# On invariant asymptotic observers

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**Abstract**

For dynamics $\dot{x} = f(x)$ with output $y = h(x)$ invariant with respect to a transformation group $G$, we define invariant asymptotic observer of the form $\dot{\hat{x}} = \hat{f}(\hat{x}, y)$ where $y = h(x)$ is the measured output and $\hat{x}$ an estimation of the unmeasured state $x$. Such a definition is motivated by a class of chemical reactors, treated in details, when the group of transformations corresponds to unit changes and the output $y$ to ratio of concentrations. We propose a constructive method that guaranties automatically the observer invariance $\dot{\hat{x}} = \hat{f}(\hat{x}, y)$: it is based on invariant vector fields and scalar functions, called invariant estimation errors, that can be computed via Darboux-Cartan moving frame methods. The observer convergence remains, in the general case, an open problem. But for the class of chemical reactors considered here, the invariant observer convergence is proved by showing that, in a Killing metric associated to the action of $G$, the symmetric part of the Jacobian matrix $\partial \hat{f}/\partial \hat{x}$ is definite negative (contraction).
**Key words**: asymptotic observers, moving-frame method, invariant, symmetries, contraction, chemical reactors.

## 1 Introduction

In this paper we show how to exploit symmetry for the design of asymptotic observer for nonlinear systems. The main contribution of the paper is to introduce the notion of invariant estimation errors and to construct them via the Darboux-Cartan moving-frame method. Although we do not have general results on the convergence of such invariant design techniques, we are able to prove global asymptotic convergence of such invariant observers for a class of chemical reactors.

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Let us first consider the following simple example of a continuous stirred tank of volume $V$ with two concentrations $c_1$ an $c_2$ and one measured output $y$:

$$
\begin{aligned}
V\frac{d}{dt}c_1 &= F(c_1^{in} - c_1) \\
V\frac{d}{dt}c_2 &= F(c_2^{in} - c_2) \\
y &= \frac{c_1}{c_1 + c_2}
\end{aligned}
\tag{1}
$$

($F$ the input flow rate, $(c_1^{in}, c_2^{in})$ input concentrations). This system is invariant under the action of the scaling group. These balance equations do not depend on the fact that the concentrations $(c_1, c_2)$ are expressed in $g/L$ or in $mol/L$. This means that for any positive constants $M_1$ and $M_2$ and any scaling, $C_1 = M_1 c_1$, $C_2 = M_2 c_2$, $C_1^{in} = M_1 c_1^{in}$ and $C_2^{in} = M_2 c_2^{in}$, the equation remains unchanged

$$
\begin{aligned}
V\frac{d}{dt}C_1 &= F(C_1^{in} - C_1) \\
V\frac{d}{dt}C_2 &= F(C_2^{in} - C_2).
\end{aligned}
$$

Notice that such scaling yields for $y$ to the following transformation

$$
Y = C_1/(C_1 + C_2) = y/(y + \frac{M_2}{M_1}(1 - y)).
$$

Let us consider the following observer where $\hat{c}_1$, $\hat{c}_2$ and $\hat{y}$ are estimates of $c_1$, $c_2$ and $y$, respectively and $k$ is a design parameter:

$$
\begin{aligned}
V\frac{d}{dt}\hat{c}_1 &= F(c_1^{in} - \hat{c}_1) - k\hat{c}_1 \log\left(\frac{\hat{y}}{1-\hat{y}}\frac{1-y}{y}\right) \\
V\frac{d}{dt}\hat{c}_2 &= F(c_2^{in} - \hat{c}_2) - k\hat{c}_2 \log\left(\frac{1-\hat{y}}{\hat{y}}\frac{y}{1-y}\right) \\
\hat{y} &= \frac{\hat{c}_1}{\hat{c}_1 + \hat{c}_2}.
\end{aligned}
\tag{2}
$$

Simple computations show that this dynamics is also invariant with respect to the same scaling group, since the error term becomes

$$
\frac{\hat{Y}}{1-\hat{Y}}\frac{1-Y}{Y} = \frac{\hat{C}_1}{\hat{C}_2}\frac{C_2}{C_1} = \frac{\hat{c}_1}{\hat{c}_2}\frac{c_2}{c_1} = \frac{\hat{y}}{1-\hat{y}}\frac{1-y}{y}
$$

If, instead of $\log\left(\frac{\hat{y}}{1-\hat{y}}\frac{1-y}{y}\right)$, we use an error term of the form $(\hat{y}-y)$, then we lose invariance. That observation has motivated this paper. Indeed, the error term $(\hat{y} - y)$ does not have a physical meaning since we are comparing two molar or mass fractions: we can understand then why with this term the observer invariance is lost. For nonlinear systems, classical design methods (as described in [4]) or more recent design methods (as described in [1]) essentially relies on the error term $(\hat{y} - y)$. This paper is a first tentative to design asymptotic observer with nonlinear errors terms derived from the symmetries of the system. As we will see in section 4, we get, for our example (1), a global asymptotic observer when $k$ is positive: this comes from the fact that, the observer dynamics is strictly contracting for the metric $ds^2 = \frac{(dc_1)^2}{(c_1)^2} + \frac{(dc_2)^2}{(c_2)^2}$ in the sense of [6].

The paper is organized as follows. In section 2, we recall some basic facts on transformation groups, define invariant observers, and propose a constructive method for invariant observer design. This method is based on invariant estimation errors that, for $G$-invariant output, can be computed, as shown in section 3, via the Darboux-Cartan moving-frame method recalled in appendix. In section 4, we apply this method on a class of chemical reactors: it yields an invariant observer that is proved to be globally convergent using contraction theory [6].

This paper is, in some sense, the counter-part of [5] where the invariant tracking problem is addressed.

# 2 Transformation group and invariance

We use here some basic notions that are defined in details in [2]. For clarity sakes, we consider transformation groups acting on $x$ local coordinates associated with the state. More global and coordinate-free definitions are possible with state living on manifolds.

Consider the smooth dynamics

$$\dot{x} = f(x) \tag{3}$$

where the state $x$ belongs to an open subset $\mathcal{X}$ of $\mathbb{R}^n$. Let $G$ be a local group of transformations acting on $\mathcal{X}$ according to

$$X = \varphi_g(x), \quad g \in G,$$

where $\varphi_g$ is a local diffeomorphism. Moreover, the dependence with respect to $g$ is smooth. Denote by $r$ the dimension of the group $G$.

**Definition 1.** *The dynamics (3) is said to be* $G$-invariant *if for every* $g \in G$ *the representation of the system remains unchanged, i.e.,* $\dot{X} = f(X)$.

Alternatively, we also say that $G$ is a symmetry group of the system. The definition means that, for every $g \in G$, we have

$$\forall x \in \mathcal{X}, \quad f(\varphi_g(x)) = \frac{\partial \varphi_g}{\partial x}(x) \cdot f(x).$$

For (1), the transformation group $G$ is of dimension $r = 2$, depends on two positive parameters $M_1 > 0$ and $M_2 > 0$, and its action is

$$(C_1, C_2) = \varphi_{(M_1,M_2)}(c_1, c_2) = (M_1 c_1, M_2 c_2)$$

where we denote by $\varphi_{(M_1,M_2)}$ the element of $G$ associated to the parameters $(M_1, M_2)$. Notice that the input concentrations $(c_1^{in}, c_2^{in})$ are also transformed according to $(C_1^{in}, C_2^{in}) = (M_1 c_1^{in}, M_2 c_2^{in})$.

Consider now a smooth output map $h : x \mapsto y = h(x)$ on $\mathcal{X}$, where the output $y$ belongs to an open subset $\mathcal{Y}$ of $\mathbb{R}^m$ ($\dim y = m$).

**Definition 2 (Invariant asymptotic observer).** *Consider a $G$-invariant dynamics $\dot{x} = f(x)$ with its output $y = h(x)$ (not necessarily $G$-*invariant*). The asymptotic observer*

$$\frac{d}{dt}\hat{x} = \hat{f}(\hat{x}, y)$$

*is said $G$-invariant if and only if, for all $g \in G$, for all estimated state $\hat{x}$ and state $x$, we have*

$$\frac{\partial \varphi_g}{\partial x}(\hat{x}) \cdot \hat{f}(\hat{x}, h(x)) = \hat{f}(\varphi_g(\hat{x}), h(\varphi_g(x))).$$

This definition just means that the observer equations remain unchanged

$$\frac{d}{dt}\hat{X} = \hat{f}(\hat{X}, h(X))$$

where $\hat{X} = \varphi_g(\hat{x})$ and $X = \varphi_g(x)$. The asymptotic observer (2) is clearly invariant.

Assume that we have a set of $p$ vector fields $w_i(x)$, $i = 1, ..., p$ on the state space that are invariant with respect to $G$. This means that for any $g \in G$ and $i \in \{1, ..., p\}$, we have $w_i(\varphi_g(x)) = \frac{\partial \varphi_g}{\partial x}(x) \cdot w_i(x)$. Consider now a set of $p$ scalar functions of the form $J_i(\hat{x}, h(x))$, $i \in \{1, ..., p\}$. Assume that they are invariant, i.e., for all $g \in G$, for all $\hat{x}$ and $x$, we have

$$J_i(\varphi_g(\hat{x}), h(\varphi_g(x))) = J_i(\hat{x}, h(x)), \quad i \in \{1, ..., p\}.$$

Then the following system

$$\frac{d}{dt}\hat{x} = f(\hat{x}) + \sum_{i=1}^{p} (J_i(\hat{x}, y) - J_i(\hat{x}, \hat{y}) w_i(\hat{x}) \tag{4}$$

is an invariant observer. This is a direct application of the definition. Notice that we do not address here the convergence of $\hat{x}$ towards $x$. We just consider the invariance.

# 3 Invariant estimation errors

**Definition 3 (invariant estimation errors).** *Assume the smooth dynamics $\dot{x} = f(x)$ is $G$-invariant. Take an output $y = h(x)$ of dimension $m$. An invariant estimation error is a set of $m$ smooth functions of the estimated state $\hat{x}$ and of the measured output $y$, $I(\hat{x}, y) = (I_1(\hat{x}, y), ..., I_m(\hat{x}, y))$, such that:*

*1. for all $g \in G$, for all $\hat{x}$ and all $x$, we have*

$$I(\varphi_g(\hat{x}), h(\varphi_g(x))) = I(\hat{x}, h(x))$$

*2. for each $\hat{x}$, the map $y \mapsto I(\hat{x}, y)$ is a diffeormorphism with $I(\hat{x}, h(\hat{x})) = 0$.*

Let us see now how to construct such invariant functions $I$. First we need the following definition.

**Definition 4.** *Assume the smooth dynamics $\dot{x} = f(x)$ is $G$-invariant. Then the output $y = h(x)$ is* $G$-invariant *if the action of $G$ on $x$ admits a well-defined restriction on $y$, i.e., for $g \in G$, there exists an output transformation $\varrho_g$ on $\mathcal{Y}$ such that $h \circ \varphi_g = \varrho_g \circ h$.*

With $X = \varphi_g(x)$ and $Y = \varrho_g(y)$, the definition reads $Y = h(X)$. The output $y$ of (1) is *$G$-invariant* since $\varphi_{(M_1,M_2)}$ yields to

$$Y = \frac{C_1}{C_1 + C_2} = \frac{M_1 c_1}{M_1 c_1 + M_2 c_2} = \frac{y}{y + \frac{M_2}{M_1}(1-y)}$$

which defines an action of $G$ on the output space.

This definition means that the action of $G$ on the state-space and the output map $h$ must be compatible. Only special output maps $h$ yield $G$-invariant output: for example (1), the map $h(c_1, c_2) = c_1 + c_2$ does not define a $G$-invariant output.

**Theorem 1.** *Take a $G$-invariant dynamics $\dot{x} = f(x)$ and a $G$-invariant output $y = h(x)$. Assume that for some $x_0$, the smooth map*

$$G \ni g \mapsto \varphi_g(x)$$

*is of rank $r = \dim G$ around $g = Id$ with $r \leq n = \dim x$. Then, locally around $(x_0)$, there exist $m = \dim y$ invariant smooth functions $I_i(\hat{x}, y)$, $i = 1, \ldots, m$ that form an invariant estimation error.*

The assumption on the action of $G$ implies that $G$ acts effectively (i.e., the isotropy group is trivial or discrete). This is not really a limitation. The local character of this result is not a strong limitation either. When $G$ is an analytic connected group with an analytic action, the $I_i$'s are analytic functions when $h$ is analytic.

*Proof.* We use here the moving frame method recalled in appendix A. In our case the manifold $\Sigma$ corresponds to the $(\hat{x}, y)$-space and the local coordinates $\xi$ to the components of $(\hat{x}, y)$. The action of $G$ on this space is well defined since $y$ is a $G$ invariant output. To any element $g \in G$ corresponds the following transformation

$$(\hat{x}, y) \mapsto (\varphi_g(\hat{x}), \varrho_g(y)).$$

The constant rank assumption implies that elements of $G$ close to identity act regularly around $(x_0, y)$. The fiber coordinates $\xi_f$ can be formed via subset of the components of $\hat{x}$. Since $a = \psi_f(\hat{x}, \Xi_f)$, this means that the transformation $Y = \varrho(y, a)$, corresponding to $g \in G$ associated to the parameter $a$, reads

$$Y = \varrho(y, \psi_f(\hat{x}, \Xi_f)) := \Upsilon(y, \hat{x}, \Xi_f).$$

It is obvious that for any normalization $\Xi_f^0$, the $m$ scalar functions in

$$\Upsilon(y, \hat{x}, \Xi_f^0)$$

are invariant. Moreover, the rank of $\Upsilon$ with respect to $y$ is maximal and equal to $m$. It will do to use

$$I(y, \hat{x}) = \Upsilon(y, \hat{x}, \Xi_f^0) - \Upsilon(\hat{y}, \hat{x}, \Xi_f^0)$$

to obtain a set of local invariant output errors. □

For example (1), the invariant error is

$$I(\hat{x}, y) = \log\left(\frac{\hat{y}}{1-\hat{y}} \frac{1-y}{y}\right)$$

# 4 A class of chemical reactors

## 4.1 The system

We consider a continuous stirred tank of volume $V$ with $n$ species of concentrations $(c_1, \ldots, c_n)$ :

$$\begin{aligned} V\frac{d}{dt}c_i &= F(c_i^{in} - c_i) + r_i(c_1, \ldots, c_n) \\ y_i &= \frac{c_i}{\sum_{h=1}^n c_h} \qquad\qquad , \quad i = 1, ..., n \end{aligned}$$

($F$ the input flow rate, $(c_1^{in}, \ldots, c_n^{in})$ input concentrations). The $n$ species react according the reaction terms $r_i(c_1, \ldots, c_n)$: these $n$ functions are supposed to be homogenous of degree one. We measure the fractions $y_i$ and we want to reconstruct the state $(c_1, \ldots, c_n)$.

The example (1) belongs to this class of chemical reactors : we have in this case $n = 2$ and no reaction terms $r \equiv 0$.

## 4.2 Invariant function

Indeed, the system equations do not depend on the units of the concentrations $c_i$. This system is not strictly invariant with respect to the definition here above. To be invariant according to the definition, we have also to consider the action of the scaling group on the input concentrations $c_i^{in}$, and on the parameters hidden in the chemical kinetics $r_i$. With such natural extension, we can say that such system is invariant under the action of the scaling group defined by

$$C_i = M_i c_i, \quad i = 1, ..., n$$

where the $M_i$ are positive constants (group parameters).

To build the observer, we consider the invariant functions of the transformation group acting on the $((\widehat{c_i}), (y_j))$-space. The normalization equations write:

$$C_i^o = M_i \widehat{c_i}, \quad i = 1, ..., n$$

and the solution is given by

$$M_i = \frac{C_i^o}{\widehat{c_i}}, \quad i = 1, ..., n$$

where $(C_i^o)$ is our reference.

As the group action on the output space is given by

$$Y_i = \frac{M_i y_i}{\sum_{h=1}^n M_h y_h}, \quad i = 1, ..., n$$

we get then the following invariant functions:

$$H_i(\widehat{c}, y) = \frac{\frac{C_i^o}{\widehat{c_i}} y_i}{\sum_{h=1}^n \frac{C_h^o}{\widehat{c_h}} y_h}, \quad i = 1, ..., n$$

where $\widehat{c} = (\widehat{c_i})$ and $\widehat{y} = (\widehat{y_i})$.

A simple combination of these invariant functions yields the following ones (more symmetric):

$$I_{ij}(\hat{c}, y) = \log\left(\frac{\widehat{y_i}}{\widehat{y_j}} \frac{y_j}{y_i}\right)$$

which gives invariant estimation error terms, for all $i$ and $j$ in $\{1 \dots n\}$.

## 4.3 The invariant observer

To design the observer, we use the invariant estimation error terms $I_{ij}$ and the $n$ infinitesimal generators of the transformation group acting on the system state:

$$v_i(c_1, \dots, c_n) = c_i \frac{\partial}{\partial c_i}, \quad \forall i \in \{1, \dots, n\}$$

since these vector fields are here invariant. We get then the following observer ($k > 0$ is a design parameter):

$$\begin{aligned} V\frac{d}{dt}\widehat{c_i} =& F(c_i^{in} - \widehat{c_i}) + \widehat{c_i} r_i((\frac{y_j}{y_i})_{1 \le j \le n}) \\ & - k\widehat{c_i} \log\left(\prod_{h=1, h\neq i}^{n} \frac{\widehat{y_i}}{\widehat{y_h}} \frac{y_h}{y_i}\right) \\ \widehat{y_i} =& \frac{\widehat{c_i}}{\sum_{h=1}^n \widehat{c_h}} \qquad , \ \forall i \in \{1, \dots, n\} \end{aligned} \tag{5}$$

which is invariant under the action of the scaling group. Notice that when $\hat{c}_i = c_i$, we recover the original dynamics, $\frac{d}{dt}\widehat{c_i} = \frac{d}{dt}c_i$.

### 4.4 Observer convergence

To prove convergence, we make a change of coordinates:

$$\widehat{\xi_i} = \log(\widehat{c_i}) \qquad i = 1, \dots, n$$

The observer equations become :

$$\begin{aligned} V\frac{d}{dt}\widehat{\xi_i} =& F(C_i^{in} exp(-\widehat{\xi_i}) - 1) + r_i((\frac{y_j}{y_i})_{1\le j\le n}) \\ & - k\left(\sum_{h=1,h\neq i}^{n}(\widehat{\xi_i} - \widehat{\xi_h}) + \log(\prod_{h=1,h\neq i}^{n}\frac{y_h}{y_i})\right) \end{aligned}$$

The symmetric part of the jacobian matrix (with respect to the $\widehat{\xi_i}$ only) is given by:

$$\begin{aligned} A =& \frac{1}{V}\begin{pmatrix} -FC_1^{in} exp(-\widehat{\xi_1}) & & \\ \mathbf{0} & \ddots & \mathbf{0} \\ & & -FC_n^{in} exp(-\widehat{\xi_n}) \end{pmatrix} \\ & + \frac{1}{V}\begin{pmatrix} -(n-1)k & k & \cdots & k \\ k & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & k \\ k & \cdots & k & -(n-1)k \end{pmatrix} \end{aligned}$$

The second matrix is negative: it has $-nk$ as eigenvalue of multiplicity $n-1$, and 0 as eigenvalue associated to the eigenvector with all coordinates equal to 1.

The first matrix is negative. Let J be the set of indices $\{\imath_1, \dots, i_r\}$ such that for every $k \in J$, $C_k^{in} = 0$. Let $\mathbf{B} = (e_1, \dots, e_n)$ be the canonical basis of $\mathbb{R}^n$. As the second matrix is definite negative on the subspace spanned by $\{e_k,\ k \in J\}$, we get that A is definite negative. This result shows that the observer is a global contraction (in the sense of [6]) which gives its global convergence. The observer (5) is thus a global invariant asymptotic observer.

To summarize: the observer is shown to be a contraction when the equations are written in the coordinates $(\widehat{\xi_i})$ and we use the Euclidian metric to define the symmetric part of the Jacobian matrix. It is equivalent to say that the observer (5) is a contraction with respect to the following metric:

$$ds^2 = \sum_{i=1}^{n}\left(\frac{dc_i}{c_i}\right)^2 = \sum_{i=1}^{n}(d\xi_i)^2$$

that is to say a Killing metric of the transformation group acting on the system state space.

## 5 Conclusion

The convergence proof for this class of chemical reactors suggests the following question: are there links between the following two facts: the observer is invariant; the observer defines a contraction for a Killing metric (a metric where the group transformations are isometries)?

## References


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## A The moving-frame method

To determine the invariant functions we use the normalization technique, which is also called the moving frame technique of Darboux Cartan (see, e.g., the very nice presentation in Chapter 8 of [3]).

**An example in $\mathbb{R}^3$** The idea is quite simple as we are going to see when the manifold $\Sigma$ corresponds to $\mathbb{R}^3$, when the transformation group $G$ acting on $\Sigma$ admits two parameters and when the orbits are surface. The normalization technique relies on the following idea. Under good regularity conditions of the transformations group, we can find a set of coordinates $(\xi_1, \xi_2, \xi_3)$ such that, as shown on figure 1, in the directions $\xi_f = (\xi_1, \xi_2)$, we are in an orbit, whereas the transverse direction $\xi_b = \xi_3$ determines in which orbit we are. So, to know if two points $P$ and $S$ are in the same orbit, that is, if there is an element $g$ of $G$ such that $S = \varphi_g(P)$, it is sufficient to check that the two points have the same coordinate $\xi_3$: $\xi_3$ is an invariant function. Any other invariant scalar function is a function of $\xi_3$.

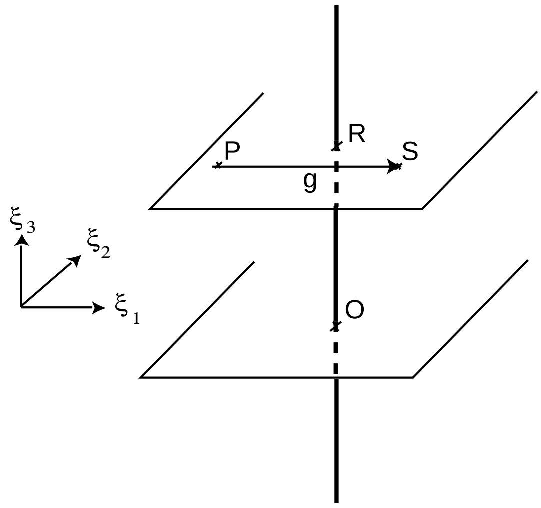


Figure 1: Orbits of the transformation group $G$ acting on $\Sigma$ in rectifying coordinates

Now, we suppose we don't know this set of coordinates. Denote by $(p_1, p_2, p_3)$ (resp. $(s_1, s_2, s_3)$, $(r_1, r_2, r_3)$) the coordinates of a point $P$ (resp. $S$, $R$) in some referential. If $P$ and $S$ belong to the same orbit, there exists an element $g \in G$ such that (in coordinates) :

$$\begin{aligned} s_1 &= (\varphi_g(P))_1 \\ s_2 &= (\varphi_g(P))_2 \\ s_3 &= (\varphi_g(P))_3 \end{aligned}$$

The problem here is that the three coordinates of $S$ depend on a particular set of the group parameters. To find the invariant functions, we have to eliminate the two parameters associated to $g$. We introduce a third point $R$ on the same orbit that we will take as a reference. We can then determine the element $g(P)$ of the group such that $R = \varphi_{g(P)}(P)$. This element is solution of the following system of equations (normalization equation):

$$\begin{aligned} r_1 &= (\varphi_g(P))_1 \\ r_2 &= (\varphi_g(P))_2 \end{aligned}$$

If the implicit functions theorem applies (the rank of this system with respect to the two parameters associated to $g$ must be 2), then we have :

$$g(P) = \gamma(P, r_1, r_2)$$

The invariant is then given by:

$$I(P) = (\varphi_{\gamma(P,r_1,r_2)}(P))_3$$

Indeed, $I(P) = r_3$ and as $R$ and $S$ are on the same orbit, we have also: $I(S) = r_3 = I(P)$, since $\varphi_{\gamma(S,r_1,r_2)}(S) = R$.

**The general case** Take a group $G$ acting regularly on a manifold $\Sigma$ of dimension $\sigma$ with $r$-dimensional orbits, $r < \sigma$. In local coordinates on $\Sigma$, $\xi$, denote by $\Xi = \varphi(\xi, a)$ the transformation associated to the element of $G$ with parameter $a$. Then the orbit equations (a set of fundamental scalar invariants) are given via elimination of $a$ once the coordinates $\xi = (\xi_b, \xi_f)$ have been decomposed into the base coordinates, $\xi_b$, of dimension $\sigma - r$ and the fiber coordinates, $\xi_f$, of dimension $r$. The transformation $\Xi = \varphi(\xi, a)$ then reads

$$\Xi_b = \varphi_b(\xi, a)$$
$$\Xi_f = \varphi_f(\xi, a)$$

with $a \mapsto \varphi_f(\xi, a)$ invertible for every $\xi$. Denoting by $\psi_f(\xi, \Xi_f)$ the inverse map $(\varphi_f(\xi, \psi_f(\xi, \Xi_f)) \equiv \Xi_f)$, one has

$$\Xi_b = \varphi_b(\xi, \psi_f(\xi, \Xi_f)) := \psi_b(\xi, \Xi_f).$$

Assume that $\zeta$ and $\xi$ belong to the same orbit. Then they have the same base coordinates. This means that, once $\Xi_f$ has been fixed, to $\Xi_f^0$ say (normalization), one has

$$\psi_b(\xi, \Xi_f^0) = \psi_b(\zeta, \Xi_f^0).$$

In other words, the $\sigma - r$ independent scalar functions $\xi \mapsto \psi_b(\xi, \Xi_f^0)$ are invariant, i.e., for any transformation $\Xi = \varphi(\xi, a)$, we have

$$\psi_b(\xi, \Xi_f^0) \equiv \psi_b(\varphi(\xi, a), \Xi_f^0).$$