ESAIM: COCV
January 2004, Vol. 10, 1–13
DOI: 10.1051/cocv:2003037

ESAIM: Control, Optimisation and Calculus of Variations

INVARIANT TRACKING *

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Abstract. The problem of invariant output tracking is considered: given a control system admitting a symmetry group G, design a feedback such that the closed-loop system tracks a desired output reference and is invariant under the action of G. Invariant output errors are defined as a set of scalar invariants of G; they are calculated with the Cartan moving frame method. It is shown that standard tracking methods based on input-output linearization can be applied to these invariant errors to yield the required "symmetry-preserving" feedback.

Mathematics Subject Classification. 53A55, 93C10, 93D25, 70Q05.

Received December 19, 2002. Revised April 2, 2003.

1. Introduction

Consider the control system

$$m_1\ddot{x}_1 = u_1$$
$$m_2\ddot{x}_2 = u_2$$

which may serve as a (simplified!) model of a two-axis machine-tool moving in a horizontal plane; here m_1 and m_2 are the masses of the axes, (x_1, x_2) are the coordinates of the position of the tool, and (u_1, u_2) are the forces applied. The goal is to build a controller such that the tool position, *i.e.*, the output $(y_1, y_2) := (x_1, x_2)$, tracks a desired reference trajectory $t \mapsto \big(y_{r1}(t), y_{r2}(t)\big)$. Obviously, this is achieved with the state feedback

$$u_1 = m_1\Big(\ddot{y}_{r1}(t) - \lambda_1\big(x_1 - y_{r1}(t)\big) - \mu_1\big(\dot{x}_1 - \dot{y}_{r1}(t)\big)\Big)$$
$$u_2 = m_2\Big(\ddot{y}_{r2}(t) - \lambda_2\big(x_2 - y_{r2}(t)\big) - \mu_2\big(\dot{x}_2 - \dot{y}_{r2}(t)\big)\Big),$$

Keywords and phrases. Symmetries, invariants, nonlinear control, output tracking, decoupling.

* *Work partially supported by the French-German cooperation program Procope and by the "Nonlinear Control Network" (E.C.'s training and mobility of researchers (TMR) contract # ERBFMRX-CT970137).*

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where $\lambda_1, \lambda_2, \mu_1, \mu_2$ are positive design parameters. Applying this feedback results in the closed-loop system

$$\ddot{x}_1 = \ddot{y}_{r1}(t) - \lambda_1\big(x_1 - y_{r1}(t)\big) - \mu_1\big(\dot{x}_1 - \dot{y}_{r1}(t)\big)$$
$$\ddot{x}_2 = \ddot{y}_{r2}(t) - \lambda_2\big(x_2 - y_{r2}(t)\big) - \mu_2\big(\dot{x}_2 - \dot{y}_{r2}(t)\big).$$

On the other hand, notice that the control system is invariant with respect to the group $SE(2)$ of planar rotations and translations: indeed, given a rotation of angle θ and a translation by a vector (a, b), the state coordinate change

$$\begin{pmatrix} X_1 \\ X_2 \\ \dot{X}_1 \\ \dot{X}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} + \begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}$$

and the invertible static feedback

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

yield the same system

$$m_1\ddot{X}_1 = U_1$$
$$m_2\ddot{X}_2 = U_2.$$

Moreover, the state transformation induces a transformation of the output by

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix},$$

which is extended on the reference output and its derivatives by

$$\begin{pmatrix} Y_{r1} \\ Y_{r2} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} y_{r1} \\ y_{r2} \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$
$$\begin{pmatrix} \dot{Y}_{r1} \\ \dot{Y}_{r2} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \dot{y}_{r1} \\ \dot{y}_{r2} \end{pmatrix}$$
$$\begin{pmatrix} \ddot{Y}_{r1} \\ \ddot{Y}_{r2} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \ddot{y}_{r1} \\ \ddot{y}_{r2} \end{pmatrix}.$$

From an engineering point of view it is very natural to require that the closed-loop system be invariant, too. But this is not the case with the feedback designed above, unless $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$, in which case the closed-loop system

$$\ddot{x} = \ddot{y}_r(t) - \lambda_2\big(x - y_r(t)\big) - \mu_2\big(\dot{x} - \dot{y}_r(t)\big),$$

where we have set $x := (x_1, x_2)$, $y := (y_1, y_2)$ and so on, transforms into

$$\ddot{X} = \ddot{Y}_r(t) - \lambda_2\big(X - Y_r(t)\big) - \mu_2\big(\dot{X} - \dot{Y}_r(t)\big).$$

This restriction on the design parameters is often implemented in practice and is sometimes called "gain alignment" in the machine-tool lingo. The main drawback, beside reducing the number of design parameters, is to debase the overall performance of the machine. Indeed the machine usually consists of a "heavy" axis carrying a "light" axis, and the "gain alignment" imposes the light axis to behave as slowly as the heavy axis.

The lack of invariance is not too surprising, because the feedback is built from the "output error" $y - y_r$ which itself is not invariant. Nevertheless, for $\dot{y}_r \neq 0$ there is an "invariant output error" defined by

$$\begin{aligned} I_1(y, y_r, \dot{y}_r) &:= \langle y - y_r, \dot{y}_r \rangle = (y_1 - y_{r1})\dot{y}_{r1} + (y_2 - y_{r2})\dot{y}_{r2} \\ I_2(y, y_r, \dot{y}_r) &:= |y - y_r, \dot{y}_r| \ = (y_1 - y_{r1})\dot{y}_{r2} - (y_2 - y_{r2})\dot{y}_{r1}, \end{aligned}$$

i.e., the error components tangent and orthogonal to the reference velocity vector.

We differentiate this new output in order to introduce a linear error dynamics (*i.e.*, to input-output linearize):

$$\begin{aligned} \dot{I}_1 &= \langle \dot{y} - \dot{y}_r, \dot{y}_r \rangle + \langle y - y_r, \ddot{y}_r \rangle \\ \dot{I}_2 &= |\dot{y} - \dot{y}_r, \dot{y}_r| + |y - y_r, \ddot{y}_r| \end{aligned}$$

$$\begin{aligned} \ddot{I}_1 &= \langle \ddot{y} - \ddot{y}_r, \dot{y}_r \rangle + 2\,\langle \dot{y} - \dot{y}_r, \ddot{y}_r \rangle + \langle y - y_r, \dddot{y}_r \rangle \\ \ddot{I}_2 &= |\ddot{y} - \ddot{y}_r, \dot{y}_r| + 2\,|\dot{y} - \dot{y}_r, \ddot{y}_r| + |y - y_r, \dddot{y}_r| \,. \end{aligned}$$

The last two equations can be solved with respect to u when $\dot{y}_{r1}^2 + \dot{y}_{r2}^2 \neq 0$:

$$\begin{pmatrix} \frac{u_1}{m_1} \\ \frac{u_2}{m_2} \end{pmatrix} = \begin{pmatrix} \ddot{y}_{r1} \\ \ddot{y}_{r2} \end{pmatrix} + \begin{pmatrix} \dot{y}_{r1} & \dot{y}_{r2} \\ \dot{y}_{r2} & -\dot{y}_{r1} \end{pmatrix} \left[\begin{pmatrix} \ddot{I}_1 \\ \ddot{I}_2 \end{pmatrix} - \begin{pmatrix} 2\,\langle \dot{y} - \dot{y}_r, \ddot{y}_r \rangle + \langle y - y_r, \dddot{y}_r \rangle \\ 2\,|\dot{y} - \dot{y}_r, \ddot{y}_r| + |y - y_r, \dddot{y}_r| \end{pmatrix} \right],$$

so we can impose the tracking error dynamics

$$\begin{aligned} \ddot{I}_1 &= -\lambda_1 I_1 - \mu_1 \dot{I}_1 \\ \ddot{I}_2 &= -\lambda_2 I_2 - \mu_2 \dot{I}_2, \end{aligned}$$

which is invariant whatever the design parameters $\lambda_1, \lambda_2, \mu_1, \mu_2$. Notice the invariant output error has the same relative degree as the original output y.

The goal of the paper is to show that the situation depicted in the example can be generalized to every "invariant" control system endowed with a "compatible" output: if output tracking can be achieved by input-output linearization, then *invariant* output tracking can also be achieved without reducing the number of design parameters. The main idea is that it is always possible to find a (local) "invariant output error" for which input-output linearization can also be used. This "invariant output error" can be explicitly computed with Cartan's *moving frame* method. Some preliminary definitions and motivating examples can be found in [11, 18].

Symmetries are important in physics and in control theory, see [1, 5, 6, 8–10, 14, 17, 19]. A related idea can be found in [2], where an "invariant" method for tracking is proposed for fully actuated mechanical systems. Nevertheless, the role of symmetries when designing tracking controllers does not seem to have been widely studied.

2. Invariant systems and compatible outputs

Definition 1. Let G be a Lie Group with identity e and Σ an open set (or more generally a manifold). A *transformation group* $(\phi_g)_{g \in G}$ on Σ is a smooth map

$$(g, \xi) \in G \times \Sigma \mapsto \phi_g(\xi) \in \Sigma$$

such that:

- $\phi_e(\xi) = \xi$ for all ξ;
- $\phi_{g_2}\big(\phi_{g_1}(\xi)\big) = \phi_{g_2 g_1}(\xi)$ for all g_1, g_2, ξ.

Notice ϕ_g is by construction a diffeomorphism on Σ for all g. The transformation group is *local* if $\phi_g(\xi)$ is defined only when g lies sufficiently near e. In this case the transformation law $\phi_{g_2}\big(\phi_{g_1}(\xi)\big) = \phi_{g_2 g_1}(\xi)$ is imposed only when it makes sense. All the results of the paper being local, since based on constant rank assumptions, we consider in the sequel only local transformation groups acting on open sets. When we say "for all g" we thus mean "for all g sufficiently near the identity e of G"; in the same way "for all ξ" usually means "for all generic ξ in Σ". We systematically use these stylistic shortcuts in order to improve readability.

Consider now the smooth control system $\dot{x} = f(x,u)$ where the state x belongs to an open subset $\mathcal{X}$ of $\mathbb{R}^n$ and the control u belongs to an open subset $\mathcal{U}$ of $\mathbb{R}^m$. Let $(\varphi_g \times \psi_g)_{g\in G}$ be the local group of transformations on $\mathcal{X} \times \mathcal{U}$ defined by

$$(X,U) = \big(\varphi_g(x), \psi_g(x,u)\big),$$

where φ_g is a local diffeomorphism and ψ_g is invertible with respect to u for all x. In other words a transformation $\varphi_g \times \psi_g$ consists of a coordinate change and a regular static state feedback.

Definition 2. The control system $\dot{x} = f(x,u)$ is *G-invariant* if for all g, x, u

$$f\big(\varphi_g(x), \psi_g(x,u)\big) = D\varphi_g(x) \cdot f(x,u).$$

The property also reads $\dot{X} = f(X,U)$, *i.e.*, the system is left unchanged by the transformation. Alternatively, we say G is a symmetry group of the system. Around a generic point, this definition is equivalent to the definitions in [8, 17].

Consider now the smooth output map $y = h(x,u) \in \mathcal{Y} \subset \mathbb{R}^m$.

Definition 3. The output $y = h(x,u)$ is *G-compatible* if there exists a transformation group $(\varrho_g)_{g\in G}$ on $\mathcal{Y}$ such that $h \circ (\varphi_g \times \psi_g) = \varrho_g \circ h$ for all g.

With $(X,U) = \big(\varphi_g(x), \psi_g(x,u)\big)$ and $Y = \varrho_g(y)$, the definition means $Y = h(X,U)$. Notice that if h is invariant, *i.e.*, $h\big(\varphi_g(x), \psi_g(x,u)\big) = h(x,u)$ for all g, x, then y trivially is a G-compatible output with $\varrho_g(y) = y$ for all g, y.

The definitions of a G-invariant system and a G-compatible output can be pictured by the commutative diagram

$$\begin{array}{ccc}
T\mathcal{X} & \xrightarrow{D\varphi_g} & T\mathcal{X} \\
f\uparrow & & f\uparrow \\
\mathcal{X}\times\mathcal{U} & \xrightarrow{\varphi_g\times\psi_g} & \mathcal{X}\times\mathcal{U} \\
h\downarrow & & h\downarrow \\
\mathcal{Y} & \xrightarrow{\varrho_g} & \mathcal{Y}
\end{array}$$

In the sequel we will manipulate also derivatives of the variables and we need to consider *prolongations* of transformation groups. The prolongation of order ν of $(\varrho_g)_{g\in G}$ is the transformation group $(\varrho_g^{[\nu]})_{g\in G}$ on $\mathcal{Y} \times (\mathbb{R}^m)^\nu$ defined by

$$\varrho_g^{[\nu]}(y, \dot{y}, \ldots, y^{(\nu)}) := \left(\varrho_g(y), \ldots, \frac{\mathrm{d}^\nu \varrho_g}{\mathrm{d}t^\nu}\left(y, \ldots, y^{(\nu)}\right)\right).$$

Recall that if the smooth map k

$$(y, \dot{y}, \ldots, y^{(\nu)}) \mapsto k(y, \dot{y}, \ldots, y^{(\nu)})$$

defined on the "jet space" $\mathcal{Y}\times(\mathbb{R}^m)^\nu$, the total derivative of k, somewhat abusively called the time derivative and denoted $\frac{\mathrm{d}k}{\mathrm{d}t}$, is the map

$$\frac{\mathrm{d}k}{\mathrm{d}t}(y,\dot y,\dots,y^{(\nu+1)}) := \partial_y k(y,\dots,y^{(\nu)})\cdot\dot y+\cdots+\partial_{y^{(\nu)}}k(y,\dots,y^{(\nu)})\cdot y^{(\nu+1)}$$

defined on the jet space $\mathcal{Y}\times(\mathbb{R}^m)^{\nu+1}$.

Likewise the total derivative along the control system $\dot x=f(x,u)$ of the map

$$(x,u,\dot u,\dots,u^{(\nu)})\mapsto k(x,u,\dot u,\dots,u^{(\nu)})$$

is the map

$$\begin{aligned}\frac{\mathrm{d}k}{\mathrm{d}t}(x,u,\dot u,\dots,u^{(\nu+1)}) := \partial_x k(x,u,\dots,u^{(\nu)})\cdot f(x,u)+\partial_u k(x,u,\dots,u^{(\nu)})\cdot\dot u\\ +\cdots+\partial_{u^{(\nu)}}k(x,u,\dots,u^{(\nu)})\cdot u^{(\nu+1)}.\end{aligned}$$

The prolongation of order ν of $(\varphi_g\times\psi_g)_{g\in G}$ is then the transformation group $(\varphi_g\times\psi_g^{[\nu]})_{g\in G}$ on $\mathcal{X}\times\mathcal{U}\times(\mathbb{R}^m)^\nu$ defined by

$$\varphi_g\times\psi_g^{[\nu]}(x,u,\dot u,\dots,u^{(\nu)}) := \Big(\varphi_g(x),\psi_g(x,u),\dots,\frac{\mathrm{d}^\nu\psi_g}{\mathrm{d}t^\nu}\big(x,u,\dots,u^{(\nu)}\big)\Big).$$

In the sequel we often use the shorthand notations $\bar y:=(y,\dots,y^{(\nu)})$, $\bar{\mathcal{Y}}:=\mathcal{Y}\times(\mathbb{R}^m)^\nu$, $\bar\varrho_g:=\varrho_g^{[\nu]}$ and so on, where ν is some large enough integer whose exact value is of no importance. With this notation, when $y=h(x,u)$ is the output of the control system $\dot x=f(x,u)$, the map

$$(\bar y,\bar y_r)\mapsto J(\bar y,\bar y_r)$$

induces the map

$$(x,\bar u,\bar y_r)\mapsto\mathcal{J}(x,\bar u,\bar y_r) := J\big(\bar h(x,\bar u),\bar y_r\big).$$

If J is invariant

$$J\big(\bar\varrho_g(\bar y),\bar\varrho_g(\bar y_r)\big)=J(\bar y,\bar y_r).$$

If moreover $\dot x=f(x,u)$ is G-invariant and $y=h(x,u)$ is G-compatible, then

$$\mathcal{J}\Big(\varphi_g(x),\psi_g\big(x,\alpha(x,\bar y_r)\big),\bar\varrho_g(y_r)\Big)=\mathcal{J}\big(x,\alpha(x,\bar y_r),\bar y_r\big).$$

3. Invariant output errors

In general the "usual" error $y-y_r$, where y_r corresponds to the reference output trajectory, is not invariant, *i.e.*, $\varrho_g(y)-\varrho_g(y_r)\neq y-y_r$. Hence, a controller based on this error will not yield closed-loop invariance. The key ingredient for invariant tracking is to build the controller from an "invariant output error" obtained from a suitable combination of y, y_r, and derivatives of y_r. We have introduced the obvious transformation group $(\varrho_g)_{g\in G}$ on $\mathcal{Y}_r:=\mathcal{Y}$, and will also consider the prolongation of $(\varrho_g)_{g\in G}$ on $\bar{\mathcal{Y}}_r$.

Definition 4. The smooth map $(y,\bar y_r)\mapsto I(y,\bar y_r)$ is an *invariant output error* if

- the map $y\mapsto I(y,\bar y_r)$ is invertible for all $\bar y_r$;
- $I(y_r,\bar y_r)=0$ for all $\bar y_r$;
- $I\big(\varrho_g(y),\bar\varrho_g(\bar y_r)\big)=I(y,\bar y_r)$ for all $g,y,\bar y_r$.

The first property in the definition ensures that if $y = h(x,u)$ is an output with m independent components, then also $I\big(h(x,u), \bar{y}_r\big)$ is an output with m independent components; together with the second property, it means I is an "output error" in the sense it is zero if and only if the actual output y lies on the reference output y_r. The third property, which also reads $I(Y, \bar{Y}_r) = I(y, \bar{y}_r)$, means I is invariant under the action of the transformation group $(\varrho_g)_{g\in G}$.

The following theorem ensures the existence of a (local) invariant error under simple regularity conditions. The proof is constructive and relies on the Cartan moving frame method.

Theorem 1. *Assume the prolonged transformation group* $(\bar{\varrho}_g)_{g\in G}$ *is such that* $\partial_g \bar{\varrho}_g$ *has full rank* $r := \dim G$ *at the point* $(e, \bar{y}_r^0) \in G \times \bar{\mathcal{Y}}_r$. *Then there exists an invariant output error (defined locally around* $(y_r^0, \bar{y}_r^0)$*).*

Proof. The result is an application of the *moving frame* method. We follow the very nice presentation of [15](Th. 8.25). Consider the transformation group $(\phi_g)_{g\in G}$ on $\Sigma \subset \mathbb{R}^s$ and assume $\partial_g \phi_g$ has full rank $r := \dim G$ at the point $(e, \xi^0) \in G \times \Sigma$. We can then split ϕ_g into (ϕ_g^a, ϕ_g^b) with respectively r and $s - r$ components so that ϕ_g^a is invertible with respect to g around (e, ξ^0). The *normalization equations* are obtained by setting

$$\phi_g^a(\xi) = c,$$

with c a constant in the range of ϕ. The implicit function theorem ensures the existence of the local solution $g = \gamma(\xi)$ (the map $\gamma : \Sigma \to G$ is known as the *moving frame*). Finally, we get a complete set I of $s - r$ functionally independent invariants by plugging $g = \gamma(\xi)$ into the remaining components,

$$J(\xi) := \phi^b_{\gamma(\xi)}(\xi).$$

The invariance property means $J\big(\phi_g(\xi)\big) = J(\xi)$ for all g, ξ.

In our case $\Sigma = \mathcal{Y} \times \bar{\mathcal{Y}}_r$, and $\phi_g = \varrho_g \times \bar{\varrho}_g$ is the composite transformation

$$\varrho_g \times \bar{\varrho}_g(y, \bar{y}_r) := \big(\varrho_g(y), \bar{\varrho}_g(\bar{y}_r)\big).$$

By the rank assumption we can split $\varrho_g \times \bar{\varrho}_g$ into $(\bar{\varrho}_g^a, \varrho_g \times \bar{\varrho}_g^b)$ with respectively r and $m(\nu+1) - r$ components so that $\bar{\varrho}_g^a$ is invertible with respect to g around $(e, \bar{y}_r^0)$. The r normalization equations

$$\bar{\varrho}_g(\bar{y}_r) = c$$

can then be solved into $g = \gamma(\bar{y}_r)$, and plugged into the remaining equations to yield the complete set of $m(\nu+1) - r$ functionally independent invariants

$$J(y, \bar{y}_r) := \varrho_{\gamma(\bar{y}_r)}(y)$$
$$J_r(\bar{y}_r) := \bar{\varrho}^b_{\gamma(\bar{y}_r)}(\bar{y}_r).$$

The required invariant output error is then given by

$$I(y, \bar{y}_r) := J(y, \bar{y}_r) - J(y_r, \bar{y}_r).$$

Notice any combination $\mathcal{J}(J, J_r)$ with full rank with respect to J also leads to the invariant output error

$$I(y, \bar{y}_r) := \mathcal{I}\big(J(y, \bar{y}_r), J_r(y, \bar{y}_r)\big) - \mathcal{I}\big(J(y_r, \bar{y}_r), J_r(y_r, \bar{y}_r)\big).$$

□

Remark. The rank assumption in the theorem (which means the prolonged action of G is regular and free) can be weakened: it is in fact enough to assume that $\partial_g \varrho_g^{[\nu]}$ has constant rank $r' < r$ at the point $(e, y_r^0, \ldots, y_r^{(\nu)0}) \in$

$G\times\mathcal{Y}_r\times(\mathbb{R}^m)^\nu$ for ν such that $m(\nu+1)>r'$. Indeed, it is then possible to solve the normalization equations for r' parameters and go along with the proof, the rank assumption meaning the remaining $r-r'$ parameters do not explicitly occur in the final formulae, see the remark after [15] (Th. 8.25) for more details.

Example 1. We compute the invariant output error for $SE(2)$ discussed in the introduction according to the proof of the theorem. We find

$$\begin{aligned}Y_1&=y_1\cos\theta-y_2\sin\theta+a\\Y_2&=y_1\sin\theta+y_2\cos\theta+b\end{aligned}$$

$$\begin{aligned}Y_{r1}&=y_{r1}\cos\theta-y_{r2}\sin\theta+a\\Y_{r2}&=y_{r1}\sin\theta+y_{r2}\cos\theta+b\end{aligned}$$

$$\begin{aligned}\dot{Y}_{r1}&=\dot{y}_{r1}\cos\theta-\dot{y}_{r2}\sin\theta\\\dot{Y}_{r2}&=\dot{y}_{r1}\sin\theta+\dot{y}_{r2}\cos\theta,\end{aligned}$$

and the last four equations have full rank with respect to the three group parameters θ,a,b. We normalize by choosing $Y_{r1}=0$, $Y_{r2}=0$, $\dot{Y}_{r1}=0$, which yields

$$\begin{aligned}\sin\theta&=\frac{\dot{y}_{r1}}{\sqrt{\dot{y}_{r1}^2+\dot{y}_{r2}^2}}\\\cos\theta&=\frac{\dot{y}_{r2}}{\sqrt{\dot{y}_{r1}^2+\dot{y}_{r2}^2}}\\a&=\frac{y_{r2}\dot{y}_{r1}-y_{r1}\dot{y}_{r2}}{\sqrt{\dot{y}_{r1}^2+\dot{y}_{r2}^2}}\\b&=-\frac{y_{r1}\dot{y}_{r1}+y_{r2}\dot{y}_{r2}}{\sqrt{\dot{y}_{r1}^2+\dot{y}_{r2}^2}}.\end{aligned}$$

We plug these values into the remaining equations to find the complete set of invariants

$$\begin{aligned}Y_1=y_1\cos\theta-y_2\sin\theta+a&=\frac{(y_1-y_{r1})\dot{y}_{r2}-(y_2-y_{r2})\dot{y}_{r1}}{\sqrt{\dot{y}_{r1}^2+\dot{y}_{r2}^2}}\\Y_2=y_1\sin\theta+y_2\cos\theta+b&=\frac{(y_1-y_{r1})\dot{y}_{r1}+(y_2-y_{r2})\dot{y}_{r2}}{\sqrt{\dot{y}_{r1}^2+\dot{y}_{r2}^2}}\\\dot{Y}_{r2}=\dot{y}_{r1}\sin\theta+\dot{y}_{r2}\cos\theta\quad&=\sqrt{\dot{y}_{r1}^2+\dot{y}_{r2}^2}.\end{aligned}$$

Multiplying the first two invariants by the third one we get the invariant output error used in the introduction:

$$\begin{aligned}I_1&=Y_2\dot{Y}_{r2}=\langle y-y_r,\dot{y}_r\rangle\\I_2&=Y_1\dot{Y}_{r2}=|y-y_r,\dot{y}_r|\,.\end{aligned}$$

Example 2. For $SL(2)$ acting on a scalar output according to the projective map

$$Y=\frac{ay+b}{cy+d},\qquad ad-bc=1,$$

an invariant output error is

$$I(y,y_r,\dot{y}_r,\ddot{y}_r):=\frac{2\dot{y}_r(y-y_r)}{\ddot{y}_r(y-y_r)+2\dot{y}_r^2}.$$

Indeed,

$$\begin{aligned}(cy+d)Y &= ay+b\\(cy_r+d)Y_r &= ay_r+b\\c\dot y_rY_r+(cy_r+d)\dot Y_r &= a\dot y_r\\c\ddot y_rY_r+2c\dot y_r\dot Y_r+(cy_r+d)\ddot Y_r &= a\ddot y_r,\end{aligned}$$

and the last three equations have full rank with respect to three independent group parameters (the four parameters a, b, c, d satisfy $ad-bc=1$). We normalize by setting for instance $Y_r=0$, $\dot Y_r=1$ and $\ddot Y_r=0$ (notice $\dot Y_r$ cannot be normalized to 0 otherwise the rank is not full), which yields

$$\begin{aligned}b &= -ay_r\\c &= a\frac{\ddot y_r}{2\dot y_r}\\d &= a\left(\dot y_r-\frac{y_r\ddot y_r}{2\dot y_r}\right).\end{aligned}$$

Plugging these values in the remaining equation, we find the invariant output error

$$Y=\frac{ay+b}{cy+d}=\frac{2\dot y_r(y-y_r)}{\ddot y_r(y-y_r)+2\dot y_r^2}.$$

4. Invariant tracking by static state feedback

Definition 5. Let $\dot x=f(x,u)$ be a G-invariant system with a G-compatible output $y=h(x,u)$, and $t\mapsto y_r(t)$ be a smooth reference trajectory. The static state feedback $u=\alpha\big(x,\bar y_r(t)\big)$ is a *G-invariant tracking controller* if:

- for every solution of the closed-loop system $\dot x=f\big(x,\alpha(x,\bar y_r(t))\big)$ defined on $[0,+\infty[$, the output $y(t)=h(x(t),\alpha(x,\bar y_r(t)))$ tends to $y_r(t)$ as t tends to $+\infty$;
- $\alpha\big(\varphi_g(x),\bar\varrho_g(\bar y_r)\big)=\alpha(x,\bar y_r)$ for all $g, x, \bar y_r$.

A G-invariant tracking controller applied to a G-invariant system yields a G-invariant closed-loop system in the sense that for all g, x, t

$$f\Big(\varphi_g(x),\alpha\big(\varphi_g(x),\bar\varrho_g(\bar y_r(t))\big)\Big)=D\varphi_g(x)\cdot f\big(x,\alpha(x,\bar y_r(t))\big).$$

The following result states that input-output decoupling by static state feedback yields an invariant tracking controller, provided it is built from an invariant output error.

Theorem 2. *Let $\dot x=f(x,u)$ be a G-invariant system with a G-compatible output $y=h(x,u)$, and $t\mapsto y_r(t)$ be a smooth reference trajectory. Assume the output has a well-defined relative degree, and that the prolonged transformation group $(\bar\varrho_g)_{g\in G}$ is such that $\partial_g\bar\varrho_g$ has full rank $r:=\dim G$ at the point $(e,\bar y_r^0)\in G\times\bar{\mathcal{Y}}_r$.*

Then there is an invariant output error with the same well-defined relative degree, and the feedback obtained by input-output linearization is a G-invariant tracking controller.

Recall the *relative degree* of the output $z=k(x,u,\bar z_r)\in\mathbb{R}^m$ is the m-tuple $(\sigma_1,\dots,\sigma_m)$ such that the i-th component z_i of z depends on u only at the σ_i-th time differentiation (see for instance [7, 13] for details), in other words, for $i=1,\dots,m$,

$$\begin{aligned}\partial_u\frac{\mathrm{d}^\nu z_i}{\mathrm{d}t^\nu}&=0, \qquad \nu=0,\dots,\sigma_i-1\\\partial_u\frac{\mathrm{d}^{\sigma_i} z_i}{\mathrm{d}t^\sigma}&\neq 0.\end{aligned}$$

Moreover, the relative degree is *well-defined* if the so-called *decoupling matrix*

$$\partial_u \begin{pmatrix} \dfrac{\mathrm{d}^{\sigma_1} z_1}{\mathrm{d}t^{\sigma_1}} \\ \vdots \\ \dfrac{\mathrm{d}^{\sigma_m} z_m}{\mathrm{d}t^{\sigma_m}} \end{pmatrix}$$

is (generically) invertible.

Proof. The rank assumption implies

$$\mathcal{I}(x,u,\bar{y}_r) := I\big(h(x,u),\bar{y}_r\big) = \varrho_{\gamma(\bar{y}_r)}(y) - \varrho_{\gamma(\bar{y}_r)}(y_r)$$

is an invariant output error, see the proof of Theorem 1. Moreover, $y = h(x,u)$ is G-compatible,

$$\varrho_g\big(h(x,u)\big) = h\big(\varphi_g(x), \psi_g(x,u)\big)$$

and, since the relative degree is not affected by a coordinate change and a regular static feedback, $\varrho_g \circ h$ obviously has the same well-defined relative degree as h. As a consequence, $\mathcal{I}$ has the same well-defined relative degree, too.

Since the relative degree is well-defined, we can input-output linearize the system by static feedback with $\mathcal{I}$ as the output. This means we may choose the static state feedback $u := \alpha(x,\bar{y}_r)$ so that

$$\frac{\mathrm{d}^{\sigma_i}\mathcal{I}_i}{\mathrm{d}t^{\sigma_i}}\big(x,\alpha(x,\bar{y}_r),\bar{y}_r\big) = \sum_{j=0}^{\sigma_i-1} \lambda_i^j \frac{\mathrm{d}^j\mathcal{I}_i}{\mathrm{d}t^j}(x,\bar{y}_r), \qquad i = 1,\dots,m,$$

where the λ_i^j are design parameters.

On the one hand, using $\varphi(x)$ and $\bar{\varrho}_g(y_r)$ instead of x and $\bar{y}_r$, we of course have

$$\frac{\mathrm{d}^{\sigma_i}\mathcal{I}_i}{\mathrm{d}t^{\sigma_i}}\Big(\varphi_g(x),\alpha\big(\varphi_g(x),\bar{\varrho}_g(y_r)\big),\bar{\varrho}_g(y_r)\Big) = \sum_{j=0}^{\sigma_i-1} \lambda_i^j \frac{\mathrm{d}^j\mathcal{I}_i}{\mathrm{d}t^j}\big(\varphi_g(x),\bar{\varrho}_g(y_r)\big).$$

On the other hand invariance of the $\frac{\mathrm{d}^j\mathcal{I}_i}{\mathrm{d}t^j}$ implies

$$\begin{aligned} \frac{\mathrm{d}^{\sigma_i}\mathcal{I}_i}{\mathrm{d}t^{\sigma_i}}\Big(\varphi_g(x),\psi_g\big(x,\alpha(x,\bar{y}_r)\big),\bar{\varrho}_g(y_r)\Big) &= \frac{\mathrm{d}^{\sigma_i}\mathcal{I}_i}{\mathrm{d}t^{\sigma_i}}\big(x,\alpha(x,\bar{y}_r),\bar{y}_r\big) \\ &= \sum_{j=0}^{\sigma_i-1} \lambda_i^j \frac{\mathrm{d}^j\mathcal{I}_i}{\mathrm{d}t^j}(x,\bar{y}_r) \\ &= \sum_{j=0}^{\sigma_i-1} \lambda_i^j \frac{\mathrm{d}^j\mathcal{I}_i}{\mathrm{d}t^j}\big(\varphi_g(x),\bar{\varrho}_g(y_r)\big). \end{aligned}$$

Comparing the two expressions, and since the decoupling matrix is invertible, we conclude

$$\alpha\big(\varphi_g(x),\bar{\varrho}_g(y_r)\big) = \psi_g\big(x,\alpha(x,\bar{y}_r)\big),$$

i.e., the feedback $u := \alpha(x,\bar{y}_r)$ is invariant. □

The result means that not only the closed-loop error dynamics is invariant, but also the remaining part of the closed-loop dynamics (the so-called internal or zero dynamics).

5. Invariant tracking for general invertible systems

If the system $\dot{x} = f(x,u)$ together with the G-compatible output $y = h(x,u)$ has a singular decoupling matrix but is nevertheless invertible, the results in the previous section can be extended by using *time-varying* static feedbacks (or *quasi-static* feedbacks, see [3]). Indeed (see [4,7,12,16]), there exists a set of m positive numbers $n_1,\dots,n_m$ such that if we introduce

$$v_j := y_j^{(n_j)} = \beta_j(x,\bar{u}), \quad j = 1,\dots,m = \dim y$$

this relation can be solved for u and its derivatives:

$$u^{(k)} = \alpha_j(x,\bar{v}), \; k > 0.$$

Therefore, introducing

$$v_j = y_j^{(n_j)} = y_{r,j}^{(n_j)}(t) - \sum_{k=0} \lambda_{j,k}(y_j^{(k)} - y_{r,j}^{(k)}(t)), \quad j = 1,\dots,m = \dim y$$

we are back in the situation of a *time-varying* static state feedback as discussed in the previous section. As a result, basing the feedback design on an invariant error instead of y yields invariant tracking. The class of feedbacks (regular *time-varying* static state feedback) is the same as before, the only difference is in the way how the feedback is introduced. This can be summed up in the following result:

Theorem 3. *Let $\dot{x} = f(x,u)$ be an invertible G-invariant system with a G-compatible output $y = h(x,u)$, and $t \mapsto y_r(t)$ be a smooth reference trajectory. Assume that the prolonged transformation group $(\bar{\varrho}_g)_{g\in G}$ is such that $\partial_g \bar{\varrho}_g$ has full rank $r := \dim G$ at the point $(e, \bar{y}_r^0) \in G \times \bar{\mathcal{Y}}_r$.*

Then there is an invariant output error such that the system is invertible also w.r.t. this invariant output error, and the feedback obtained by input-output linearization is a G-invariant tracking controller.

Remark. Notice that the result of Theorem 2 is obtained if $(n_1,\dots,n_m)$ coincides with the relative degree.

Proof. The proof mimics the proof of Theorem 2. Instead of the relative degree now the set $(n_1,\dots,n_m)$ is considered, and the $u^{(j)} := \alpha_j(x,\bar{y}_r)$ are substituted in the derivatives of the $\mathcal{I}_j$, as well as the prolongation of ψ_g. □

Again, the result means that both the closed-loop error dynamics and the so-called internal or zero dynamics are invariant.

Example 3. Consider the planar motion of a rigid body. The dynamical system model in an inertial Cartesian frame reads

$$\ddot{y}_1 = u_1 \cos\theta + g_1, \qquad \ddot{y}_2 = u_1 \sin\theta + g_2, \qquad \ddot{\theta} = u_2$$

where (y_1,y_2) is the position of the so-called center of percussion, θ describes the orientation of the body, and (g_1,g_2) the gravitational acceleration. The inputs are the linear and the rotational acceleration. A state representation is easily introduced with $x = (y_1,\dot{y}_1,y_2,\dot{y}_2,\theta,\dot{\theta})$.

With

$$\vec{Y} = (y_1,y_2), \; \vec{\tau} = (\cos\theta,\sin\theta), \; \vec{\nu} = (-\sin\theta,\cos\theta), \; \vec{g} = (g_1,g_2), \; \omega = \dot{\theta}$$

a coordinate free representation of the model reads

$$\ddot{\vec{Y}} = u_1\vec{\tau} + \vec{g}, \; \ddot{\theta} = u_2, \; \dot{\vec{\tau}} = \dot{\theta}\vec{\nu}, \; \dot{\vec{\nu}} = -\dot{\theta}\vec{\tau}.$$

Obviously, there is an *invariance* under changes of the reference frame (planar rotations and translations). For any $(a, b, \alpha) \in \mathbb{R}^3$, the transformation (a subgroup of $SE(2) \times SO(2) \times \mathrm{id}$)

$$\begin{aligned}
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} Y_1 \cos\alpha - Y_2 \sin\alpha + a \\ Y_1 \sin\alpha + Y_2 \cos\alpha + b \end{pmatrix}, \quad \theta = \Theta - \alpha, \\
\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} &= \begin{pmatrix} G_1 \cos\alpha - G_2 \sin\alpha \\ G_1 \sin\alpha + G_2 \cos\alpha \end{pmatrix}, \quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}
\end{aligned}$$

leaves the representation unchanged:

$$\ddot{Y}_1 = U_1 \cos\Theta + G_1, \quad \ddot{Y}_2 = U_1 \sin\Theta + G_2, \quad \ddot{\Theta} = U_2.$$

The same holds for the state representation with state $(y_1, \dot{y}_1, y_2, \dot{y}_2, \theta, \dot{\theta}, g_1, g_2)$, input (u_1, u_2), and output $y = (y_1, y_2)$.

Differentiate the output equations:

$$\begin{aligned}
\ddot{y}_1 &= u_1 \cos\theta + g_1 \\
y_1^{(3)} &= \dot{u}_1 \cos\theta - u_1 \dot{\theta} \sin\theta \\
y_1^{(4)} &= \ddot{u}_1 \cos\theta - 2\dot{u}_1 \dot{\theta} \sin\theta - u_1 \dot{\theta}^2 \cos\theta - u_1 u_2 \sin\theta \\
\ddot{y}_2 &= u_1 \sin\theta + g_2 \\
y_2^{(3)} &= \dot{u}_1 \sin\theta + u_1 \dot{\theta} \cos\theta \\
y_2^{(4)} &= \ddot{u}_1 \sin\theta + 2\dot{u}_1 \dot{\theta} \cos\theta - u_1 \dot{\theta}^2 \sin\theta + u_1 u_2 \cos\theta.
\end{aligned}$$

Introducing $v_1 = \ddot{y}_1$ and $v_2 = y_2^{(4)}$ (exact) feedback linearization is achieved. Iff $\cos\theta \neq 0$ and $u_1 \neq 0$, the preceding equations can be solved as

$$u_1 = \varphi_1(\theta, v_1), \quad u_2 = \varphi_2(\theta, \dot{\theta}, v_1, \dot{v}_1, \ddot{v}_1, v_2)$$

and exponentially stable tracking error dynamics may be obtained by injection of

$$\begin{aligned}
v_1 &= \ddot{y}_{r1} - k_{1,1}\,(\dot{y}_1 - \dot{y}_{r1}) - k_{1,0}\,(y_1 - y_{r1}) \\
v_2 &= y_{r2}^{(4)} - k_{2,3}\,(y_2^{(3)} - y_{r2}^{(3)}) - \cdots - k_{2,0}\,(y_2 - y_{r2}).
\end{aligned}$$

However, this design bears two problems: on the one hand, one is confronted with singularities of two different types (at $\cos\theta = 0$ and $u_1 = 0$), and, on the other hand, the invariance property is lost!

Both, the loss of invariance and the singularity at $\cos\theta = 0$, can be tackled by introducing a moving frame. To this end, two independent scalar invariant output errors are introduced by projection on a frame moving with $\vec{Y}_r = (y_{r1}, y_{r2})$:

$$e_\tau = \langle (\vec{Y} - \vec{Y}_r), \vec{\tau}_r \rangle, \quad e_\nu = \langle (\vec{Y} - \vec{Y}_r), \vec{\nu}_r \rangle.$$

The use of these invariant errors eliminates the singularities at $\theta = \pm\pi/2$. In order to see this, consider the implicit mathematical model again. Differentiations of the defining equations yield

$$\dot{e}_\tau = \left\langle \frac{\mathrm{d}}{\mathrm{d}t}(\vec{Y} - \vec{Y}_r), \vec{\tau}_r \right\rangle + \dot{\theta}_r \left\langle (\vec{Y} - \vec{Y}_r), \vec{\nu}_r \right\rangle = \left\langle \frac{\mathrm{d}}{\mathrm{d}t}(\vec{Y} - \vec{Y}_r), \vec{\tau}_r \right\rangle + \dot{\theta}_r e_\nu$$
$$\dot{e}_\nu = \left\langle \frac{\mathrm{d}}{\mathrm{d}t}(\vec{Y} - \vec{Y}_r), \vec{\nu}_r \right\rangle - \dot{\theta}_r \left\langle (\vec{Y} - \vec{Y}_r), \vec{\tau}_r \right\rangle = \left\langle \frac{\mathrm{d}}{\mathrm{d}t}(\vec{Y} - \vec{Y}_r), \vec{\nu}_r \right\rangle - \dot{\theta}_r e_\tau$$
$$\ddot{e}_\tau = \left\langle \frac{\mathrm{d}}{\mathrm{d}t}^2 (\vec{Y} - \vec{Y}_r), \vec{\tau}_r \right\rangle + 2\dot{\theta}_r \left\langle \frac{\mathrm{d}}{\mathrm{d}t}(\vec{Y} - \vec{Y}_r), \vec{\nu}_r \right\rangle + \ddot{\theta}_r e_\nu - \dot{\theta}_r^2 e_\tau$$

and, therefore,

$$\ddot{e}_\tau = u_1 \langle \vec{\tau}, \vec{\tau}_r \rangle - u_{r1} + \ddot{\theta}_r e_\nu + \dot{\theta}_r^2 e_\tau + 2\dot{\theta}_r \dot{e}_\nu.$$

For simplicity of notations, introduce the *orientation error* δ as the angle between the axes of the reference frame and the corresponding axes of the actual frame, *i.e.*, $\langle \vec{\tau}, \vec{\tau}_r \rangle = \cos\delta$. Now $v_1 = \ddot{e}_\tau$ can be solved for u_1 if the orientation error $\delta = \theta - \theta_r \neq \pm\pi/2$. Analogous calculations for e_ν and its derivatives yield an equation of the form $e_\nu^{(4)} = u_2 u_1 \cos\delta + \cdots$ which, away from $u_1 = 0$, can be solved for u_2. Therefore, choosing

$$0 = \ddot{e}_\tau - k_{1,1}\, \dot{e}_\tau - k_{1,0}\, e_\tau$$
$$0 = e_\nu^{(4)} - k_{2,3}\, e_\nu^{(3)} - \cdots - k_{2,0}\, e_\nu$$

and solving for u completes the computation of the invariant feedback.

Remark. This moving frame approach should be compared with analogous results on the nonholonomic car in [18, 20]: while the orientation of the moving frame is defined by the velocity (tangent of $\vec{\mathcal{C}}_r$) for the car, here it follows from the acceleration $\ddot{\vec{Y}}_r - \vec{g}$ (related to the curvature of $\vec{\mathcal{C}}_r$).

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