta, \dot{\theta}, i_s) \mapsto (0, \dot{\theta}, \frac{\partial L_m}{\partial \theta} - t_L, \bar{C} - R_s i_s, i_s)
\]
where \(L_m\) depends on \(\theta\) and \(i_s\). Its rank should be maximum, i.e., equal to 5. This is not the case since its rank is obviously equal to 4. For induction machines described by (15), the mapping is
\[
(t_L, \theta, \dot{\theta}, \dot{i_r}, i_r) \mapsto (0, \dot{\theta}, \frac{\partial L_m}{\partial \theta} - t_L, -R_s i_s, \bar{C} - R_s i_s, i_r)
\]
where \(L_m\) depends on \(\theta\), \(i_r\), and \(i_s\). Its rank is equal to 6 whereas the maximum rank is 7. The above arguments yield following proposition:

Proposition 1: Any dynamical model of permanent-magnet machines (4) (resp. induction machines (15)) is unobservable around zero stator frequency regime when the measured output is the stator current \(i_s\) and the load torque is constant but unknown. By unobservable we mean that:
- to any constant input and output \(\bar{u}_s\) and \(\bar{r}_s\) satisfying \(\bar{u}_s = R_s \bar{i}_s\) correspond a one dimensional family of steady states parameterized by the scalar variable \(\xi\) with
\[
- \tau_L = \frac{\partial L_m}{\partial \theta}(\xi, \bar{i}_s, \bar{r}_s), \quad \theta = \xi, \quad i_s = \bar{r}_s \quad \text{for permanen-
- \tau_L = \frac{\partial L_m}{\partial \theta}(\xi, 0, 0, \bar{i}_s, \bar{r}_s), \quad \theta = \xi, \quad i_s = 0, \quad i_r = \bar{r}_s \quad \text{for induction machines};
\]
- the linear tangent systems around such steady-states are not ob-

V. CONCLUSION

The models proposed in this note, (10) for permanent-magnet machines and (21) for induction drives, are based on variational princi-
les and Lagrangian formulation of the dynamics. Such formulations are particularly efficient to preserve the physical insight while main-
taining a synthetic view without describing all the technological and material details (see [10] for an excellent and tutorial overview of vari-
ational principles in physics). Extensions to network of machines and
generators connected via long lines can also be developed with similar variational principles and Euler-Lagrange equations with complex cur-
rents and voltages.

In this note we have put the emphasis on currents and thus La-
grangian modelling. Since flux variables are conjugated to current variables, Hamiltonian modelling is also possible when fluxes are used instead of currents. For permanent-magnet machines, the Hamiltonian counterpart of Lagrangian models (4) reads
\[
\frac{d}{dt}(J\dot{\theta}) = -\frac{\partial H_m}{\partial \theta} + \tau_L, \quad \frac{d}{dt}\phi_r = u_s - 2R_s \frac{\partial H_m}{\partial \phi_r^*}
\]
where the magnetic energy \(H_m\) is considered as a function of the rotor angle \(\theta\), the stator flux \(\phi_r\) and its complex conjugate \(\phi_r^*\). The stator current \(i_s\) corresponds then to \(2\partial H_m/\partial \phi_r^*\). For induction machines, the Hamiltonian formulation associated to (15) becomes
\[
\frac{d}{dt}(J\dot{\theta}) = -\frac{\partial H_m}{\partial \theta} + \tau_L, \quad \frac{d}{dt}\phi_r = -2R_s \frac{\partial H_m}{\partial \phi_r^*}, \quad \frac{d}{dt}\phi_r = u_s - 2R_s \frac{\partial H_m}{\partial \phi_r^*}
\]

The magnetic energy \(H_m\) now depends on \(\theta\), the rotor flux \(\phi_r\) and its complex conjugate \(\phi_r^*\), the stator flux \(\phi_s\) and its complex conjugate \(\phi_s^*\). The rotor (resp. stator) current is then given by \(2\partial H_m/\partial \phi_r^*\) (resp. \(2\partial H_m/\partial \phi_s^*\)). As for Lagrangian modelling, one can modify the magnetic energies of the standard models (1) and (12) to include, for example, magnetic-saturation or space-harmonics effects. This yields new formula expressing \(H_m\) as function of angle and fluxes. The corresponding flux-based models are then given by the above equations.

APPENDIX

Lagrangian and Hamiltonian With Complex Variables: It is explained in [4, page 87] how to use complex coordinates for Lagrangian and Hamiltonian systems. Here we propose a straightforward exten-
sion where the complexification procedure is only partial. Such exten-
sion cannot be found directly in any of the standard textbooks. In the context of electrical drives, such complexification applies only on elec-
trical quantities whereas mechanical ones remain untouched.

Assume we have a Lagrangian system with generalized positions \(q \in \mathbb{R}^n, n \geq 3\) and an analytic Lagrangian \(L(q, \dot{q})\). Let us decompose \(q\) into two sets of components:
- the first set \(q^c = (q_1, \ldots, q_{3n+c})\) with \(0 < 2n^c \leq n\) will be identified with \(n^c\) complex numbers \(q_k^c = q_{k-1} + j q_k, k = 1, \ldots, n^c\);
- the second set \(q^r = (q_{3n+c+1}, \ldots, q_n)\) gathers the \(n^r = n - 2n^c\) components that will remain untouched and real.

Thus we can identify \(q\) with \((q^c, q^r)\) where \(q^c \in \mathbb{C}^{n^c}\) and \(q^r \in \mathbb{R}^{n^r}\). Since the Lagrangian \(L\) is a real-value and analytic function, it can be seen as an analytic function of the complex variables \(q^c, q^c*, q^c, q^c*, q^r, q^r*, q^r, q^r*, q^r\) and of the real variables \(q^c\) and \(q^r\) (\(q^c*\) corresponds to the complex conjugate of \(q^c\)). This function will be denoted by \(\tilde{L}(q^c, q^c*, q^r, q^r*, q^r, q^r*, q^r)\) and is equal to \(L(q, \dot{q})\) where
\[
q = \frac{(q_1^c + q_1^c*)}{2}, \quad \frac{(q_1^c - q_1^c*)}{2j}, \ldots, \quad \frac{(q_{3n+c}^c + q_{3n+c}^c*)}{2}, \ldots, \frac{(q_{3n+c}^c - q_{3n+c}^c*)}{2j}, \ldots, q_n^r
\]
\[
\dot{q} = \frac{\dot{q}_1^c + \dot{q}_1^c*}{2}, \quad \frac{\dot{q}_1^c - \dot{q}_1^c*}{2j}, \ldots, \frac{\dot{q}_{3n+c}^c + \dot{q}_{3n+c}^c*}{2}, \ldots, \frac{\dot{q}_{3n+c}^c - \dot{q}_{3n+c}^c*}{2j}, \ldots, \frac{\dot{q}_n^r}{2j}, \ldots, \frac{\dot{q}_n^r*}{2j}, \ldots, \frac{\dot{q}_n^r*}{2j}.
\]
Let us consider the Euler-Lagrange equations
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) = \frac{\partial L}{\partial q_k} + S_k, \quad k = 1, \ldots, n
\]
where the \(S_k\)-terms correspond to non conservative energy ex-
changes with the environment. Similarly to \(q\), we decompose \(S = (s_1, \ldots, s_n)\) into \(S^c \in \mathbb{C}^{n^c}\) and \(S^r \in \mathbb{R}^{n^r}\); \(S\) is identified with \((S^c, S^r)\). We will reformulate these equations with \(\tilde{L}\) and its partial derivatives. For \(k = 2n^c + k'\) with \(k' = 1, \ldots, n^r\), they remain unchanged
\[
\frac{d}{dt} \left( \frac{\partial \tilde{L}}{\partial q_{3n+c+k'}} \right) = \frac{\partial \tilde{L}}{\partial q_{3n+c+k'}} + S_k', \quad \text{the two scalar equations corresponding to } k = 2k' + 1, \ldots, n^r \text{ yield the following single complex equation}
\]
\[
\frac{d}{dt} \left( 2\frac{\partial \tilde{L}/\partial q_{3n+c+k'}}{\partial q_{3n+c+k'}} \right) = 2\frac{\partial \tilde{L}}{\partial q_{3n+c+k'}} + S_k'' \quad \text{since we have the identities}
\]
\[
\frac{2 \partial \tilde{L}}{\partial q_{3n+c+k'}} = \frac{\partial \tilde{L}}{\partial q_{3n+c+k'}} + \frac{\partial \tilde{L}}{\partial q_{3n+c+k'}}
\]
and similarly
\[ 2 \frac{\partial \xi}{\partial \xi_{r+1}} = \left( \frac{\partial \mathbf{L}}{\partial \mathbf{q}_k} \right)_{\xi_{r+1}} = \left( \frac{\partial \mathbf{L}}{\partial \mathbf{q}_k} \right)_{\xi_{r+1}} + j \left( \frac{\partial \mathbf{L}}{\partial \mathbf{q}_k} \right)_{\xi_{r+1}}. \]

This provides the following complex formulation of the real Euler-Lagrange (24)
\[
\frac{d}{dt} \left( 2 \frac{\partial \xi}{\partial \xi_{r+1}} \right) = 2 \frac{\partial \xi}{\partial \xi_{r+1}} + S_k, \quad k = 1, \ldots, n^r. \tag{25}
\]
\[
\frac{d}{dt} \left( \frac{\partial \xi}{\partial \xi_{r+1}} \right) = \frac{\partial \xi}{\partial \xi_{r+1}} + S_k, \quad k = 1, \ldots, n^r. \tag{26}
\]

In the usual complicxification procedure ([4, page 87]) the coefficient 2 appearing in the above equations is not present. This is due to our special choice \( q_k = q_{2k} + j q_{2k+1} \) instead of the usual choice \( q_k = q_{2k+1} + j q_{2k} \). This special choice preserves the correspondence, commonly used in electrical engineering, between complex and real electrical quantities.

Let us assume that, for each \( q \), the mapping \( \dot{q} \rightarrow \frac{\partial \mathbf{L}}{\partial \mathbf{q}_k} \) is a smooth bijection. Then the Hamiltonian formulation (24) reads
\[
\frac{d}{dt} q_k = \frac{\partial \mathbf{H}}{\partial \mathbf{p}_k}, \quad \frac{d}{dt} p_k = -\frac{\partial \mathbf{H}}{\partial \mathbf{q}_k} + S_k, \quad k = 1, \ldots, n \tag{27}
\]
with \( \mathbf{H} = \mathbf{L}/\mathbf{q}_k - \mathbf{L} \) and \( p = \mathbf{L}/\mathbf{q}_k \). Let us decompose \( p \) into \( p^r \in \mathbb{C}^r \) and \( p^r \in \mathbb{R}^r \). Then \( p^r = \mathbf{L}/\mathbf{q}_k^r \) and \( p^r = \mathbf{L}/\mathbf{q}_k^r \). Simple computations yield another derivation of the Hamiltonian from \( \mathbf{L} \)
\[
\frac{\partial \mathbf{H}}{\partial \mathbf{q}_k^r} = \frac{\partial \mathbf{H}}{\partial \mathbf{q}_k^r} + \frac{\partial \mathbf{H}}{\partial \mathbf{q}_k^r} \frac{\partial \mathbf{H}}{\partial \mathbf{q}_k^r} + \frac{\partial \mathbf{H}}{\partial \mathbf{q}_k^r} - \mathbf{L} \tag{28}
\]
where \( \mathbf{H} \) denotes the Hamiltonian \( \mathbf{H} \) when is considered as a function of \( (q^r, q^*, \dot{q}^*, p^r, p^*, \dot{p}^*) \). Then (27) becomes
\[
\frac{d}{dt} q_k^r = \frac{\partial \mathbf{H}}{\partial \mathbf{p}_k}, \quad \frac{d}{dt} p_k^r = -\frac{\partial \mathbf{H}}{\partial \mathbf{q}_k} + S_k, \quad k = 1, \ldots, n^r \tag{29}
\]
\[
\frac{d}{dt} q_k^r = \frac{\partial \mathbf{H}}{\partial \mathbf{p}_k} + S_k, \quad k = 1, \ldots, n^r. \tag{30}
\]

REFERENCES


Simultaneous Design of Controllers and Instrumentation: ILQR/ILQG
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Abstract—The instrumentation, i.e., sensors and actuators, in feedback control systems often contain nonlinearities, such as saturation, deadzone, quantization, etc. Standard synthesis techniques, however, assume that the actuators and sensors are linear. This technical note is intended to modify the LQR/LQG methodology into the so-called Instrumented LQR/LQG (referred to as ILQR/ILQG), which allows for simultaneous synthesis of optimal controllers and instrumentation.

Index Terms—Linear plant/nonlinear instrumentation (LPNI).

I. INTRODUCTION

LQR/LQG is a widely used methodology for designing linear controllers for linear plants. Within this methodology, the instrumentation, i.e., actuators and sensors, are also assumed to be linear. In reality, however, the instrumentation is often nonlinear, e.g., having saturation, deadzones, quantization, etc. This leads to the so-called Linear Plant/Nonlinear Instrumentation (LPNI) system. Is it possible to extend LQR/LQG to such systems? A positive answer to this question was provided in [1], where systems with saturating actuators were considered and a methodology, referred to as SLQR/SLQG (with S standing for ‘saturating’), has been developed.

The results of [1] have been obtained using the method of stochastic linearization [2], which is a global quasi-linearization technique that reduces an LPNI system to a linear one with the instrumentation gains being functions of all systems blocks, including functional blocks and exogenous signals. The results of [1] have been extended in [3] to LPNI systems with nonlinearities in actuators and sensors simultaneously.

In [1] and [3] the instrumentation was assumed to be given prior to the controller design. The goal of this Technical Note is to develop a method for simultaneous design of controllers and instrumentation. To accomplish that, we parameterize the instrumentation by the severity of its nonlinearities, e.g., levels of saturations, steps of quantization, etc. Then, we introduce a performance index, which includes both the system behavior and the parameters of the instrumentation. Assuming that this performance index is quadratic, we derive synthesis equations for designing optimal controllers and instrumentation simultaneously.


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