$2k\pi$, the juggling robot

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Abstract

The $2k\pi$ robot, developed at École des Mines de Paris, has five degrees of freedom (dof's): 3 actuated angles for the manipulator and 2 non-actuated angles for the pendulum. This underactuated system is flat, the oscillation center of the pendulum being the flat output. Steering the pendulum from the lower to the upper equilibrium requires to cross a singularity where the first order approximation of the system is not controllable. This difficulty is investigated in details: flatness, invariant calculations and time-scaling arguments yield a robust control scheme. Its real-time implementation is reported.

1 Introduction

$2k\pi$ is a robot, developed at École des Mines de Paris, consisting of a manipulator carrying a pendulum, see figure 1. There are five degrees of freedom (dof's): 3 angles for the manipulator and 2 angles for the pendulum. The 3 dof's of the manipulator are actuated by electric drives, while the 2 dof's of the pendulum are *not* actuated.

The goal is to design a control law steering the pendulum from the lower equilibrium to the upper equilibrium, and more generally able to stabilize the pendulum while the manipulator is moving around.

This system is typical of underactuated nonlinear and unstable mechanical systems such as the Vtol [8], the ducted fan [10, 9], the crane [3], champaign flyer [?]. As shown in [7, 3, 9] the $2k\pi$-robot is flat, the flat output being Huygens oscillation center of the pendulum. The contribution of the paper is twice:

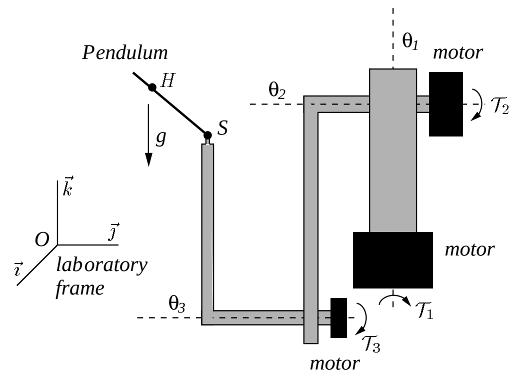

Figure 1: The robot $2k\pi$.

1. planification of open-loop trajectories steering from the lower to the upper equilibrium and the design of a tracking loop controller.

2. the real-time implementation of this control scheme.

Point 1 requires to deal with a singularity both for the open-loop and the closed-loop. This singularity is similar to those of non-holonomic system around equilibrium point. It can be over-come via the same technique: time-scaling.

For the pendulum dynamics, all the computation are done in an intrinsic way via vectors and geometrical constructions, in order to preserve invariance with respect to the Galilean group. This reduces computations to a minimum, and preserves the physics of the system. This leads to a robust control scheme depending essentially on two parameters (the length of the equivalent punctual pendulum and the acceleration of gravity) and avoid the precise knowledge of manipulator parameters (excepted its geometry for conversion between angular to Cartesian coordinates).

In section 2 equations of motion are sketch pointing output their structures and the splitting between the pendulum dynamics (the 2 unactuated dof's) and the manipulator dynamics (the 3 actuated dof's).

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In section 3 we show that, whatever the friction effort and inertia in the manipulator are, the system is flat with oscillation center of the pendulum as flat-output. Section 4 is devoted to the design of open-loop trajectories steering the pendulum from the lower to the upper equilibrium. In section 5, the closed-loop tracking is designed using hierarchical control between the pendulum and manipulator dynamics and time-scaling arguments. Finally, section 6 relates the real-time implementation and the main characteristics of the robot.

2 The equations of motion

The pendulum is modeled as a one-dimensional rod Σ connected at the end point S of the manipulator. In the inertial reference frame $\{O, \vec{\imath}, \vec{\jmath}, \vec{k}\}$ fixed with respect to Earth, with $\vec{k}$ the upward vertical, Newton's second law for the pendulum reads

$$m\dot{\vec{V}}_C = m\vec{g} + \vec{F} \tag{1}$$

$$\dot{\vec{\sigma}} = \vec{CS} \times \vec{F}, \tag{2}$$

where $\vec{F}$ is the reaction of the manipulator on the pendulum, $\vec{V}_C$ is the velocity vector of the center of mass C and $\vec{\sigma}$ is the kinetic momentum about C,

$$\vec{\sigma} := \int_\Sigma \vec{CM} \times \vec{V}_M \; d\mu(M).$$

As Σ is one-dimensional, any point M of Σ satisfies $\vec{SM} = l_M \vec{u}$, where $\vec{u} := \frac{\vec{SC}}{\|\vec{SC}\|}$. Hence,

$$\vec{CM} = (l_M - l_C)\; \vec{u}$$
$$\vec{V}_M - \vec{V}_C = (l_M - l_C)\; \dot{\vec{u}},$$

which implies, since $\int_\Sigma \vec{CM} d\mu(M) = \vec{0}$,

$$\vec{\sigma} = I \vec{u} \times \dot{\vec{u}},$$

where $I := \int_\Sigma (l_M - l_C)^2 d\mu(M)$ is the moment of inertia of Σ about C.

For the manipulator, we will only need to know it can be described by

$$\dot{\vec{V}}_S = \vec{A}(\vec{OS}, \vec{V}_S, \vec{F}, \mathcal{T}), \tag{3}$$

where $\mathcal{T}$ is the 3-dimensional input corresponding to the 3 actuated dof's $((\theta_1, \theta_2 \theta_3)$ on figure 1).

Notice (1)-(2)-(3) is an implicit differential system. To put it in explicit form the coupling force $\vec{F}$ must be eliminated. The resulting explicit system would then have a state of dimension 10 (5 generalized positions and 5 generalized velocities).

3 The $2k\pi$-robot is flat

The important role of the oscillation center for control design has been first noticed in [1]. We recall in this section that the robot enjoys a nice property, namely all its variables can be known *without integration* as soon as the motion of a single special point H of the pendulum (the so-called center of oscillation [11]) is known. In other words the system is *flat* [3], which will be a key feature for the design of the the control law.

Indeed, using (1)-(2) and noticing $\dot{\vec{\sigma}} = I\vec{u} \times \ddot{\vec{u}}$, we find

$$\begin{aligned}
\vec{u} \times \dot{\vec{V}}_H &= \vec{u} \times \left(\dot{\vec{V}}_C + (l_H - l_C)\ddot{\vec{u}}\right) \\
&= \vec{u} \times \dot{\vec{V}}_C + (l_H - l_C)\dot{\vec{\sigma}} \\
&= \vec{u} \times \vec{g} + \left(\frac{1}{m} + (l_H - l_C)\frac{l_C}{I}\right) \vec{u} \times \vec{F}.
\end{aligned}$$

Therefore, the only point H of the pendulum such that

$$l_H := l_C + \frac{I}{m l_C}$$

satisfies $\vec{u} \times (\dot{\vec{V}}_H - \vec{g}) = \vec{0}$. In other words $\dot{\vec{V}}_H - \vec{g}$ is colinear to the direction of the pendulum.

It is now clear that the motion of the center of oscillation completely defines the evolution of all the system variables: indeed, when the motion of H, i.e., $t \mapsto \vec{OH}(t)$, is known, $\vec{V}_H$ and $\dot{\vec{V}}_H$ are known at all times. The orientation of the pendulum is then computed from $\dot{\vec{V}}_H$ by

$$\vec{u} = \pm \frac{\dot{\vec{V}}_H - \vec{g}}{\|\dot{\vec{V}}_H - \vec{g}\|}$$

(at least when $\dot{\vec{V}}_H \neq \vec{g}$), and $\dot{\vec{u}}, \ddot{\vec{u}}, \ldots$ follow by differentiation. Therefore, we obviously know the position, velocity, acceleration, ... of every point of the pendulum, and finally obtain $\vec{F}$ and $\mathcal{T}$ from the system equations (1) and (3).

Hence any motion of the center of oscillation determines a *trajectory* of the system (the converse is obvious). This property is clearly very helpful to plan reference trajectories; it is also very helpful to design a controller able to track desired reference trajectory [3, 9, 4]

The orientation of the pendulum is a priori not defined from $\dot{\vec{V}}_H$ when $\dot{\vec{V}}_H = \vec{g}$, i.e., when the center of oscillation H is *freely falling*. Notice any system trajectory connecting the lower and upper rest points necessarily cross this singularity. Indeed, $\dot{\vec{V}}_H = \vec{0}$ at rest and

$$\vec{u} = -\vec{k} = +\frac{\dot{\vec{V}}_H - \vec{g}}{\|\dot{\vec{V}}_H - \vec{g}\|}$$

when the pendulum is down, while

$$\vec{u} = \vec{k} = -\frac{\dot{\vec{V}}_H - \vec{g}}{||\dot{\vec{V}}_H - \vec{g}||}$$

when the pendulum is up.

4 Design of a suitable reference trajectory

As shown previously, any smooth trajectory from the lower to the upper equilibrium crosses the singularity $\dot{\vec{V}}_H = \vec{g}$. This singularity is intrinsic: around $\dot{\vec{V}}_H = \vec{g}$, the first order approximation is not controllable. This implies that the algebraic equations between the system variables and H with its derivatives are singular. Nevertheless, we will see that it is possible to design a wide family of smooth trajectories for H such that this algebraic systems admits a smooth solution. A careful inspection of computations of section 3, shows that the difficulty comes from sub-system defining S. It corresponds to the intersection of the sphere of center H and radius l_H with the straight line passing through H and of direction $\dot{\vec{V}}_H - \vec{g}$, direction that becomes undefined at the singularity. The problem reduces thus to smoothly define the direction of $\dot{\vec{V}}_H - \vec{g}$.

A natural path for H is a circle of radius $l_H = r$ in a vertical plane (see figure 2). It is defined by $||\vec{OH}|| = r$ and $\vec{OH}.\vec{\jmath} = 0$ We assume that the system is a steady-state at the beginning $t = 0$ and at the end $t = T > 0$ with S is at the origin:

$$\vec{OS}(O) = \vec{V}_S(O) = \dot{\vec{V}}_S(0) = \vec{0}$$

$$\vec{OS}(T) = \vec{V}_S(T) = \dot{\vec{V}}_S(T) = \vec{0}$$

Let $[0,1] \ni s \mapsto \alpha(s)$ be a smooth function, symmetric with respect to $s = 1/2$, strictly increasing on $]0,1[$ and subject to the boundary conditions

$$\begin{array}{ccc} \alpha(0) = -\frac{\pi}{2} & & \alpha(1) = \frac{\pi}{2} \\ \dot{\alpha}(0) = 0 & & \dot{\alpha}(1) = 0 \\ \ddot{\alpha}(0) = 0 & \text{and} & \ddot{\alpha}(1) = 0 \\ \alpha^{(3)}(0) = 0 & & \alpha^{(3)}(1) = 0 \\ \alpha^{(4)}(0) = 0 & & \alpha^{(4)}(1) = 0 \end{array}$$

(we used for instance a polynomial of degree 9). We consider derivatives up to order 4, since the reference torques $\mathcal{T}$ must be continuous and they depend on $(H, \ldots, H^{(4)})$.

Writing $\vec{OH}$ in coordinates, $\vec{OH} = \chi_x\vec{\imath} + \chi_y\vec{\jmath} + \chi_z\vec{k}$, we have the desired motion for H by taking

$$\begin{aligned} \chi_x(t) &:= r\cos(\alpha(t/T)) \\ \chi_y(t) &:= 0 \\ \chi_z(t) &:= r\sin(\alpha(t/T)) \end{aligned}$$

where T is the yet to be defined duration of the motion.

We now prove that a suitable choice of T provides a t_c such that $\ddot{\chi}_x(t_c) = \ddot{\chi}_z(t_c) + g = 0$ while $\frac{\ddot{\chi}_x}{\ddot{\chi}_z + g}$ is non-zero and smooth around t_c. This will ensure the direction of the pendulum is smoothly defined when crossing the singularity. For that consider the functions

$$\begin{aligned} f_x(s) &:= \cos(\alpha(s)) \\ f_z(s) &:= \sin(\alpha(s)). \end{aligned}$$

Notice f_x (resp. f_z) is symmetric (resp. antisymmetric) with respect to $s = 1/2$ and that for all $n \in \mathbb{N}$,

$$\begin{aligned} \chi_x^{(n)}(t) &:= \frac{r}{T^n} f_x^{(n)}(t/T) \\ \chi_z^{(n)}(t) &:= \frac{r}{T^n} f_z^{(n)}(t/T). \end{aligned}$$

For s small (and positive),

$$\alpha(s) \sim -\frac{\pi}{2} + as^k, \quad k \geq 5,$$

with $a > 0$ since α is strictly increasing on $]0,1[$. This implies

$$f_x(s) \sim \cos(-\frac{\pi}{2} + as^k) \sim as^k,$$

hence $\ddot{f}_x(s) > 0$ for s (positive) small. But $\ddot{f}_x(1/2) = -\dot{\alpha}^2(1/2) < 0$, and there must be $r_c \in]0,1/2[$ such that $\ddot{f}_x(r_c) = 0$. By symmetry $\ddot{f}_x(s_c) = 0$ for $s_c := 1 - r_c \in]1/2,1[$.

On the other hand $\ddot{f}_z(s_c) < 0$; indeed,

$$\begin{aligned} \ddot{f}_z(s_c) &= \frac{\ddot{f}_x(s_c)\cos\alpha(s_c) + \ddot{f}_z(s_c)\sin\alpha(s_c)}{\sin\alpha(s_c)} \\ &= -\frac{\dot{\alpha}^2(s_c)}{\sin\alpha(s_c)} < 0, \end{aligned}$$

since $\sin\alpha > 0$ on $]1/2,1[$. We can thus define

$$T := \sqrt{-\frac{g}{r\ddot{f}_z(s_c)}} \qquad \text{and} \qquad t_c := s_c T,$$

to get get $\ddot{\chi}_x(t_c) = \ddot{\chi}_z(t_c) + g = 0$.

Assuming moreover $f_x^{(3)}(s_c).f_z^{(3)}(s_c) \neq 0$ (this is true for almost all α satisfying the above requirements), Taylor's theorem gives

$$\begin{aligned} \ddot{\chi}_x(\tau + t_c) &= -\frac{r}{T^2}\tau f_x^{(3)}(s_c) + \tau^2 F_x(\tau) \\ \ddot{\chi}_z(\tau + t_c) + g &= -\frac{r}{T^2}\tau f_z^{(3)}(s_c) + \tau^2 F_z(\tau), \end{aligned}$$

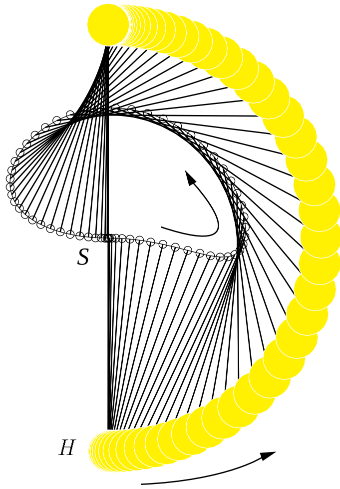

Figure 2: reference trajectory steering the pendulum from the lower equilibrium to the upper one.

where F_x and F_z are smooth mappings. Hence $\ddot{\chi}_x$ and $\ddot{\chi}_z + g$ change signs at t_c while their ratio is nonzero and smooth. In other words, by Taylor's theorem,

$$\frac{\dot{\vec{V}}_H(\tau + t_c/T) - \vec{g}}{||\dot{\vec{V}}_H(\tau + t_c/T) - \vec{g}||} = \text{sign}(\tau)\big(\vec{u}_c + \tau \vec{U}(\tau)\big),$$

where $\vec{u}_c$ is the unit vector

$$\vec{u}_c = \frac{f_x^{(3)}(s_c)\ \vec{\imath} + f_z^{(3)}(s_c)\vec{k}}{\sqrt{\left(f_x^{(3)}(s_c)\right)^2 + \left(f_z^{(3)}(s_c)\right)^2}}$$

and $\tau \mapsto \vec{U}(\tau)$ is smooth.

This means that the orientation of the pendulum is smoothly defined on $[0, T]$ from the motion of the center of oscillation by

$$\vec{u}(t) = \text{sign}(t - t_c/T)\frac{\dot{\vec{V}}_H(t) - \vec{g}}{||\dot{\vec{V}}_H(t) - \vec{g}||}$$

(we assume without restriction that t_c is the only point such that $\dot{\vec{V}}_H(t_c) = \vec{g}$). It is now obvious to build a smooth trajectory of (1)-(2)-(3) steering the robot from the lower equilibrium to the upper equilibrium. Figure 2 displays the motions of H and S obtained with α the polynomial of degree 8, $s_c \approx 0.630$, $T \approx 1.238\ s$ for $g = 9.81\ ms^{-2}$ and $l_H = r = 0.35\ m$.

5 The tracking feedback law

We assume full state measures. The tracking controlled is decomposed into two levels:

1. the low-level corresponds to the manipulator; it is fast and ensures a velocity control of the manipulator; for each actuated axis, the correction from the reference torque (deduced from the steering trajectory of previous section) is computed via a simple high-gain and proportional angular velocity-loop.

2. the high-level corresponds to the pendulum with the velocity of S as control; it is slow and provide, after a transformation into angular velocity, the velocity set-point of the low-level.

This two-time scale structure ensures the robustness with respect to modeling error in the manipulator dynamics: with $\vec{V}_S$ as input, the pendulum dynamics is precisely known and admits only two well defined parameter, g and l_H.

We start with the precise validation of this two-time-scale structure. A careful choice of the state variables used for the pendulum dynamics must be done. The use of position/velocity variables in the design of the high-level controller will destroy, in general such structure. Roughly speaking, position/impulsion variables are necessary at this stage. This phenomenon is comparable to a classical computation due to Kapitsa and relative to the motion of a particle under highly oscillating forces [6].

We will use the following notation: whenever $\vec{X}$ is a vector, we decompose it into $\vec{X} = x\vec{u} + \vec{X}^-$, where $x = \vec{X}.\vec{u}$ is the component along the pendulum and $\vec{X}^-.\vec{u} = 0$

5.1 Time-scale reduction

The low-level high-gain loop transforms the manipulator dynamics into

$$\varepsilon\dot{\vec{V}}_S = \vec{V} - \vec{V}_S + \mathcal{O}(\varepsilon)$$

with ε a "small" positive parameter.

To correctly perform the reduction we rewrite the system in state form in the adapted coordinates $\vec{OH}, \vec{V}_H^-, \vec{OS}, \vec{V}_S$.

Writing $\vec{V}_H^- = \vec{V}_H - V_H.\vec{u}$, we then find

$$\begin{aligned}\dot{\overbrace{\vec{V}_H^-}} &= \dot{\vec{V}}_H - (\dot{\vec{V}}_H.\vec{u} + \vec{V}_H.\dot{\vec{u}})\vec{u} - (\vec{V}_H.\vec{u})\dot{\vec{u}} \\ &= \dot{\vec{V}}_H - \vec{g} - (\dot{\vec{V}}_H - \vec{g}).\vec{u} + \vec{g}^- - (\vec{V}_H.\dot{\vec{u}})\vec{u} - (\vec{V}_H.\vec{u})\dot{\vec{u}} \\ &= \vec{g}^- - (\vec{V}_H^-.\dot{\vec{u}})\vec{u} - v_S\dot{\vec{u}},\end{aligned}$$

using the fact that $\dot{\vec{V}}_H - \vec{g}$ is colinear to $\vec{u}$ and

$$\begin{aligned}\dot{\vec{u}}.\vec{V}_H &= \dot{\vec{u}}.(v_H\vec{u} + \vec{V}_H^-)\\ &= \dot{\vec{u}}.\vec{V}_H^-\\ \vec{V}_H.\vec{u} &= (\vec{V}_S + l_H\dot{\vec{u}}).\vec{u}\\ &= \vec{V}_S.\vec{u}.\end{aligned}$$

We then have the state form

$$\begin{aligned}\dot{\vec{OH}} &= \vec{V}_H^- + v_S\vec{u}\\ \dot{\vec{V}}_H^- &= \vec{g}^- - (\vec{V}_H^-.\dot{\vec{u}})\vec{u} - v_S\dot{\vec{u}}\\ \dot{\vec{OS}} &= \vec{V}_S\\ \varepsilon\dot{\vec{V}}_S &= \vec{V} - \vec{V}_S + \mathcal{O}(\varepsilon)\end{aligned}$$

where $\vec{u}$ and $\dot{\vec{u}}$ are defined in terms of the state variables by

$$\begin{aligned}l_H\vec{u} &= \vec{OH} - \vec{OS}\\ l_H\dot{\vec{u}} &= \vec{V}_H^- - \vec{V}_S^-.\end{aligned}$$

Notice this system is not controllable due to overparametrization (we use $3+3=6$ position variables in the state though there are only 5 dof's). Nevertheless, the theorem of singular perturbations [5] ensures it can be approximated with an error of order ε in the "low" frequency range by the "slow" system

$$(4)\qquad \begin{cases}\dot{\vec{OH}} = \vec{V}_H^- + v_S\vec{u}\\ \dot{\vec{V}}_H^- = \vec{g}^- - (\vec{V}_H^-.\dot{\vec{u}})\ \vec{u} - v_S\dot{\vec{u}}\\ \dot{\vec{OS}} = \vec{V}\\ \\ l_H\vec{u} = \vec{OH} - \vec{OS}\\ l_H\dot{\vec{u}} = \vec{V}_H^- - \vec{V}^-.\end{cases}$$

5.2 The high-level controller

There follows

$$\begin{aligned}\dot{\vec{V}}_H &= \dot{\vec{V}}_H^- + \dot{v}\vec{u} + v\dot{\vec{u}}\\ &= \vec{g} + (\dot{v} - \vec{V}_H^-.\dot{\vec{u}} - \vec{g}.\vec{u})\vec{u},\end{aligned}$$

so that the globally defined dynamic feedback

$$\begin{aligned}\vec{V} &= \zeta\vec{u} + \vec{V}_H^- - l_H\vec{W}^-\\ \dot{\zeta} &= \vec{g}.\vec{u} + \vec{V}_H^-.\vec{W}^- - l_H w,\end{aligned}$$

where $\vec{W}$ is the new control, transforms (4), hence (1)-(2)-(3), into

$$(5)\qquad \begin{cases}\dot{\vec{V}}_H = \vec{g} - l_H w\vec{u}\\ \dot{\vec{u}} = \vec{W}^-.\end{cases}$$

Notice $\vec{u}(t)$ will be a unit vector as soon as $\vec{u}(0)$ is a unit vector. The controllable state of this system has thus dimension 3+3+2=8 obtained from the original controllable system of dimension 10 with a dynamic feedback adding 1 state and after time-scale reduction of 3 states. This global dynamic feedback has been obtained via a state representation of the dynamics that is not minimal.

The time varying change of coordinates

$$H \mapsto P = H - 1/2t^2\vec{g}, \quad \vec{V}_H \mapsto \vec{V}_H - t\vec{g}$$

, put (5) into the so-called second order nonholonomic form

$$\ddot{P} = -l_H w\vec{u} \qquad \dot{\vec{u}} = \vec{W}^-.$$

with $\vec{u} \in \mathbb{S}^2$ and 3 controls $w \in \mathbb{R}$ and $\vec{W}^- \in T\mathbb{S}^2_{\vec{u}}$. Lower and upper equilibria correspond here to $\vec{W}^- = 0$ and $|l_H w| = g$

Let us concentrate on the sub-system

$$\dot{\vec{V}}_P = -l_H w\vec{u} \qquad \dot{\vec{u}} = \vec{W}^-.$$

ignoring $\dot{P} = \vec{V}_P$. This sub-system is the analog of a nonholonomic car in $\mathbb{R}^4$ rolling without slipping on $\mathbb{R}^3$ with the velocity $-l_H w \in \mathbb{R}$ and two steering angles represented by $\vec{u} \in \mathbb{S}^2$. The singularity $\dot{\vec{V}}_H = \vec{g}$ corresponds to zero velocity: its crossing coincides here with the switch between backward and forward motion. Such a similarity enable us to mimic the time-varying tracking controller already used for trajectory tracking of such non-holonomic flat systems [2]. Denoting reference by superscript ^{r}, we have the following dynamics feedback ($\xi \approx 1$):

$$\begin{aligned}\dot{\xi} &= (\vec{u}\cdot\vec{\eta})\ \dot{s}_r(t)\\ -l_H w &= \xi\ \dot{s}_r(t)\\ \vec{W}^- &= (\vec{\eta} - (\vec{u}\cdot\vec{\eta})\vec{u})\ \dot{s}_r(t)\end{aligned}$$

with

$$(6)\qquad \vec{\eta} = [\vec{V}_P^r]'' + \lambda_1(\xi\vec{u} - [\vec{V}_H^r]') + \lambda_2(\vec{V}_H - \vec{V}_H^r).$$

Operator $'$ denotes derivatives with respect to s_r the arc length of the curve followed by $\vec{V}_P^r$ and defined by $ds_r = \pm\left|\dot{\vec{V}}_H(t) - \vec{g}\right|\ dt$. The two design parameters $\lambda_1 = (1/\sigma_1 + 1/\sigma_2)$ and $\lambda_2 = -1/(\sigma_1\sigma_2)$ correspond to the two tracking poles σ_1, σ_2 in scale s_r: their sign depends on the sign of $\dot{s}_r$.

The above tracking controller is a velocity tracking controlled for H. To obtain a position tracking controller we just have replaced in (6) $\vec{V}_P^r$ by

$$(7)\qquad \vec{V}_P^r - (H - H^r)\dot{s}_r/\sigma_3$$

with $\sigma_3\dot{s}_r < 0$ and $|\sigma_3| \ll |\sigma_1|, |\sigma_2|$ in order to ensure stability for time-scale reasons.

6 Experiments

6.1 The experimental setup

The main geometrical and mechanical characteristics of manipulator and pendulum are:

- pendulum length : 0.40 m with $l_H = 0.35$ m;
- manipulator bodies length : 0.32 m and 0.28 m;
- max acceleration capacity for the pendulum oscillation center 7 g;

The pendulum is brought by a two axis Cardan joint, each one being equipped with an incremental single turn encoder (angular resolution between electrical transitions : 0.000628 rd and 0.000785 rd). The three angles of the manipulator bodies are measured by incremental single turn encoder (angular resolution between electrical transitions : 0.000314 rd).

The encoder electrical transitions are sampled by a high frequency clock. This allows to give the accurate time of the angular transitions to the numerical filter (derivator) which computes the angular speeds. Therefore the precision of speed measurement remains high at very low speed, which is the required quality in order to stabilize the pendulum.

The time scaling of the whole control system is shared as below :

- open-loop pendulum characteristic time, 189 ms;
- angular transitions sampling clock, 800 ns;
- measurement period, 4 ms (typical);
- derivator time constant, 4 ms;
- manipulator control period, 12 ms (typical);
- manipulator angular speed control time constant, 66 ms (the "high-gain" low-level controller);
- pendulum cartesian speed tracking time constants (σ_1 and σ_2 used in (6)) are 150 ms and 225 ms;
- pendulum cartesian position tracking time constant (σ_3 used in (7)) is of 675 ms;

The real-time controller software is implemented under SUN Unix 4.1.3 Operating System on a SUN Sparc 2.

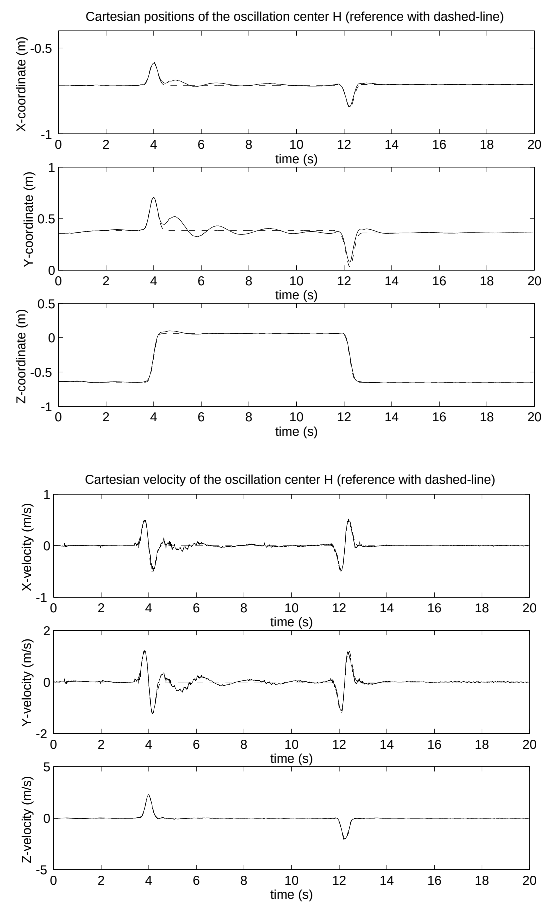

Figure 3: real data corresponding to steering from the lower equilibrium to the upper one and back to the lower.

6.2 Experimental results

The experimental data of figure 3 correspond to the above tracking controller with a reference trajectory made of the concatenation of the one of figure 2. Notice the good tracking performance for $\vec{V}_H$. The low frequency oscillation during the intermediate stabilization phase (between the rise and fall phases) is due to imperfection not taking into account in the model.

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