ACC paper number

Computing the Flat Outputs of Engel Differential Systems The Case Study of the Bi-steerable Car

S. Sekhavat P. Rouchon J. Hermosillo

Abstract

Flatness is an interesting structural property of many engineering systems. Typically, the knowledge of the Flat Outputs of a system allows the design of open loop control and helps the design of control loops. When the literature includes many works on finding families of flat systems or proposing tools to check the flatness of a system, there has been very few works on the actual computation of the Flat Outputs which remains an open practical problem. In this paper, we present a necessary condition for the flat coordinates change of any system represented by two differential 1-forms in a state space of dimension 4. This condition is expressed by a partial differential equation and implies an approach to help the computation of the flat outputs. Then, we apply our approach to a specific system, namely the Bi-steerable car for which we explicitely compute the flat outputs. Hence, we solve the problem of the open loop control design for bi-steerable cars which was our initial and main motivation.

1 Introduction

This work has been originally motivated by the problem of path planning for a Bi-steerable car. Indeed among people working on mobile robots and Intelligent Transportation Systems, there has been lately an increasing interest for Bi-steerable plateforms which offer a better maneuvrability. Examples of such vehicles are the "Cycab" platform commercialized by Robosoft[1] company or the "πCar" prototype of IEF[2]. A bi-steerable car is a car such that the steering of its front wheels by an angle ϕ induces a steering of its rear wheels by an angle $f(\phi)$. This feature not only increases the upper bound of the rotational velocity of the vehicle but also it reduces the sweeping volume of the vehicle in motion and therefore enhances its maneuvrability in cluttered environments. However to our knowledge there is no existing motion planning method for such systems. One way of designing an open loop control for this system is to use its *flatness* property.

A system $\dot{X} = f(X, u)$ is said *differentially flat* [1] if there exist *flat (or linearizing) outputs* $Y = (y_1, ..., y_m)$ differentially independent such that:

- any system variable (state, controls,...) can be expressed only from the linearizing outputs and their successive derivatives.
- the flat outputs can be expressed as a function of the system variables and their successive derivatives,

which roughly means that there is a one-to-one correspondance between arbitrary maps $Y(t)$ and the solutions $(X(t), u(t))$ for the system.

The interesting point here is that, unlike the state coordinates, $(y_1, ..., y_m)$ are differentially independent. Therefore unlike the state space, any smooth curve in $(y_1, ..., y_m)$ space corresponds to an admissible path for the system. Hence, path planning becomes easier in the linearizing space since we do not have to take into account any kinematic constraint along the path. The only constraints that have to be considered are those on the successive time derivatives of the curve at its both ends (Y_{init}, Y_{goal}). These constraints are imposed by the starting and goal configurations and their successive derivatives $(X_{init}, X_{goal}, \dot{X}_{init}, \dot{X}_{goal}, \cdots)$.

There is therefore an evident advantage in exploiting the differential flatness of a nonholonomic system (such as the bi-steerable car) for trajectory planning purposes: arbitrary curves (e.g. polynomials) in the

[1] See e.g. www.robosoft.fr.

[2] "Institut d'Electronique Fondamentale" of Paris-Sud University.

flat output space (having only initial and final constraints on their time derivatives) could be employed in order to connect Y_{init} and Y_{goal}.

however, in order to use the flatness property of the system for path planning a main problem remains: the computation of the flat outputs. indeed, even if determining the flatness of a system remains an open practical problem in general, the problem has been already solved for some family of systems such as the driftless 2-inputs systems [2] or Pfaffian systems of dimension 2 [3]. however, the literature includes very few works on the actual computation of the flat outputs. In this paper, we aim at bringing a contribution in this direction.

Concerning the bi-steerable car, the system proved to be flat [4] and we present in this paper its actual flat outputs and the way to compute them (section 4). Our approach has been inspired from the study of Engel form systems [5] for which we obtain a necessary condition on the flat coordinates change. Therefore to some extends, our approach can be applied to any differential system of dimension 2 on a Manifold of dimension 4 (section 3). but let us start with a quick recall of some exterior calculus notions (section 2) that will be used later.

2 Few concepts of the Exterior Differential Systems

3 Flat outputs of an Engel system

Consider a manifold M of dimension 4 and an exterior differential system on M defined by :

$$\omega_1 = 0 \qquad \omega_2 = 0 \tag{1}$$

where $\omega_1, \omega_2 \in \Omega^1(M)$ are independant 1-forms. It has been shown [2] that such a system is flat if and only if its derived flag satisfies :

$$\begin{array}{l} I = \{\omega_1, \omega_2\} \\ dimI^{(1)} = 1 \\ dimI^{(2)} = 0 \end{array} \tag{2}$$

Notice that other possibilities for the derived flag ($dimI^{(1)} = 2$ or $dimI^{(1)} = 1$, $dimI^{(2)} = 1$) correspond to a non controllable system (e.g. from the Lie Algebra Rank Condition [?]) and each instance of the problem is actually equivalent to a system of lower dimension. Now if we come back to the flat case, the Engel theorem states ([5]):

Theorem 1 - Engel : *For a system (1) verifying the condition (2) on its derived flag, there exists coordinates y_1, y_2, y_3, y_4 such that the system can be put in Engel form :*

$$I = \{dy_4 - y_3 dy_1, dy_3 - y_2 dy_1\}$$

If we consider the Engel form of the system (1), (y_1, y_4) are clearly the flat outputs of the system. Indeed in $\{y_1, y_2, y_3, y_4\}$ coordinates, the system (1) can be equivalently written in the chained form :

$$\left\{ \begin{array}{lcl} \dot{y_1} & = & u_1 \\ \dot{y_2} & = & u_2 \\ \dot{y_3} & = & y_2 u_1 \\ \dot{y_4} & = & y_3 u_1 \end{array} \right.$$

where obviously all variables of the system can be obtained from (y_1, y_4) and their derivatives. Now generally, ω_1 and ω_2 are not expressed in the $\{y_i\}$ coordinates. Therefore, in order to take a practical benefit of the flatness we have to compute the coordinates change which put the system into Engel form and explicitly obtain the flat outputs expressions. What follows is a scheme to help the computation of this coordinates change which is obviously function of $\{w_1, \omega_2\}$ and the original coordinates in which they are expressed.

The main point to compute the coordinates change $\{x_i\} \longrightarrow \{y_i\}$ is to notice that the Engel form of the system is adapted to the derived flag (see the proof of the Engel theorem [5]). Therefore:

$$I^{(1)} = \{dy_4 - y_3 dy_1\}$$

Let us compute a 1-form of $I^{(1)}$ in the original coordinates system (x_1, x_2, x_3, x_4) of M, in which $\{\omega_1, \omega_2\}$ are expressed. Given $\{dx_1, dx_2, dx_3, dx_4\}$ the associated basis of $\Omega^1(M)$, there are scalar functions θ^i_j on M such that:

$$\omega_i = \sum_{j=1}^{4} \theta^i_j dx_j \qquad i = 1, 2 \tag{3}$$

For $\eta \in \Omega^1$:

$$\eta \in I^{(1)} \Longrightarrow \eta \in I \;\; and \;\; d\eta = 0 \; mod \; I$$

Therefore there are scalar functions α_i on M such that:

$$\eta = \alpha_1 \omega_1 + \alpha_2 \omega_2$$

which implies

$$\begin{array}{rcl} d\eta & = & d\alpha_1 \wedge \omega_1 + \alpha_1 d\omega_1 + d\alpha_2 \wedge \omega_2 + \alpha_2 d\omega_2 \\ & = & \alpha_1 d\omega_1 + \alpha_2 d\omega_2 \;\; mod\ I \end{array} \tag{4}$$

Now from (3):

$$d\omega_i = \sum_{j=1}^{4} d\theta_j^i \wedge dx_j \qquad i = 1, 2$$

and expressing $d\theta_j^i$ in $\{dx_i\}$ basis using their partial derivatives one gets:

$$d\omega_i = \sum_{1 \leq j < k \leq 4} \left(\frac{\partial \theta_j^i}{\partial x_k} - \frac{\partial \theta_k^i}{\partial x_j}\right) dx_j \wedge dx_k \qquad i = 1, 2 \tag{5}$$

Since ω_1 and ω_2 are independant, from (3) one can express two of the $\Omega^1(M)$ basis vectors in function of ω_1, ω_2 and the other vectors. Without loss of generality, assume those vectors are dx_1, dx_2, (3) leads to:

$$dx_1 = \beta_3^1 dx_3 + \beta_4^1 dx_4 \;\; mod\ \{\omega_1, \omega_2\}$$

$$dx_2 = \beta_3^2 dx_3 + \beta_4^2 dx_4 \;\; mod\ \{\omega_1, \omega_2\}$$

where β_i^j's are scalar functions: $M \longrightarrow R$ involving only θ_j^k's. Injecting these expressions in (5), the exterior product properties give:

$$d\omega_1 = \gamma_1 dx_3 \wedge dx_4 \quad mod\ \{\omega_1, \omega_2\}$$

$$d\omega_2 = \gamma_2 dx_3 \wedge dx_4 \quad mod\ \{\omega_1, \omega_2\}$$

where γ_1, γ_2 are computable functions of θ_i^j's and their partial derivatives. Therefore (4) becomes:

$$d\eta = (\alpha_1\gamma_1 + \alpha_2\gamma_2) dx_3 \wedge dx_4 \quad mod\ I$$

Therefore a suffisiant condition for η to belong to $I^{(1)}$ is:

$$\alpha_1\gamma_1 + \alpha_2\gamma_2 = 0$$

Hence:

$$\eta = \gamma_2\omega_1 - \gamma_1\omega_2 = \sum_{j=1}^{4} (\gamma_2\theta_j^1 - \gamma_1\theta_j^2) dx_j \tag{6}$$

is a 1-form of $I^{(1)}$ and therefore colinear to $dy_4 - y_3 dy_1$. Now suppose:

$$y_1 = P^1(x) \quad y_4 = P^4(x)$$

Then:

$$dy_i = \sum_{j=1}^{4} \frac{\partial P^i}{\partial x_j} dx_j \qquad i = 1, 4$$

Again since dy_1 and dy_4 are independant, expressing two of the dx_i's (e.g. dx_1, dx_2) in function of the others and dy_1, dy_4 and injecting the result in (6), one gets:

$$\eta = g_1 dy_1 + g_4 dy_4 + f_3 dx_3 + f_4 dx_4$$

where f_3, f_4 are expressions of known functions ($\theta_j^i(x)$'s and their partial derivatives) and unknown functions that we are looking for ($P^1(x), P^4(x)$) and their partial derivatives. Therefore we have the following lemma:

Lemma 1 *The coordinates change giving the flat outputs:*

$$y_1 = P^1(x) \quad y_4 = P^4(x)$$

is solution of the partial differential equation:

$$f_3(x, P^1, P^4, \partial P^1, \partial P^4) = 0$$

$$f_4(x, P^1, P^4, \partial P^1, \partial P^4) = 0$$

In the next section we study the specific case of the bi-steerable car and show how this approach allows us the explicite computation of the flat outputs.

4 The flat outputs of the Bi-steerable Car

A bi-steerable car is a car with both front and rear orientable wheels such that the rear wheels steering angle α_r is a function $f(\alpha_f)$ of the front steering angle α_f. We represent a configuration of the system by a point (x, y, θ, α) of the manifold $M = R^2 \times (S^1)^2$ (of dimension 4) where x, y are the Cartesian coordinates of the middle point of the rear axle, θ is the orientation of the car in the reference frame and α is the angle of the front wheels with respect to the car (see figure (2). The kinematic constraints imposed on the system are due to the rolling without slippage of the wheels which means that the instantaneous velocity of each wheel is parallel to its orientation:

$$\left\{ \begin{array}{rcl} \dot{y}\cos(\theta+\alpha) - \dot{x}\sin(\theta+\alpha) & = & 0 \\ \dot{y}_{rear}\cos(\theta+f(\alpha)) - \dot{x}_{rear}\sin(\theta+f(\alpha)) & = & 0 \end{array} \right.$$

4.1 The systematic approach

The equivalent exterior differential system on M is $I = \{w_1, w_2\}$ with:

$$\omega_1 = \cos(\theta+\alpha)dy - \sin(\theta+\alpha)dx$$

$$\omega_2 = \cos(\theta+f(\alpha))dy - \sin(\theta+f(\alpha))dx - L\cos(f(\alpha))d\theta$$

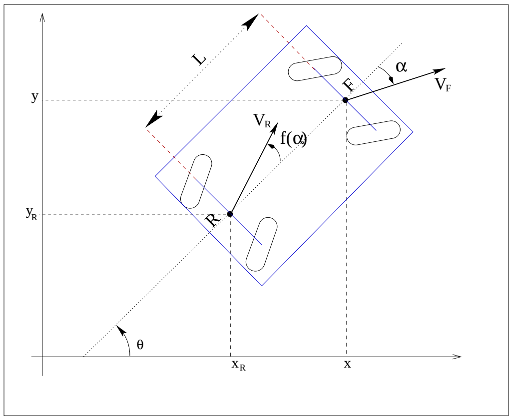

Figure 1: Model of the Bi-steerable car.

where L is the distance between the front and rear axle. Following the scheme of Section (3) the differentiation leads to:

$$\begin{aligned} d\omega_1 &= -\sin(\theta+\alpha)(d\theta+d\alpha)\wedge dy \\ &\quad -\cos(\theta+\alpha)(d\theta+d\alpha)\wedge dx \\ d\omega_2 &= -\sin(\theta+f(\alpha))(d\theta+f'(\alpha)d\alpha)\wedge dy \\ &\quad -\cos(\theta+f(\alpha))(d\theta+f'(\alpha)d\alpha)\wedge dx \\ &\quad +Lsin(f(\alpha))f'(\alpha)d\alpha\wedge d\theta \end{aligned}$$

And later on:

$$d\omega_1 = \gamma_1(\alpha)d\theta\wedge d\alpha \quad mod\ I$$

$$d\omega_2 = \gamma_2(\alpha)d\theta\wedge d\alpha \quad mod\ I$$

with

$$\gamma_1(\alpha) = \frac{-L\cos(f(\alpha))}{\sin(\alpha - f(\alpha))}$$

$$\gamma_2(\alpha) = \frac{-L\cos(\alpha)f'(\alpha)}{\sin(\alpha - f(\alpha))}$$

Here and from now on, for any h, the notation $h'(\alpha)$ implies that h is a scalar function of the unique variable α and $h'(\alpha)$ is its derivative with respect to α; as it is the case of $f'(\alpha)$ above. Notice also that unlike the general case, here γ_i's are functions of only the coordinate α. Always following the scheme of Section (3) we get a vector of $I^{(1)}$:

$$\begin{aligned} \eta &= [\gamma_2\sin(\theta+\alpha) - \gamma_1\sin(\theta+f(\alpha))]dx \\ &\quad -[\gamma_2\cos(\theta+\alpha) - \gamma_1\cos(\theta+f(\alpha))]dy \\ &\quad -L\gamma_1\cos(f(\alpha))d\theta \end{aligned}$$

and we know that if we find variables y_1 and y_2 such that at each point $p = \{x, y, \theta, \alpha\}$:

$$\eta = k_1(p)dy_1 + k_2(p)dy_2$$

for some scalar functions k_1, k_2 then y_1, y_2 are the flat outputs. One can prove that for our system the flat outputs are only function of the state variables [3]. Considering our specific system and the invariances of the problem (with respect to the translations and rotations of the car) it is sound to consider (y_1, y_2) as the Cartesian coordinates of a point whose relative position with respect to the robot does not depend on the position and the orientation of the vehicle. Therefore the coordinates of such a general point in the vehicle frame can be expressed as follows:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + P(\alpha)\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} + Q(\alpha)\begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \tag{7}$$

with P and Q the unknow functions that we have to determine. By computing dx, dy in function of $dy_1, dy_2, d\theta, d\alpha$ and subtituting them in the expression of η above we get the PDE of the Lemma (1). After some computations and simplifications all coordinates but α disappear in the PDE and we get:

$$\begin{aligned} &P(\alpha)[\cos^2(\alpha)f'(\alpha) - \cos^2(f(\alpha))] + \\ &Q(\alpha)[\cos(\alpha)\sin(\alpha)f'(\alpha) - \cos(f(\alpha))\sin(f(\alpha))] - \\ &L\cos 2(f(\alpha)) = 0 \end{aligned} \tag{8}$$

$$\begin{aligned} &P'(\alpha)[\cos(\alpha)\sin(\alpha)f'(\alpha) - \cos(f(\alpha))\sin(f(\alpha))] + \\ &Q'(\alpha)[\cos^2(\alpha)f'(\alpha) - \cos^2(f(\alpha))] = 0 \end{aligned} \tag{9}$$

Now from (8) we can express P' in function of Q and Q' and subtitute it in (9) in order to obtain a first order ODE which we can theoretically solve using the method of the variation of the constant to obtain Q. Thence we get P and y_1, y_2.

However, from a practical point of view, such a solution is not yet quite satisfactory. Indeed, Q will be computed through a double (enclosed) numerical integration which gives no hint on how to compute the inverse transformation (i.e. the expressions of the original coordinates in function of the flat outputs and their derivatives).

4.2 Further computations

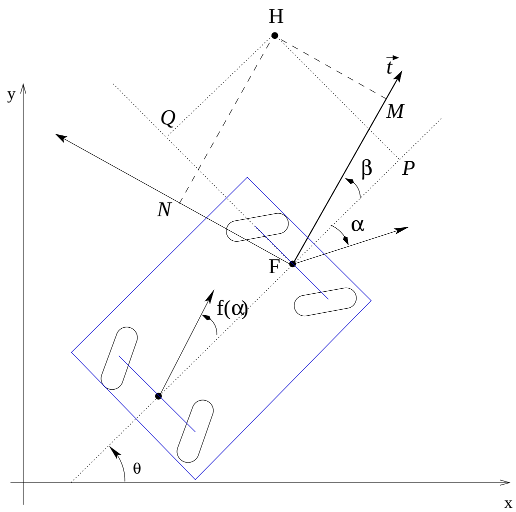

Figure 2: Frames and coordinates.

Let us work out the equations to obtain more tractable expressions for P and Q. For an arbitrary angle ϕ let us denote by $\vec{u}_\phi$ the unitary vector of direction ϕ. Let us define the vector:

$$\vec{t} = \cos(\alpha)f'(\alpha)\vec{u}_{\theta+\alpha} - \cos(f(\alpha))\vec{u}_{\theta+f(\alpha)}$$

Which is a linear combination of a vector parallel to the front wheel and a second one parallel to the rear wheel. Let us call F (resp. H) the point of the Cartesian coordiante (x, y) the middle point of the rear axle (resp. (y_1, y_2) the flat outputs), see Fig. (**??**). Expressing $\vec{t}$ and $\vec{FH}$ in the robot frame we get:

$$\vec{FH} = P(\alpha)\vec{u}_\theta + Q(\alpha)\vec{u}_{\theta+\frac{\pi}{2}}$$

$$\vec{t} = A(\alpha)\vec{u}_\theta + B(\alpha)\vec{u}_{\theta+\frac{\pi}{2}}$$

With

$$\begin{aligned} A(\alpha) &= \cos^2(\alpha)f'(\alpha) - \cos^2(f(\alpha)) \\ B(\alpha) &= \cos(\alpha)\sin(\alpha)f'(\alpha) - \cos(f(\alpha))\sin(f(\alpha)) \end{aligned}$$

Then (8) and (9) imply:

$$\vec{FH}.\vec{t} = L\cos^2(f(\alpha)) \tag{10}$$

$$\left[\frac{d\vec{FH}}{d\alpha}\right]_{\mathcal{R}_\theta} \; // \; \vec{t} \tag{11}$$

Which show the interest of the vector $\vec{t}$. Indeed, $\vec{t}$ varies in function of α and θ however the projection of $H(y_1, y_2)$ on $\vec{t}$ is known (see (10)). Moreover, as we have seen H position with respect to the car is only function of α and its infinitessimal variation in the car frame $\mathcal{R}_\theta$ is parallel to $\vec{t}$ for any α (see (11)).

Therefore it seems more interesting to express H in the frame attached to $\vec{t}$. With the follwing notations:

$$\beta(\alpha) = (\widehat{\vec{u}_\theta, \vec{t}}) = \tan^{-1}\frac{B(\alpha)}{A(\alpha)}$$

$$\vec{FH} = M\vec{u}_\beta + N\vec{u}_{\beta+\frac{\pi}{2}}$$

One can prove using (10) and (11):

$$M(\alpha) = \frac{L\cos^2(f(\alpha))}{(A^2(\alpha) + B^2(\alpha))^{\frac{3}{2}}}$$

$$N(\alpha) = -\int_0^\alpha \frac{L\cos^2(f(u))(B'(u)A(u) - A'(u)B(u))}{(A^2(u) + B^2(u))^{\frac{3}{2}}}$$

Thus the projection of the problem in the turning frame attached to $\vec{t}$ allows us to have tractable expressions of the H coordinates. Typically M is expressed analytically and N is the primitive of a simple expression. Hence:

$$P(\alpha) = M(\alpha)\cos(\beta(\alpha)) - N(\alpha)\sin(\beta(\alpha))$$

$$Q(\alpha) = M(\alpha)\sin(\beta(\alpha)) + N(\alpha)\cos(\beta(\alpha))$$

4.3 The inverse expressions

The formulation of the problem in the new frame also allows the computation of the original coordinates (x, y, θ, α) in function of the flat outputs and their derivatives.

Considering the invariances of the problem, one can prove that the curvature $\kappa(t)$ of the curve $H(t)$ during the motion is only function of α. Then we can compute the relation $\kappa(\alpha)$ by considering the case where the car does not move and only turns its wheels at a speed $\dot{\alpha} = 1$ (i.e $\alpha = t$) inducing a motion of H. In this specific case the absolute velocity of H is equal to its relative velocity with respect to the car. This velocity has an angle β relatively to the car (see 11). Eventually one gets:

$$\kappa(\alpha) = \frac{\beta'(\alpha)}{(M'(\alpha) - \beta'(\alpha)N(\alpha))}$$

Also by finding the right trigonometric simplifications one can prove that:

$$\frac{dy_2}{dy_1} = \tan(\theta + \beta)$$

In other words, knowing the curve $y_1(t), y_2(t)$ of the flat outputs during the motion we can compute $\alpha(t)$ through $\kappa(t)$ (by inversing the expression $\kappa(\alpha)$). Then from $\beta(\alpha)$, and the orientation of the velocity of $H(t)$ we get θ. Finally, we compute x, y using (7).

Fig. (**??**) and Fig. (**??**) are examples of paths followed by the car when the flat outputs follow the path $y_2 = y_1^2$. In Fig. (**??**), $f(\alpha) = -0.6\alpha$ whereas in Fig. (**??**), $f(\alpha) = sin(\alpha)$.

5 Conclusions

In this paper we aimed at stressing the difficulty of the practical computation of the linearizing outputs of a flat system. We suggested a strategy for this computation for a family of systems (namely the Engel Systems) based on the fact that the flat outputs are necessary solutions of a certain PDE. Then we study in details the case of the bi-steerable car which illustrate our strategy and also allows to present some typical obstacles in the computation of the flat outputs and some hints to overcome them. Obtaining the flat outputs for the bi-steerable car solves the problem of open loop control for this system (e.g. using the curves suggested in [6]) and leads to the first path planner for this system.

References

[1] M. Fliess, J. Lévine, P. Martin and P. Rouchon. Flatness and defect of nonlinear Systems : Introductory theory and examples. *Int. Journal of Control*, 61(6):1327-1361, 1995.

[2] P. Rouchon. Necessary condition and genericity of dynamic feedback linearization. *J. Math. Syst. Estimation Control*, 4(2), 1994.

[3] P. Martin and P. Rouchon. Any (controllable) driftless system with m inputs and $m+2$ states is flat.*In Proc. of Conf. on Decision and Control*, CDC'95.

[4] S. Sekhavat, J. Hermosillo. The Cycab robot : a differentially flat system. *In Proc. of Intelligent Robots and Systems*, IROS'00, Japan 2000.

[5] R.L Bryant, S.S. Chern, R.B Gardner, H.L.Goldschmidt, P.A. Griffiths. Exterior Differential Systems. Springer-Verlag, 1991.

[6] S. Sekhavat, F. Lamiraux, JP. Laumond, G. Bauzil and A. Ferrand. Motion Planning and Control for Hilare pulling a trailer: experimental issues. *In Proc. of IEEE Int. Conf. Robotics and Autom.*, Albuquerque, NM (US), April 1997.