TWO REMARKS ON INDUCTION MOTORS

Philippe Martin [*] **Pierre Rouchon** [**]

[*] *Centre Automatique et Systèmes, École des Mines de Paris*
35, rue Saint-Honoré, 77305 Fontainebleau Cedex, France
martin@cas.ensmp.fr
[**] *Centre Automatique et Systèmes, École des Mines de Paris*
60, Bd. Saint-Michel, 75272 Paris, France
rouchon@cas.ensmp.fr

Abstract: This paper addresses two points: first, induction motors are shown to be flat, which implies they can be easily controlled by full-state feedback (for instance with a linearizing (dynamic) feedback); second, an exponential observer, relying on easily measurable variables (stator current and rotor velocity) is proposed.

Keywords: Nonlinear control, induction motors, flatness, observers.

1. INTRODUCTION

The induction motor has very good qualities –reliability, ruggedness, relatively low cost, etc– for industrial applications. The reason why is that there is no mechanical commutation: the rotor consists simply of closed windings in which currents are induced by a rotating magnetic field set up by the stator, hence creating a torque. But the control of the induction motor is not so easy, mainly because of the three following points: the model is rather nonlinear, some variables (in particular the magnetic fluxes, or equivalently the rotor current) cannot easily be measured, and some parameters (rotor resistance, load torque) vary a lot in operation. This has motivated a growing literature in the control community, see e.g. [6,13,7,8,12,10,1,3,2,9] and the references therein.

The goal of this paper is twofold: first we show that an induction motor is *flat*, i.e., its system dynamics, hence its control, is not as complicated as it may look; in particular it could easily be controlled with a linearizing (dynamic) feedback if all the variables were measured. Second, assuming measurements of the stator current and rotor velocity, we propose a flux observer which exponentially converges as fast as desired. This extends and simplifies two already known results: on the one hand, the induction motor is linearizable by dynamic feedback [1,2] –which is essentially equivalent to the system being flat–; on the other hand, exponential flux observers exist in some cases [13]. An interesting point in our approach is that it remains very simple and is easily understandable in terms of the physics of the system. It is interesting to notice that the computations are straightforward, mainly because they are carried out using the complex variable notation (time-varying phasor) model of the induction motor, contrary to what is often done.

These two results show that an induction motor is rather easy to control, provided the stator current and rotor velocity are measured. A high level of performance can be achieved. It is fair to mention that in many industrial applications, the rotor velocity is in fact not measured (though velocity (or position) sensors are reliable and rather cheap). In this case, one has to make do with only stator current measurements ("sensorless" control), which is much for difficult, especially when a high level of performance is desired.

2. MODEL OF THE INDUCTION MOTOR

We recall here the two-phase equivalent machine representation of a standard balanced three-phase induction motor. The reader is referred for instance to [6] for a complete derivation of the equations, as well as the general theory of motors and related control problems.

The electrical equations describing the stator and rotor circuits are

$$R_s i_s + \frac{d\psi_s}{dt} = u_s \tag{1}$$

$$R_r i_r + \frac{d\psi_r}{dt} = 0, \tag{2}$$

where R_s is the stator resistance, i_s the stator current, ψ_s the stator flux, u_s the voltage applied to the stator, R_r the rotor resistance, i_r the rotor current, and ψ_r the rotor flux. We use the complex representations for currents, fluxes and voltages. For instance $i_s = i_{sa} + j i_{sb}$, where i_{sa} and i_{sb} denote the currents in each of the two stator phases and $j^2 = -1$. Since the motor can be actuated through u_s, there are two independent (real) controls u_{sa} and u_{sb}.

Under the assumptions of linearity of the magnetic circuits (valid as long as the stator currents are not too large) and neglecting iron losses, fluxes and currents are related by

$$\psi_s = L_s i_s + M e^{jn\theta} i_r \tag{3}$$

$$\psi_r = M e^{-jn\theta} i_s + L_r i_r, \tag{4}$$

where θ is the rotor position, L_s the stator inductance, L_r the rotor inductance, M the mutual inductance between the stator and the rotor, and n the number of pairs of poles.

As a consequence of the Lorenz force law, the torque produced by the motor is

$$n\Im(i_r^\star \psi_r),$$

where $\Im(z)$ stands for the imaginary part of the complex number z. The motion of the rotor is thus given by

$$J\frac{d^2\theta}{dt^2} = n\Im(i_r^\star \psi_r) - \tau_L, \tag{5}$$

with J the moment of inertia of the rotor and τ_L the load torque. The load torque usually depends on θ and $\dot\theta$.

3. THE INDUCTION MOTOR IS FLAT

We show here that the model of the induction motor is *flat*; briefly (see [5,11,4] for a detailed exposition), a control system $\dot x = f(x,u)$ is said to be *flat* if there exists a map $y := h(x,\overline{u})$ which is "differentially invertible". This means that in the one hand the components of h are differentially independent (i.e., $(h, \dot h, \ddot h, \ldots, h_{(k)})$ is full rank for every integer k), and in the other hand $x = \varphi(\overline{y})$ and $u = \alpha(\overline{y})$ for some maps φ and α. We use the notation $\overline{z}$ to denote $(z, \dot z, \ddot z, \ldots, z_{(l)})$ for some finite but otherwise arbitrary integer l. Such a differentially invertible map is called a *flat output* of the system. In other words, there is a one to one correspondence between *trajectories* $(x(t), u(t))$ of the system and *arbitrary* curves $y(t)$. A flat system enjoys two essential properties: first the motion planning is trivial; second it can be linearized by (dynamic) feedback (around a regular point). Therefore, it is rather easy to control.

Though flatness is obviously a highly non-generic property, many systems encountered in engineering happen to be flat; moreover the flat outputs usually have a nice physical interpretation, as it is the case for the induction motor. It is also worth mentioning that the notion of flatness is not restricted to system in "state form" $\dot x = f(x,u)$, but is also meaningful for undertermined system of differential equations of the form $F(\overline{\xi}) = 0$ (in simple words: a bunch of equations constraining a bunch of variables). For instance, we will not have to write state equations for the induction motor.

It was already proved in [2,1] that the induction motor was linearizable by (dynamic) feedback. What we do here is in a way a very simple and natural restatement of this result which moreover gives physical insight on the motor dynamics. To start with, we rewrite the mechanical equation (5) in a suitable form: using the electrical equation (2) and setting $\psi_r = \rho e^{j\alpha}$, the torque produced by the motor is

$$\begin{aligned} -n\Im(i_r \psi_r^\star) &= \frac{n}{R_r}\Im(\psi_r^\star \frac{d\psi_r}{dt}) \\ &= \frac{n}{R_r}\rho^2\dot\alpha. \end{aligned}$$

The mechanical equation (5) then reads

$$J\frac{d^2\theta}{dt^2} = \frac{n}{R_r}\rho^2\dot\alpha - \tau_L. \tag{6}$$

We claim that $y := (\theta, \alpha)$ is a flat output –assuming the load torque τ_L does not depend on ψ_s nor i_s (but possibly depends on θ and $\dot\theta$)–. Indeed from (6), ρ is readily seen to be a function of $\theta, \dot\theta, \ddot\theta$ and $\dot\alpha$ (notice there is a singularity when $\dot\alpha = 0$),

$$\rho = \sqrt{\frac{R_r(J\ddot\theta + \tau_L}{n\dot\alpha}},$$

hence we can write

$$\psi_r = A(y, \dot y, \ddot y). \tag{7}$$

From (2) and (7),

$$i_r = -\frac{1}{R_r}\frac{d\psi_r}{dt} = -\frac{1}{R_r}\dot A(y, \dot y, \ddot y, y^{(3)}). \tag{8}$$

From (4), (7) and (8),

$$i_s = -\frac{1}{M}(\psi_r - L_r i_r)e^{jn\theta} = B(y, \dot{y}, \ddot{y}, y^{(3)}). \quad (9)$$

From (3) and (7)–(9),

$$\psi_s = L_s i_s + M e^{jn\theta} i_r = C(y, \dot{y}, \ddot{y}, y^{(3)}). \quad (10)$$

Finally from (1) and (7)–(10),

$$u_s = R_s i_s + \frac{d\psi_s}{dt} = D(y, \dot{y}, \ddot{y}, y^{(3)}, y^{(4)}),$$

which ends the proof. Since α, the argument of the complex rotor flux ψ_r is simply (up to a rotation of θ) the angle of the magnetic field, we have thus established the following fact:

> *the angles of the rotor position and of the magnetic field form a flat output, i.e., they completely determine* all *the variables describing the motor.*

4. AN OBSERVER WITH ARBITRARY EXPONENTIAL SPEED OF CONVERGENCE

As mentioned in the introduction, the only easily measured electrical variable is the stator current. To build a controller, it is thus necessary to somehow estimate the rotor current (or equivalently the stator or rotor fluxes). We propose in this section an exponential observer of the rotor flux, assuming the stator current i_s and rotor speed $\omega := \frac{d\theta}{dt}$ are measured. The observer relies only on the electo-magnetic equations, hence is independant of the mechanical parameters. Our approach is quite similar to [13], but with a rather different treatment which leads in particular to a very simple proof of the exponential decay of the error, not restricted, like in [13], to gains giving real eigenvalues.

It is convenient for our purpose to use a state-form model with the stator current i_s and the rotor flux $\tilde{\psi}_r := e^{jn\theta}\psi_r$ as state variables. We first eliminate ψ_s and i_r in (3)-(4) to get

$$\psi_s = \sigma L_s i_s + \frac{M}{L_r}\tilde{\psi}_r$$
$$e^{jn\theta} i_r = \frac{\tilde{\psi}_r - M i_s}{L_r},$$

where $\sigma := 1 - \frac{M^2}{L_s L_r}$ is the so-called *leakage factor*. Noticing that

$$\frac{d\tilde{\psi}_r}{dt} = e^{jn\theta}\frac{d\psi_r}{dt} + jn\omega\tilde{\psi}_r,$$

and using (2), we then deduce

$$\frac{d\tilde{\psi}_r}{dt} = \frac{R_r M}{L_r} i_s + (jn\omega - \frac{R_r}{L_r})\tilde{\psi}_r.$$

Eventually, it follows from (1) that

$$\sigma\frac{di_s}{dt} + \left(\frac{R_s}{L_s} + \frac{R_r}{L_r}(1-\sigma)\right) i_s + \frac{1-\sigma}{M}(jn\omega - \frac{R_r}{L_r})\tilde{\psi}_r = \frac{u_s}{L_s}.$$

Introducing the time constants

$$\tau_r := \frac{L_r}{R_r}$$
$$\tau_s := \frac{1}{\sigma}\left(\frac{R_s}{L_s} + \frac{R_r}{L_r}(1-\sigma)\right)$$

and the dimensionless complex quantity $z := 1 - jn\tau_r\omega$, we can rearrange the two last equations as

$$\frac{d\tilde{\psi}_r}{dt} = \frac{M}{\tau_r} i_s - \frac{1}{\tau_r} z\tilde{\psi}_r \quad (11)$$
$$\frac{di_s}{dt} = -\frac{1}{\tau_s} i_s - \frac{1-\sigma}{M\sigma\tau_r} z\tilde{\psi}_r + \frac{u_s}{L_s\sigma}. \quad (12)$$

A fundamental remark for the design of an observer is that the only unknown variable $\tilde{\psi}_r$ always enters the equations with a z factor; this quantity will be related later to a sort of “complex time”.

We claim that the system

$$\frac{d\hat{\psi}_r}{dt} = \frac{M}{\tau_r} i_s - \frac{1}{\tau_r} z + \xi_1(i_s - \hat{i}_s)z$$
$$\frac{d\hat{i}_s}{dt} = -\frac{1}{\tau_s} i_s - \frac{1-\sigma}{M\sigma\tau_r} z\hat{\psi}_r + \frac{u_s}{L_s\sigma} + \xi_2(i_s - \hat{i}_s)z,$$

with ξ_1, ξ_2 complex gains yet to be chosen, is an exponential observer for the flux (notice the driving term $i_s - \hat{i}_s$ enters the equations with a z factor). Indeed, setting $\varepsilon_\psi := \psi_s - \hat{\psi}_s$ and $\varepsilon_i := i_s - \hat{i}_s$, the error equation is given by

$$\frac{d}{dt}\begin{pmatrix}\varepsilon_\psi \\ \varepsilon_i\end{pmatrix} = -z\begin{pmatrix}\frac{1}{\tau_r} & \xi_1 \\ \frac{1-\sigma}{M\sigma\tau_r} & \xi_2\end{pmatrix}\begin{pmatrix}\varepsilon_\psi \\ \varepsilon_i\end{pmatrix},$$

or, with self-evident vector notations,

$$\frac{d\varepsilon}{dt} = -zA\varepsilon,$$

where A is a constant matrix whose eigenvalues can be freely assigned by ξ_1 and ξ_2.

Now, if we perform the “complex change of time” defined by

$$\frac{ds}{dt} := z,$$

or, more explicitely, by

$$s := t - j\, n\tau_r\tilde{\theta}(t)$$

(notice s is indeed a time-like quantity), where

$$\tilde{\theta}(t) := \int_0^t \omega(\tau) d\tau, \quad (\text{notice } \theta \equiv \tilde{\theta} \bmod 2\pi),$$

and set $\tilde{\varepsilon}(s) := \varepsilon(t)$, the error equation in time s

$$\frac{d\tilde{\varepsilon}}{ds} = A\tilde{\varepsilon}.$$

is time-invariant. This immediately gives an explicit formula for the evolution of the error:

$$\varepsilon(t) = \exp[-A(t - j\ \tau_r \tilde{\theta}(t) L_r)]\ \varepsilon(0).$$

Denoting by $\lambda_1 = \lambda_{1a} + j\ \lambda_{1b}$ and $\lambda_2 = \lambda_{2a} + j\ \lambda_{2b}$ the eigenvalues of A (they need not be complex conjugates since A is complex), the error $\varepsilon(t)$ is a linear combination of the complex exponentials

$$\begin{array}{c} \exp\left[-\lambda_{1a}t - \lambda_{1b} n \tau_r \tilde{\theta}(t) - j\ (\lambda_{1b}t - \lambda_{1a} n \tau_r \tilde{\theta}(t))\right] \\ \exp\left[-\lambda_{2a}t - \lambda_{2b} n \tau_r \tilde{\theta}(t) - j\ (\lambda_{2b}t - \lambda_{2a} n \tau_r \tilde{\theta}(t))\right], \end{array}$$

hence its growth is bounded by $\exp\left[-\lambda_{1a}t - \lambda_{1b} n \tau_r \tilde{\theta}(t)\right]$ and $\exp\left[-\lambda_{2a}t - \lambda_{2b} n \tau_r \tilde{\theta}(t)\right]$. To ensure an exponential decay, it thus suffices to choose the gains ξ_1 and ξ_2 so that λ_{1a}, λ_{2a}, $\lambda_{1b}\text{sign}(\omega)$, $\lambda_{2b}\text{sign}(\omega)$ are positive. Notice the gains depend on $\text{sign}(\omega)$, in order to make $\lambda_{1b} n \tau_r \tilde{\theta}(t)$ and $\lambda_{2b} n \tau_r \tilde{\theta}(t)$ increase with time.

In practice, it is worth choosing ξ_1, ξ_2 such that λ_1, λ_2 are not real. Indeed, as soon as the rotor speed is not too small, $\tau_r \tilde{\theta}(t)$ is much larger than t, hence the observer converges much faster and is much less sensitive to perturbations or modeling errors –remember that the rotor resistance R_r is not very well-known– than with gains of the same magnitude leading to real eigenvalues. Since the variable $\tilde{\theta}$ corresponds to the number of rotor turns, the observer is fast at high rotor speeds and slower at lower speeds, which is perfectly sensible from a physical point of view. In [13] the authors had to restrict to real eigenvalues in order to prove the convergence of their observer, hence did not make use of this interesting property.

ACKNOWLEDGMENTS

We are thankful to J. Chiasson for interesting remarks about this paper.

5. REFERENCES

[1] J. Chiasson. Dynamic feedback linearization of the induction motor. *IEEE Trans. Automat. Control*, 38:1588–1594, 1993.

[2] J. Chiasson. A new approach to dynamic feedback linearization control of an induction motor. In *Proc. of the 34nd IEEE Conf. on Decision and Control*, pages 2173–2178, New Orleans, 1995.

[3] G. Espinosa and Ortega. State observer are unnecessary for induction motor contro. *Systems & Control Letters*, 23:315–323, 1994.

[4] M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon. Nonlinear control and Lie-B acklund transformations: towards a new differential geometric standpoint. In *Proc. of the 33nd IEEE Conf. on Decision and Control*, pages 329–344, Lake Buena Vista, 1994.

[5] M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon. Flatness and defect of nonlinear systems: introductory theory and applications. *Int. J. of Control*, 61(6):1327–1361, 1995.

[6] W. Leonhard. *Control of Electrical Drives*. Springer-Verlag, 1985.

[7] W. Leonhard. 30 years space vectors, 20 years field orientation, 10 years digital signal processing with control ac-drives, a review (part 1). *EPE Journal*, 1(1):13–20, 1991.

[8] W. Leonhard. 30 years space vectors, 20 years field orientation, 10 years digital signal processing with control ac-drives, a review (part 2). *EPE Journal*, 1(2):89–100, 1991.

[9] R. Marino, S. Peresada, and P. Tomei. Output feedback control of current-fed induction motors with unknown rotor resistance. *IEEE Trans. Automat. Control*, 1996. To appear.

[10] R. Marino, S. Peresada, and P. Valigi. Adaptive input-output linearizing control of induction motors. *IEEE Trans. Automat. Control*, 38:208–221, 1993.

[11] Ph. Martin. *Contribution à l'étude des Systèmes Différentiellement Plats*. PhD thesis, Ecole des Mines de Paris, France, 1992.

[12] R. Ortega and G. Espinosa. Torque regulation of induction motors. *Automatica*, 29:621–633, 1993.

[13] G.C. Verghese and S.R. Sanders. Observers for flux estimation in induction machines. *IEEE Trans. Industrial Electronics*, 35:85–94, 1988.