Feedback linearization and driftless systems *

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Résumé

The problem of dynamic feedback linearization is recast using the notion of dynamic immersion. We investigate here a "generic" property which holds at every point of a dense open subset, but may fail at some points of interest, such as equilibrium points. Linearizable systems are then systems that can be immersed into linear controllable ones. This setting is used to study the linearization of driftless systems : a geometric sufficient condition in terms of Lie brackets is given ; this condition is shown to be also necessary when the number of inputs equals two. Though non invertible feedbacks are not a priori excluded, it turns out that linearizable driftless systems with two inputs can be linearized using only invertible feedbacks, and can also be put into chained form by (invertible) static feedback. Most of the developments are done within the framework of differential forms and Pfaffian systems.

Key words : nonlinear systems, feedback linearization, dynamic immersion, driftless systems, Pfaffian systems.

1 Introduction

The problem of feedback linearization (see e.g., [CLM89]) of a (smooth) control system

$$\dot{x} = f(x,u)$$

defined on an open subset $X \times U$ of $\mathbb{R}^n \times \mathbb{R}^m$ consists in finding a (smooth) dynamic feedback

$$\begin{aligned} \dot{z} &= a(x,z,v) \\ u &= \sigma(x,z,v) \end{aligned}$$

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defined on an open subset $\tilde{X} \times Z \times V$ of $X \times \mathbb{R}^r \times \mathbb{R}^q$, such that the closed-loop system

$$\begin{aligned}\dot{x} &= f(x, \sigma(x, z, v))\\ \dot{z} &= a(x, z, v),\end{aligned}$$

is diffeomorphic on $\tilde{X} \times Z$ to a controllable linear system.

To be precise, we say the system is *linearizable at a point* (x_0, u_0) if $\tilde{X}$ is a neighborhood of x_0 and $\sigma(\tilde{X}, Z, V)$ is a neighborhood of u_0, and say it is *linearizable* if it is linearizable at every point of a dense open subset of $X \times U$. Clearly, when a system is linearizable at a point, it is linearizable on a neighborhood of that point.

Some remarks are in order to emphasize what we mean by "feedback linearizable", for the terminology has been used by several authors with different meanings. First we deal with *dynamic* feedbacks, unless we explicitly mention we consider static feedbacks (i.e., $u = \sigma(x, v)$ and $\dim(z) = 0$).

Second, we carefully distinguish a "point-wise" property (to be linearizable *at* a point (x_0, u_0)) from a "generic" one (to be linearizable). Also we do not implicitly assume that (x_0, u_0) is an equilibrium point of the system. For instance the driftless system

$$\begin{aligned}\dot{x}^1 &= u^1\\ \dot{x}^2 &= u^2\\ \dot{x}^3 &= x^2 u^1\end{aligned}$$

is linearizable, using the feedback

$$\begin{aligned}\dot{z}^1 &= v^1\\ u^1 &= z^1\\ u^2 &= \frac{v^2 - x^2 v^1}{z^1}\end{aligned}$$

and the diffeomorhism

$$(x^1, x^2, x^3, z^1) \longmapsto (x^1, z^1, x^3, x^2 z^1),$$

though it is not linearizable at points (x_0, u_0) such that $u_0^1 = 0$, and in particular at equilibrium points. Indeed, a linearizable driftless system $\dot{x} = \sum_{i=1}^m u^i f_i(x)$ is never linearizable at an equilibrium point (discarding the trivial case $m = n$) because its linear approximation is not controllable [CLM89]. Of course, knowing that a system is feedback linearizable may not be very useful for controlling it around such "singular" points. Nevertheless such structural information should not be discarded : from the control point of view, it may be used to easily steer the system to a neighborhood of the singular point of interest [RFLM93] ; from the mathematical point of view it is sensible to first study the "generic" case and then try to find adequate local models for the singularities.

Third, we do not a priori rule out non invertible feedbacks. By invertible feedback we mean that the closed-loop system, together with the output $y = \sigma(x, z, v)$, is input-output invertible (see e.g., [DBGM89]). Notice for instance that the system

$$\begin{aligned}\dot{x}^1 &= x^2 + (u^1)^2 (u^2)^3\\ \dot{x}^2 &= x^3 + u^2\\ \dot{x}^3 &= u^1\end{aligned}$$

is not linearizable by invertible (dynamic) feedback, since it does not satisfy the necessary condition of [Rou94]. However, the non invertible static feedback

$$\begin{aligned} u^1 &= v^1 \\ u^2 &= 0 \end{aligned}$$

is obviously linearizing. In the same way we do not a priori assume that dynamic feedbacks have some extra property leading to input-output decoupling without zero dynamics [Isi86, IML86] or leading to dynamic equivalence like flatness [FLMR92b, FLMR95, Mar92, FLMR] , absolute equivalence [Sha90, Slu92, NRM94], or related concepts [Jak92, PMA92, Jak93]. Indeed, a dynamic feedback, even invertible, may lose essential properties such as controllability or feedback linearizability as soon as it is not endogenous [Mar92, Mar93], hence may not induce any interesting equivalence relation. A trivial example is the controllable linear system

$$\dot{x} = u$$

on which acts the invertible feedback

$$\begin{aligned} \dot{z} &= v \\ u &= v. \end{aligned}$$

The closed-loop system is no longer controllable.

Checking that a system is feedback linearizable seems a very difficult problem and is up to now widely open. One major reason is that it is not known whether the dimension of the dynamic feedback can be a priori bounded. Notice that considering non invertible feedbacks adds to the complexity, even for static feedbacks. So far, only few very special cases have been investigated. In particular, there is a nice geometric characterization in terms of Lie brackets of systems linearizable by invertible static feedback [JR80, HSM83]. Links with symmetry groups associated to static feedback equivalence can be found in [GS90a, GS90b, GS92]. Let us also mention an interesting result of [CLM89] which asserts that a single-input system is linearizable by invertible dynamic feedback if and only if it is linearizable by static feedback.

The purpose of this paper is twofold : on the one hand, we propose a new formulation of the feedback linearization problem using the notion of *dynamic immersion*. One interesting feature of this notion is that it does not explicitly involve a feedback, but rather maps with some adequate properties, and is for that reason easier to manipulate. We believe that this feedback-free formulation may help to link control theory with other fields of mathematics.

On the other hand, and this is the main result of the paper, we use this formulation to give, for driftless systems, a computable condition for feedback linearization in terms of Lie brackets. This condition is sufficient for an arbitrary number of inputs. It is shown that it is also necessary when the number of inputs equals two. An interesting consequence is that a 2-input driftless system that is linearizable by (dynamic) feedback can be converted, around every point of a dense open subset, into a so-called chained system [MS93] using only *static* feedback. We are not concerned here with what happens at "singular" points : see [Mur92] for a result in this direction. We have borrowed several ideas from Cartan's paper [Car15]. A closely related paper [Car14] was already used in [Sha90, Slu92], but with a very different interpretation.

The paper is organized as follows : we recall in section 2 definitions and results we will need on Pfaffian systems. In section 3, we define dynamic immersion and relate it to feedback linearization. We then state in subsection 4.1 a sufficient condition for driftless systems with an arbitrary number of inputs, show in 4.2 that the condition is also necessary for two inputs, and illustrate the results on examples in 4.3.

A preliminary version of this paper can be found in [MR93].

2 Pfaffian systems

We recall here some facts about Pfaffian systems, which will be used mainly in section 4. We refer the reader to [BCG⁺91, chapter 2] for a modern introduction and to [AMR88, chapter 6] for a survey of differential forms and exterior calculus. See also [TMS93] for a crash course with motivations from nonholonomic mechanics and control.

Let X be an open subset of $\mathbb{R}^n$, $C^\infty(X)$ the ring of smooth functions on X, $\mathcal{X}(X)$ the $C^\infty(X)$-module of smooth vector fields on X, and $\Omega^k(X)$ the $C^\infty(X)$-module of smooth differential k-forms on X. We denote by $d\alpha$ the *exterior derivative* of a k-form α and by $i_\xi\alpha$ the *interior product* (or *contraction*) of α by a vector field ξ. If φ is a map from an open subset Y of $\mathbb{R}^m$ to X, the *pull-back* of $\alpha \in \Omega^k(X)$, $\varphi^*\alpha$, is an element of $\Omega^k(Y)$. For instance if $\alpha := \sum_{i=1}^n \alpha_i(x)dx^i$ is a 1-form on X,

$$\varphi^*\alpha = \sum_{j=1}^m \sum_{i=1}^n \alpha_i(\varphi(y)) \frac{\partial \varphi^i}{\partial x^j}(y) dy^j.$$

An important property of the pull-back is that it commutes with the exterior derivative, i.e., $d(\varphi^*\alpha) := \varphi^*(d\alpha)$ for any k-form α.

A *Pfaffian system* on X is a submodule of $\Omega^1(X)$ (Pfaffian systems are dual objects to *distributions*, i.e. submodules of $\mathcal{X}(X)$). When dealing with a Pfaffian system I, we will be interested only in local and generic results, and will therefore always implicitly work on an open subset $\tilde{X}$ of X on which the dimension of the real vector spaces $I(x) := \{\alpha(x),\ \alpha \in I\}$ is equal to the dimension of I considered as a $C^\infty(\tilde{X})$-module. The points having such a neighborhood $\tilde{X}$ from a dense open subset of X. Notice that if Y is an open subset of $\mathbb{R}^p$, a Pfaffian system on X may be considered as a Pfaffian system on $X \times Y$ by identifying it with its pullback by the canonical from $X \times Y$ to X.

The *retracting space* of a Pfaffian system I is the Pfaffian system

$$C(I) := \{\xi \in \mathcal{X}(X),\ \forall\ \alpha \in I,\ \alpha(\xi) = 0 \text{ and } i_\xi d\alpha \in I\}^\perp.$$

By construction any 1-form $\alpha \in C(I)$ satisfies Frobenius' condition $d\alpha \equiv 0 \bmod C(I)$, i.e., $C(I)$ is completely integrable. The importance of the retracting space lies in the fact that I can be rewritten using only $\dim C(I)$ variables (instead of n) :

Theorem 1 (Cartan) *Consider a Pfaffian system of dimension s, and let $r := \dim C(I)$. Then there are coordinates $(z^1, \ldots, z^r, \zeta^1, \ldots, \zeta^{n-r})$ such that*

$$I = \{b_1^k(z)dz^1 + \ldots + b_r^k(z)dz^r, \quad k = 1, \ldots, s\}.$$

The *derived flag* of I is the descending chain of Pfaffian systems $I^0 := I \supset I^1 \supset \dots$ defined by

$$I^{k+1} := \{\alpha \in I^k,\ d\alpha \equiv 0 \bmod I^k\}.$$

We say I is *totally nonholonomic* if $I^k = 0$ for k large enough. In some cases the structure of the derived flag strongly constrain the dimensions of the retracting spaces, which will be a key ingredient in section 4 :

Lemma 2 *Let I be a Pfaffian system of dimension $s \geq 2$, such that $\dim I^1 = s-1$ and $\dim I^2 = s-2$. Then $\dim C(I) = s+2$ and $\dim C(I^1) = s+1$.*

This lemma is scattered through Cartan's paper [Car15] (in a slightly less general version), and was formally stated in [KR82] (see also [Slu92]).

Proof. Performing linear combinations, we can assume $I^k = \{\alpha^1, \dots, \alpha^{s-k}\}$, $k = 0, 1, 2$ (where $I^0 := I$ and $\alpha^0 := 0$). We thus have

$$d\alpha^k \equiv 0 \bmod I^1, \quad k = 1, \dots, s-2, \tag{1}$$

$d\alpha^{s-1} \equiv 0 \bmod I^0$ and $d\alpha^{s-1} \not\equiv 0 \bmod I^1$. Hence there exists a form α^{s+1} independent of $\alpha^1, \dots, \alpha^s$ such that

$$d\alpha^{s-1} \equiv \alpha^s \wedge \alpha^{s+1} \bmod I^1. \tag{2}$$

Contracting (1) and (2) by a vector field ξ such that $\alpha^1(\xi) = \dots = \alpha^{s-1}(\xi) = 0$, we find

$$\begin{array}{rcll} i_\xi d\alpha^k & \equiv & 0 & \bmod I^1, \quad k = 1, \dots, s-2 \\ i_\xi d\alpha^{s-1} & \equiv & \alpha^s(\xi)\ \alpha^{s+1} - \alpha^{s+1}(\xi)\ \alpha^s & \bmod I^1. \end{array} \tag{3}$$

We deduce that the retracting space of I^1 is given by

$$\begin{array}{rcl} C(I^1) & = & \{\xi \in \mathcal{X}(M),\ \alpha^1(\xi) = \dots = \alpha^{s+1}(\xi) = 0\}^\perp \\ & = & \{\alpha^1, \dots, \alpha^{s+1}\}, \end{array}$$

and is of dimension $s+1$.

By construction $C(I^1)$ satisfies Frobenius' condition, and in particular $d\alpha^s \equiv 0 \bmod C(I^1)$; but $d\alpha^s \not\equiv 0 \bmod I^0$, hence there exists a form α^{s+2} independent of $\alpha^1, \dots, \alpha^{s+1}$ such that

$$d\alpha^s \equiv \alpha^{s+1} \wedge \alpha^{s+2} \bmod I^0,$$

and the contraction by a vector field ξ such that $\alpha^1(\xi) = \dots = \alpha^s(\xi) = 0$ gives

$$i_\xi d\alpha^s \equiv \alpha^{s+1}(\xi)\ \alpha^{s+2} - \alpha^{s+2}(\xi)\ \alpha^{s+1} \bmod I^{(0)}.$$

On the other hand, we find by (3) that

$$i_\xi d\alpha^k \equiv 0 \bmod I^0, \quad k = 1, \dots, s-2$$

when $\alpha^1(\xi) = \dots = \alpha^s(\xi) = 0$, hence

$$\begin{array}{rcl} C(I^0) & = & \{\xi \in \mathcal{X}(M),\ \alpha^1(\xi) = \dots = \alpha^{s+2}(\xi) = 0\}^\perp \\ & = & \{\alpha^1, \dots, \alpha^{s+2}\} \end{array}$$

and is of dimension $s+2$. $\blacksquare$

Dually, we can deal with the *derived coflag* of I, i.e. the ascending chain of distributions $E_0 := (I_0)^\perp \subset E_1 \subset \ldots$ defined by

$$E_{k+1} := E_k + [E_k, E_k],$$

where $[E_k, E_k] := \{[X,Y],\ X,Y \in E_k\}$. It is easy to show that

$$\forall\ k \geq 0,\ I^k = (E_k)^\perp \tag{4}$$

using the formula $d\eta(X,Y) = L_X(\eta(Y)) - L_Y(\eta(X)) - \eta([X,Y])$ which links the exterior derivative of a 1-form η to the Lie derivatives of two vector fields X and Y.

We will also need a simplified version of Pfaff's theorem, which could alternatively be deduced directly from Frobenius' theorem :

Lemma 3 *Let $I = \{\alpha\}$ be a Pfaffian system of dimension 1 such that $d\alpha \wedge \alpha \neq 0$. Then in suitable coordinates $I = \{dz^1 + z^2 dz^3 + \sum_{i=4}^n b_i(z) dz^i\}$.*

3 Dynamic immersion and feedback linearization

As suggested in [Her73], a control system $\dot{x} = f(x,u)$ defined on an open subset $X \times U$ of $\mathbb{R}^n \times \mathbb{R}^m$ can be regarded as a Pfaffian system on $\mathbb{R} \times X \times U$,

$$I_f = \{dx^i - f^i(x,u)dt, \quad i = 1, \ldots, n\}. \tag{5}$$

Indeed, a map $t \mapsto (x(t), u(t))$ is a solution of $\dot{x} = f(x,u)$ if and only if $\{(t, x(t), u(t))\}$ is an integral manifold of I_f. I_f is called the *associated Pfaffian system.*

In this formalism, $\dot{x} = f(x,u)$ is equivalent by invertible static feedback to $\dot{y} = g(y,v)$ (with g defined on an open subset $Y \times V$ of $\mathbb{R}^p \times \mathbb{R}^q$) at (x_0, u_0) if and only if there exists a diffeomorphism ψ defined on $\tilde{Y} \times \tilde{V} \subset Y \times V$,

$$\psi :\ (t, y, v) \longmapsto (s, x, u) := (t, \varphi(y), \kappa(y,v)),$$

with $\varphi(\tilde{Y}) \times \kappa(\tilde{Y}, \tilde{V})$ a neighborhood of (x_0, u_0), such that $\psi^*(I_f) = I_g$. If moreover $\kappa(y,v) = v$, we say the systems are *conjugate.*

This point of view clearly corresponds to the usual formulation of static feedback equivalence : time is not changed, φ is a (local) diffeomorphism of $\mathbb{R}^n$ and $\partial_v \kappa$ has full rank m. In particular, the state dimension and number of inputs of both systems must be the same, i.e. $n = p$ and $m = q$.

We can allow for non invertible static feedbacks by dropping the rank requirement on κ. Of course in that case it is no longer question of equivalence. To generalize the definition to dynamic feedback, we relax in the same way the rank requirement on φ, by asking for a submersion instead of a diffeomorphism.

Definition 1 $\dot{x} = f(x,u)$ *is* dynamically immersed in $\dot{y} = g(y,v)$ at (x_0, u_0) *if there exists a map κ from an open subset $\tilde{Y} \times \tilde{V}$ of $Y \times V$ to a neighborhood of u_0 and a submersion φ from $\tilde{Y}$ to a neighborhood of x_0 such that $\psi^* I_f \subset I_g$, where ψ is the map*

$$\psi : (t, y, v) \longmapsto (t, \varphi(y), \kappa(y, v)).$$

If this property holds at every point (x_0, u_0) of a dense open subset of $X \times U$, we say $\dot{x} = f(x,u)$ is dynamically immersed *in $\dot{y} = g(y,v)$.*

Notice that ψ is in general not a submersion (because of κ), hence the pullback by ψ is not injective. We nevertheless use the term "dynamic immersion" to stress that φ, hence the transformation on the state space, is a submersion. In view of our problem of "embedding" a system into a controllable linear one, it would be meaningless to consider a non submersive map φ, since it would induce constraints on the state space, i.e., a loss of controllability. Moreover, the definition conveys exactly the usual notion of transformation by dynamic feedback and coordinate change :

Theorem 4 *$\dot{x} = f(x,u)$ is dynamically immersed in $\dot{y} = g(y,v)$ at (x_0, u_0) if and only if there exists a (dynamic) feedback B,*

$$\begin{aligned} \dot{z} &= a(x, z, v) \\ u &= \sigma(x, z, v), \end{aligned}$$

defined around a point (x_0, z_0, v_0) with $u_0 = \sigma(x_0, z_0, v_0)$, such that the closed-loop system f_B,

$$\begin{aligned} \dot{x} &= f(x, \sigma(x, z, v)) \\ \dot{z} &= a(x, z, v), \end{aligned}$$

is conjugate to $\dot{y} = g(y,v)$ at (x_0, z_0, v_0).

Proof. Suppose there exists a feedback B defined around a point (x_0, z_0, u_0) with $u_0 = \sigma(x_0, z_0, v_0)$, and a diffeomorphism $(x, z) = \Phi(y)$, defined around y_0 with $(x_0, z_0) = \Phi(y_0)$, such that f_B and g are conjugate. We thus have $\Psi^*(I_{f_B}) = I_g$, where $\Psi(t, y, v) := (t, \Phi(y), v)$. Since Φ is a diffeomorphism, its first n components, denoted by φ, form a submersion. Setting $\kappa(y, v) := \sigma(\Phi(y), v))$, we get a map $\psi(t, y, v) := (t, \varphi(y), \kappa(y, v))$. Now, for $i = 1, \ldots, n$,

$$\begin{aligned} \psi^*(dx^i - f^i(x,u)dt) &= d\varphi^i(y) - f^i(\varphi(y), \kappa(y,v))dt \\ &= d\varphi^i(y) - f^i(\varphi(y), \sigma(\Phi(y), v)dt \\ &= \Psi^*(dX^i - f_B^i(X, v)dt), \end{aligned} \tag{6}$$

where we have set $X := (x, z)$. We thus have $\psi^*(I_f) \subset \Psi^*(I_{f_B}) = I_g$.

Suppose $\psi^*(I_f) \subset I_g$, for a map $\psi(t, y, v) = (t, \varphi(y), \kappa(y, v))$, with φ a submersion; we assume ψ defined around (t_0, y_0, v_0) with $(x_0, u_0) = (\varphi(y_0), \kappa(y_0, v_0))$. It is possible to complete φ by a map π (for instance by picking some components of y) so that the map $\Phi(y) := (\varphi(y), \pi(y))$ is a diffeomorphism. We then define a dynamic feedback B by setting

$$\begin{aligned} \sigma(x, z, v) &:= \kappa(\Phi^{-1}(x, z), v) \\ a(x, z, v) &:= D\pi(\Phi^{-1}(x, z))g(\Phi^{-1}(x, z), v), \end{aligned}$$

with $z_0 := \pi(y_0)$. Setting also $\Psi(t, y, v) := (t, \Phi(y), v)$, a computation analogous to (6) shows that $\Psi^*(dX^i - f_B^i(X, v)dt) = \psi^*(dx^i - f^i(x, u)dt) \in I_g$ for $i = 1, \ldots, n$. On the other hand, for $i = 1, \ldots, p-n$,

$$\begin{array}{rcl} \Psi^*(dX^{n+i} - f_B^{n+i}(X, v)dt) & = & \pi^*(dz^i - a^i(x, z, v))dt) \\ & = & d\pi^i(y) - D\pi^i(y)g(y, v)dt \\ & = & D\pi^i(y)(dy - g(y, v)dt), \end{array}$$

hence $\Psi^*(I_{f_B}) \subset I_g$. But I_{f_B} and I_g have by construction the same dimension and Ψ is a diffeomorphism, so we have in fact $\Psi^*(I_{f_B}) = I_g$. ■

Remark that the feedback constructed in the proof may be not invertible or not endogenous. Extra conditions must be satisfied by κ and its prolongations in order to get these properties. The theorem has an obvious corollary :

Corollary 5 *A system $\dot{x} = f(x, u)$ is feedback linearizable (resp. feedback linearizable at (x_0, u_0)) if and only if it is dynamically immersed (resp. dynamically immersed at (x_0, u_0)) in a controllable linear system.*

Without loss of generality, we can assume that a controllable linear system $\dot{y} = Ay + Bv$ defined on a subset of $\mathbb{R}^p \times \mathbb{R}^q$ is in Brunovsky form, i.e., formed of q chains of d_i integrators, with $d_1 + \ldots + d_q = p$. If we set $y := (y_0^1, \ldots, y_{d_1-1}^1, \ldots, y_0^q, \ldots, y_{d_q-1}^q)$, $v := (y_{d_1}^1, \ldots, y_{d_q}^q)$ and

$$\omega_j^i := dy_{j-1}^i - y_j^i dt, \quad i = 1, \ldots, q, \quad j = 1, \ldots, d_i,$$

then its associated Pfaffian system is

$$C_{d_1, \ldots, d_q}^q := \{\omega_1^1, \ldots, \omega_{d_1}^1, \ldots, \omega_1^q, \ldots, \omega_{d_q}^q\}.$$

In other words, it is a (partial prolongation of a) contact system. Clearly, such a system is totally nonholonomic.

4 Driftless systems

We now restrict our attention to a driftless control system

$$D: \ \dot{x} = \sum_{i=1}^{m} u^i f_i(x),$$

where $f_1, \ldots, f_m$ are vector fields on X. We assume the vectors $f_1(x), \ldots, f_m(x)$ independent at every point x of a dense open subset $\tilde{X}$ of X. To this driftless system is naturally associated the Pfaffian system on $\tilde{X}$ $I := \{f_1, \ldots, f_m\}^\perp$ obtained by eliminating the inputs in D. Notice that $I = (I_f)^1$, where I_f is the Pfaffian system associated to D : indeed, I_f has generators $\alpha^1, \ldots, \alpha^{n-m}, \beta^1, \ldots, \beta^m$ of the form

$$\begin{array}{rclr} \alpha^i & = & \sum_{i=1}^n a_k^i(x)dx^k, & i = 1 \ldots n-m \\ \beta^j & = & \sum_{i=1}^n b_k^j(x)dx^k - u^j dt, & j = 1, \ldots m. \end{array} \qquad (7)$$

The α^i's, which clearly span I, can be seen as kinematic constraints of a mechanical system, and the β^j's as a description of the inputs in terms of the velocities. Since

$$\begin{array}{rcl} d\alpha^i & = & \sum_{i=1}^n da_k^i \wedge dx^k \\ & \equiv & 0 \mod I_f \\ d\beta^j & = & \sum_{i=1}^n db_k^j \wedge dx^k - du^j \wedge dt \\ & \equiv & dt \wedge du^j \mod I_f, \end{array}$$

we have $I = (I_f)^1$. We call *derived flag (*resp. *coflag) of D*, the derived flag (resp. coflag) of I. Remember that D is *controllable* (at every point of a dense open subset) when the Lie algebra generated by $f_1, \ldots, f_m$ has dimension n (as module over smooth functions), i.e, $E_k = \mathcal{X}(X)$ for k large enough. By (4), this is equivalent to I being totally nonholonomic. When $n-m = 1$, controllability reads $d\alpha \wedge \alpha \neq 0$ (with α any generator of I), which is the early characterization of accessibility by Caratheodory [Car09], extended by Rashevsky [Ras38] and Chow [Cho40] to driftless systems with an arbitrary number of inputs.

It is remarkable that for a driftless system $(I_f)^1$ (hence the whole derived flag) depends neither on time nor inputs, i.e. is a Pfaffian system on $\tilde{X}$. The interesting point is that $(I_f)^1$ contains all that we need to study feedback linearization, which means we can directly apply to it the results of section 2. For a general system, things are more complicated since time must be carefully handled. To make clear the role of $(I_f)^1$, we first need a definition (we use the notations of the end of section 3) :

Definition 2 *A Pfaffian system I on X is* linearizable at $x_0 \in X$ *if there exists a submersion φ from an open subset $Y \subset \mathbb{R}^{q+d_1+\ldots+d_q}$ to a neighborhood of x_0 such that $\varphi^* I \subset C^q_{d_1,\ldots,d_q}$ for some positive integers q, $d_1, \ldots, d_q$. If this property holds at every point x_0 of a dense open subset of X, we say that I is* linearizable.

Notice that we consider $\varphi^* I$, which is a Pfaffian system on Y, as a system on $\mathbb{R} \times Y$ (by pulling it back by the canonical projection). This definition is consistent with definition 1 :

Proposition 6 *A driftless system $\dot{x} = \sum_{i=1}^m u^i f_i(x)$ is feedback linearizable in the sense of definition 1 if and only if the Pfaffian system $\{f_1, \ldots, f_m\}^\perp$ is linearizable in the sense of definition 2.*

Proof. If $\dot{x} = \sum_{i=1}^m u^i f_i(x)$ is feedback linearizable, there exists a map

$$\psi : \ (t, y, v) \longmapsto (t, \varphi(y), \kappa(y, v)), \tag{8}$$

with φ a submersion, such that $\psi^\star I_f \subset C^q_{d_1,\ldots,d_q}$. Hence, $\varphi^\star I = \psi^\star I \subset \psi^\star I_f \subset C^q_{d_1,\ldots,d_q}$, i.e. I is linearizable

Conversely, if I is linearizable there exists a submersion φ pulling back I into $C^q_{d_1,\ldots,d_q}$. A map ψ of the form (8)) with κ yet to be determined, will pull back the generators $\alpha^1, \ldots, \alpha^{n-m}$ and $\beta^1, \ldots, \beta^m$ of I_f (see (7)) into $\psi^\star \alpha^i = \varphi^\star \alpha^i \in C^q_{d_1,\ldots,d_q}$ and

$$\begin{array}{rcl} \psi^\star \beta^l & = & \sum_{i=1}^q \sum_{j=0}^{d_i} \sum_{k=1}^n b_k^l(\varphi(y)) \frac{\partial \varphi^k}{\partial y_j^i}(y) dy_j^i - \kappa^l(y, v) dt \\ & \equiv & \left(\sum_{i=1}^q \sum_{j=0}^{d_i} \sum_{k=1}^n y_{j+1}^i b_k^l(\varphi(y)) \frac{\partial \varphi^k}{\partial y_j^i}(y) - \kappa^l(y, v) \right) dt \mod C^q_{1+d_1,\ldots,1+d_q}. \end{array}$$

If we set $v^i := y^i_{1+d_i}$ and define κ to zero the dt term, we have $\psi^\star I_f \subset C^q_{1+d_1,\ldots,1+d_q}$ i.e., $\dot{x} = \sum_{i=1}^m u^i f_i(x)$ is feedback linearizable. ■

An immediate consequence of this proposition is that a feedback linearizable driftless system is controllable. Indeed a Pfaffian system which can the pulled back by a submersion into a totally nonholonomic system is itself totally nonholonomic because the pull-back by a submersion is one-to-one.

Notice also that a driftless system is never linearizable by invertible static feedback (except in the trivial case $m = n$) : indeed, the condition in [JR80, HSM83] requires that $\{f_1, \ldots, f_m\}$ be at the same time involutive and of dimension n.

4.1 A sufficient condition

Theorem 7 *A driftless system $\dot{x} = \sum_{i=1}^m u^i f_i(x)$ with n states and m inputs is (dynamic) feedback linearizable if its derived coflag satisfies, at every point of a dense open subset,*

$$\dim E_k(x) = m + k, \quad k = 0, \ldots, n - m$$

(or equivalently if its derived flag satisfies $\dim I^k(x) = n - m - k$ for $k = 0, \ldots, n - m$).

Proof. The conditions on the flag and the coflag are clearly equivalent by (4). By proposition 6, we have to prove that $I^0 := \{f_1, \ldots, f_m\}^\perp$, which has dimension $s := n - m$ is linearizable. The case $s = 0$ being trivial, we assume $s > 0$.

• $s = 1$. The condition reads $\dim I^0 = 1$ and $\dim I^1 = 0$, hence I^0 is spanned by a single 1-form α such that $d\alpha \wedge \alpha \neq 0$ (on a dense open subset). By lemma 3, we may assume $\alpha = dz^2 - z^1 dz^3 + \sum_{i=4}^n a^i(z) dz^i$. We get a submersion

$$(\varphi^1, \ldots, \varphi^n) : \ (y^1_0, y^2_0, \ldots, y^1_s, y^2_s) \longmapsto (z^1, \ldots, z^n)$$

pulling back I^0 into $C^s_{1,\ldots,1}$ if we set $\varphi^i(y) := y^i_0, \ i = 2, \ldots, m$, and choose φ^1 so as to zero the dt term in

$$\begin{aligned} \varphi^\star \alpha &= dy^2_0 - \varphi^1 dy^3_0 + \textstyle\sum_{i=4}^n (a^i \circ \varphi) dy^i_0 \\ &\equiv (y^2_1 - y^3_1 \varphi^1 + \textstyle\sum_{i=4}^n (a^i \circ \varphi) y^i_1) dt \quad \mod C^s_{1,\ldots,1}. \end{aligned}$$

• $s > 1$ We follow exactly Cartan's proof [Car15], with a more modern language. Using repeatedly lemma 2, we find $\dim C(I^k) = s - k + 2$ for $k = 0, \ldots, s - 1$. In particular $\dim C(I^0) = s + 2$ and $\dim C(I^{s-1}) = 3$, hence there exist by theorem 1 coordinates $(\zeta^1, \ldots, \zeta^{n-s-2}, z^1, \ldots, z^{s+2})$ in which I^0 depends only on the z variables and I^{s-1} only on (z^1, z^2, z^3). In other words I^{s-1} is a Pfaffian system of dimension 1 in 3 variables, and since $I^s = 0$ we can assume by lemma 3 (performing a coordinate change leaving all the variables but z^1, z^2, z^3 unchanged) that I^{s-1} is spanned by $\alpha^1 := dz^2 - z^3 dz^1$.

We now proceed by induction. I^{s-2} is spanned by α^1, α^2 where $\alpha^2 := a^1(z) dz^1 + \sum_{i=3}^n \alpha_i(z) dz^i$. But, by definition of I^{s-1}, $d\alpha^1 \wedge \alpha^1 \wedge \alpha^2 = 0$, which implies $a_4 = \ldots = a_n = 0$. On the other hand $d\alpha^2 \wedge \alpha^1 \wedge \alpha^2 \neq 0$, hence a_1 and a_3, are non-zero (on a dense open subset), and we may assume $a_3 = 1$; but $d\alpha^2 \wedge \alpha^1 \wedge \alpha^2 \neq 0$ now means that a_4

does not depend only on z^1, z^2, z^3 and we may assume $a_1(z) = -z^4$, i.e. $\alpha^2 = dz^3 - z^4 dz^1$. Repeating exactly the same arguments, we eventually get

$$\begin{array}{rcl} I^{s-1} & = & \{dz^2 - z^3 dz^1\} \\ I^{s-2} & = & \{dz^2 - z^3 dz^1, dz^3 - z^4 dz^1\} \\ & \vdots & \\ I^0 & = & \{dz^2 - z^3 dz^1, dz^3 - z^4 dz^1, \ldots, dz^{s+1} - z^{s+2} dz^1\}. \end{array}$$

We end the proof by building a submersion

$$(\varphi^1, \ldots, \varphi^{s+2}) : \ (y_0^1, y_0^2, \ldots, y_s^1, y_s^2) \longmapsto (z^1, \ldots, z^{s+2})$$

which pulls back I^0 into $C^2_{s,s}$. Remember that I^0 is in fact a Pfaffian system in the z and ζ variables, so we will trivially extend φ by $\zeta^i := y_0^{i+2},\ i = 1, \ldots, n-s-2$ to pull back I^0 into $C^{n-s}_{s,s,1,\ldots,1}$. We set $\varphi^1(y) := y_0^1$ and $\varphi^2(y) := y_0^2$ and construct $\varphi^3, \ldots, \varphi^{s+2}$ inductively. Since

$$\begin{array}{rcl} \varphi^\star(dz^2 - z^3 dz^1) & = & dy_0^2 - \varphi^3 dy_0^1 \\ & \equiv & (y_1^2 - y_1^1 \varphi^3) dt \quad \mod C^2_{1,1}, \end{array}$$

we get a submersion $(\varphi^1, \varphi^2, \varphi^3)$ pulling back I^{s-1} into $C^2_{1,1}$ by setting $\varphi^3(y) := \dfrac{y_1^2}{y_1^1}$. Assume then that

$$(\varphi^1, \ldots, \varphi^{k+2}) : \ (y_0^1, y_0^2, \ldots, y_k^1, y_k^2) \longmapsto (z^1, \ldots, z^{k+2}),$$

is a submersion pulling back I^{s-k} into $C^2_{k,k}$ and such that $\dfrac{\partial \varphi^{k+2}}{\partial y_k^i} = \dfrac{\partial \varphi^3}{\partial y_1^i} \neq 0$. Now

$$\begin{array}{rcl} \varphi^\star(dz^{k+2} - z^{k+3} dz^1) & = & d\varphi^{k+2} - \varphi^{k+3} dy_0^1 \\ & \equiv & (\displaystyle\sum_{i=1}^{2} \sum_{j=0}^{k} y_{j+1}^i \frac{\partial \varphi^{k+2}}{\partial y_j^i} - y_1^1 \varphi^{k+3}) dt \quad \mod C^2_{k+1,k+1}, \end{array}$$

and if we choose φ^{k+3} to zero dt term, the map

$$(\varphi^1, \ldots, \varphi^{k+3}) : \ (y_0^1, y_0^2, \ldots, y_{k+1}^1, y_{k+1}^2) \longmapsto (z^1, \ldots, z^{k+3})$$

is a submersion pulling back $I^{s-(k+1)}$ into $C^2_{k+1,k+1}$ and such that $\dfrac{\partial \varphi^{k+3}}{\partial y_{k+1}^i} = \dfrac{\partial \varphi^3}{\partial y_1^i} \neq 0$. ■

It follows form the proof that a feedback linearizable driftless system with more than 2 inputs can be transformed by invertible static feedback and coordinate change into

$$\begin{array}{rcl} \dot z & = & v^1 g_1(z) + v^2 g_2(z) \\ \dot\zeta^1 & = & v^3 \\ & \vdots & \\ \dot\zeta^{m-2} & = & v^m, \end{array}$$

where $\dot z = v^1 g_1(z) + v^2 g_2(z)$ is a feedback linearizable system with 2 inputs.

Theorem 7 has two obvious corollaries. The first one can be seen as a special case of a result on *affine* systems [CLM89] ; the second could be recovered directly from Engel's theorem.

Corollary 8 *A controllable driftless system with m inputs and $m+1$ states is feedback linearizable.*

Proof. Here $\dim E_0 = m$ and controllability implies $\dim E_1 = m+1$. ■

Corollary 9 *A driftless system with 2 inputs and 4 states is feedback linearizable if and only if it is controllable.*

Proof. A feedback linearizable is of course controllable. Conversely, $E_0 = \{f_1, f_2\}$ has by assumption dimension 2 and $E_1 = \{f_1, f_2, [f_1, f_2]\}$ has dimension 2 or 3. But $\dim E_1 = 2$ would contradict controllability. Hence $\dim E_2 = 3$ and controllability now implies $\dim E_3 = 4$. ■

4.2 Necessity of the condition for two inputs

We now show that the sufficient condition of theorem 7 is in fact necessary when the system has two inputs.

Theorem 10 *A driftless system $\dot{x} = f_1(x)u^1 + f_2(x)u^2$ with n states and 2 inputs is (dynamic) feedback linearizable if and only if its derived coflag satisfies, at every point of a dense open subset,*

$$\dim E_k(x) = 2 + k, \quad k = 0, \dots, n-2$$

(or equivalently if its derived flag satisfies $\dim I^k(x) = n - 2 - k$, $k = 0, \dots, n-2$).

Proof. Of course, by theorem 7, we only have to prove the necessity. We use some tricks from [Car15], but with a very different point of view.

Let the Pfaffian system $I = \{f_1, f_2\}^\perp$ be generated by $s := n-2$ independent 1-forms $\alpha^1, \dots, \alpha^s$. The case $s = 1$ being trivial (the condition boils down to controllability), we assume $s > 1$. By assumption I is linearizable, hence controllable, which implies $\dim C(I) = s+2 = n$ and $\dim I^1 = s-1$. Without restriction, we can complete $(\alpha^1, \dots, \alpha^s)$ by α^{s+1} and α^{s+2} to a basis $(\alpha^1, \dots, \alpha^{s+2})$ of $\Omega^1(X)$. Moreover, we may assume $I^1 = \{\alpha^1, \dots, \alpha^{s-1}\}$ with

$$\begin{array}{rcll} d\alpha^s & \equiv & \alpha^{s+1} \wedge \alpha^{s+2} & \bmod I \\ d\alpha^i & \equiv & 0 & \bmod I, \quad i = 1, \dots, s-1. \end{array}$$

Clearly we have

$$d\alpha^i \equiv \lambda_1^i \alpha^s \wedge \alpha^{s+1} + \lambda_2^i \alpha^s \wedge \alpha^{s+2} \bmod I^1, \quad i = 1, \dots, s-1 \tag{9}$$

with λ_1^i and λ_2^i functions of x. The dimension of I^2 is given by the rank of the matrix

$$L := \begin{pmatrix} \lambda_1^1 & \dots & \lambda_1^{s-1} \\ \lambda_2^1 & \dots & \lambda_2^{s-1} \end{pmatrix}.$$

More precisely, $\dim I^2 = s - 1 - r$ where r is the rank of L.

The theorem is proved if we can show that $\dim I^2 = s-2$. Indeed, if this is true then $\dim C(I^1) = s+1$ by lemma 2, and, by theorem 1, I^1 is a system of dimension $s-1$ in $s+1$ variables. Of course I^1, being included in a by assumption linearizable system, is itself linearizable, hence we are left with exactly the same problem, but now with a system of dimension $s-1$ instead of s. Repeating the same argument till the obvious dimension 1 case, we eventually obtain the necessary condition $\dim I^k = s-k$ for $k = 1, \dots, s$.

We now prove that $\dim I^2 = s-2$, i.e., that the rank of L is one. Clearly, the rank of L is not zero (otherwise $I^2 = I^1$ and I is not controllable) and it suffices to prove that $\mathrm{rank}(L) < 2$. We give here a detailed proof when the integer q in definition 2 is equal to 2. The case $q > 2$ proceeds exactly along the same line. Thus there exist by assumption integers $d_1,\ d_2 \geq 0$ and a submersion

$$\varphi : \ (y_0^1, \dots, y_{d_1}^1, y_0^2, \dots, y_{d_2}^2) \longmapsto x$$

such that $\varphi^* I \subset C^2_{d_1,d_2}$, and without loss of generality $\dfrac{\partial \varphi}{\partial (y^1_{d_1}, y^2_{d_2})} \neq 0$.

For the sake of clarity, the pullback by φ will be denoted as follows : $\tilde{I} = \varphi^* I$, $\tilde{\alpha} = \varphi^* \alpha$ for any 1-form $\alpha \in \Omega(X)$, $\tilde{a} = \varphi^* a$ for any function on X, etc.

Let r_1 and r_2 be the smallest integers such that $\tilde{I} \subset C^2_{r_1,r_2}$ (of course $r_1 \leq d_1$ and $r_2 \leq d_2$). This means that

$$\tilde{\alpha}^i = \sum_{j=1}^{r_1} b_1^{i,j} \omega^1_j + \sum_{k=1}^{r_2} b_2^{i,k} \omega^2_k, \quad i = 1, \dots, s$$

where, at least, one of the functions b_1^{i,r_1} or b_2^{i,r_2} is not zero.

The forms $\tilde{\alpha}^{s+1}, \tilde{\alpha}^{s+2}$ do not necessary belong to $C^2_{r_1,r_2}$ but we can still write them as

$$\tilde{\alpha}^{s+i} \equiv \sum_{j=1}^{d_1+1} b_1^{s+i,j} \omega^1_j + \sum_{k=1}^{d_2+1} b_2^{s+i,k} \omega^2_k \quad \mathrm{mod}\, \{dt\}, \qquad i = 1, 2.$$

Moreover the square matrix

$$B := \begin{pmatrix} b_1^{s+1,d_1+1} & b_1^{s+2,d_1+1} \\ b_2^{s+1,d_2+1} & b_2^{s+2,d_2+1} \end{pmatrix}$$

is not zero because $\dfrac{\partial \varphi}{\partial (y^1_{d_1}, y^2_{d_2})} \neq 0$.

A routine computation gives for $i = 1, \dots, s-1$,

$$\begin{aligned} d\tilde{\alpha}^i &\equiv \textstyle\sum_{j=1}^{r_1} b_1^{i,j} d\omega^1_j + \sum_{k=1}^{r_2} b_2^{i,k} d\omega^2_k \qquad \mathrm{mod}\, C^2_{r_1,r_2} \\ &\equiv b_1^{i,r_1} dt \wedge \omega^1_{r_1+1} + b_2^{i,r_2} dt \wedge \omega^2_{r_2+1} \quad \mathrm{mod}\, C^2_{r_1,r_2} \end{aligned} \tag{10}$$

(remember $d\omega^l_k = dt \wedge \omega^l_{k+1}$ for $l = 1, 2$). But $\tilde{I} \subset C^2_{r_1,r_2}$, thus $d\alpha^i \equiv 0 \bmod I$ for $i = 1, \dots, s-1$ implies $d\tilde{\alpha}^i \equiv 0 \bmod C^2_{r_1,r_2}$ for $i = 1, \dots, s-1$. Therefore, $b_1^{i,r_1} = b_2^{i,r_2} = 0$

for $i = 1, \ldots, s-1$, i.e. $\tilde{I}^1 \subset C^2_{r_1-1,r_2-1}$. Remember on the other hand that by definition $b_1^{s,r_1} \neq 0$ or $b_2^{s,r_2} \neq 0$.

Similarly, $d\alpha^s \equiv \alpha^{s+1} \wedge \alpha^{s+2} \mod I$ implies

$$d\tilde{\alpha}^s \equiv \tilde{\alpha}^{s+1} \wedge \tilde{\alpha}^{s+2} \bmod \left(C^2_{r_1,r_2} \oplus \{dt\}\right).$$

Using now $\tilde{I}^1 \subset C^2_{r_1-1,r_2-1}$, we get from (9)

$$d\tilde{\alpha}^i \equiv \tilde{\lambda}_1^i \tilde{\alpha}^s \wedge \tilde{\alpha}^{s+1} + \tilde{\lambda}_2^i \tilde{\alpha}^s \wedge \tilde{\alpha}^{s+2} \bmod C^2_{r_1-1,r_2-1}, \quad i = 1, \ldots, s-1. \tag{11}$$

But we also have by a computation analogous to (10)

$$d\tilde{\alpha}^i \equiv 0 \bmod \left(C^2_{r_1-1,r_2-1} \oplus \{dt\}\right), \quad i = 1, \ldots, s-1. \tag{12}$$

Expanding (11) with the expressions of $\tilde{\alpha}^s, \tilde{\alpha}^{s+1}, \tilde{\alpha}^{s+2}$, we find, for $i = 1, \ldots, s-1$, that the expression $\tilde{\lambda}_1^i \tilde{\alpha}^s \wedge \tilde{\alpha}^{s+1} + \tilde{\lambda}_2^i \tilde{\alpha}^s \wedge \tilde{\alpha}^{s+2}$ contains a linear combination of 4 decomposable 2-forms

$$\begin{aligned}
&b_1^{s,r_1}(\tilde{\lambda}_1^i b_1^{s+1,d_1+1} + \tilde{\lambda}_2^i b_1^{s+2,d_1+1})\, \omega^1_{r_1} \wedge \omega^1_{d_1+1} \\
+\, &b_1^{s,r_1}(\tilde{\lambda}_1^i b_2^{s+1,d_2+1} + \tilde{\lambda}_2^i b_2^{s+2,d_2+1})\, \omega^1_{r_1} \wedge \omega^2_{d_2+1} \\
+\, &b_2^{s,r_2}(\tilde{\lambda}_1^i b_1^{s+1,d_1+1} + \tilde{\lambda}_2^i b_1^{s+2,d_1+1})\, \omega^2_{r_2} \wedge \omega^1_{d_1+1} \\
+\, &b_2^{s,r_2}(\tilde{\lambda}_1^i b_2^{s+1,d_2+1} + \tilde{\lambda}_2^i b_2^{s+2,d_2+1})\, \omega^2_{r_2} \wedge \omega^2_{d_2+1}
\end{aligned}$$

that are independent $\bmod \left(C^2_{r_1-1,r_2-1} \oplus \{dt\}\right)$. Since $b_1^{s,r_1} \neq 0$ or $b_2^{s,r_2} \neq 0$, we deduce from (11) and (12) that

$$\begin{pmatrix} b_1^{s+1,d_1+1} & b_1^{s+2,d_1+1} \\ b_2^{s+1,d_2+1} & b_2^{s+2,d_2+1} \end{pmatrix} \begin{pmatrix} \tilde{\lambda}_1^i \\ \tilde{\lambda}_2^i \end{pmatrix} = 0, \quad i = 1, \ldots, s-1.$$

Otherwise stated, $B\tilde{L} = 0$ with $B \neq 0$. Thus the rank of $\tilde{L}$, which is equal to the rank of L (the pull-back by φ is one-to-one), is not maximal. Hence $\dim I^2 = s-2$. ■

4.3 Examples

We illustrate various aspects of theorems 7 and 10 on four examples.

Example 1 [The rolling hoop] A circular hoop of radius a rolls without slipping on a fixed horizontal plane. The configuration of the hoop is defined by the projections x, y of its center C on the plane, and by the Euler angles ψ, θ, φ (see figure 1). The condition that the velocity of slip is zero are

$$\begin{aligned}
\dot{x}\cos\psi + \dot{y}\sin\psi + a(\dot{\psi}\cos\theta + \dot{\varphi}) &= 0 \\
-\dot{x}\sin\psi + \dot{y}\cos\psi + a\dot{\theta}\sin\theta &= 0.
\end{aligned}$$

These constraints describe a Pfaffian system of dimension 2 in 5 variables

$$I := \{\cos\psi dx + \sin\psi dy + a\cos\theta d\theta + a d\varphi,\ -\sin\psi dx + \cos\psi dy + a\sin\theta d\theta\}$$

which is linearizable ($\dim I^1 = 1$ and $\dim I^2 = 0$). Alternatively, we could consider this Pfaffian as a driftless system with 3-inputs and 5-states (for instance by choosing 3-velocities as the inputs). If we set $X := x + a\sin\psi\cos\theta$ and $Y := y - a\cos\psi\cos\theta$,

$$I = \{dX + a\cos\psi d\varphi,\ dY + a\sin\psi d\varphi\},$$

and only 4 variables are needed. Notice that X and Y are the coordinates of the point of contact M between the hoop and the plane. □

Example 2 **[The general 1-trailer system]** This nonholonomic system (see figure 2) generalizes the 1-trailer system considered in [LS89, MS93] : here the trailer is hitched to the car not directly at the center of the car rear axle, but more realistically at a distance d of this point. The two inputs are the driving velocity u_1 of the car rear wheels, and the steering velocity u_2 of the car front wheels. The wheels are assumed to roll and spin without slipping. With notations of figure 2), the kinematic equations are

$$\begin{aligned}
\dot{x} &= u_1\cos\theta \\
\dot{y} &= u_1\sin\theta \\
\dot{\varphi} &= u_2 \\
\dot{\theta} &= \frac{u_1}{l}\tan\varphi \\
\dot{\theta}_1 &= \frac{u_1}{d_1}\left(\sin(\theta-\theta_1) - \frac{d}{l}\tan\varphi\cos(\theta-\theta_1)\right)
\end{aligned}$$

By theorem 7, the system is feedback linearizable : notice $[f_2,[f_1,f_2]]$ is colinear to $[f_1,f_2]$, hence $E_2 = \{f_1, f_2, [f_1,f_2], [f_1,[f_1,f_2]]\}$ and has dimension 4. This proves that the system is linearizable. More details and computations can be found in [RFLM93].

This is no longer true when there is more than one trailer [RFLM93] : notice the difference with the standard n-trailer system (i.e., each trailer is hitched at the center of the rear axle of the preceding vehicle), which is flat [FLMR92b], hence feedback linearizable, whatever the number of trailers and can be put into chained form by static feedback [TMS93]. □

Example 3 **[Second order Monge equation]** By corollary 9, a driftless system with 2 inputs and 4 states is feedback linearizable as soon as it is controllable. This is already no longer true in state dimension 5. The 2-input system

$$\begin{aligned}
\dot{x} &= u \\
\dot{y} &= au \\
\dot{z} &= b^2u \\
\dot{a} &= bu \\
\dot{b} &= v
\end{aligned}$$

is controllable but not linearizable ($\dim E_0 = 2$, $\dim E_1 = 3$, but $\dim E_3 = 5 \neq 4$). Notice the system is derived from the second-order Monge equation

$$\frac{dZ}{dX} - \left(\frac{d^2Y}{dX^2}\right)^2 = 0$$

by setting $x := X,\ y := Y,\ z := Z$ and $a := \frac{dY}{dX},\ b = \frac{d^2Y}{dX^2}$. Hilbert's study of this equation [Hil12] was the starting point of Cartan's paper [Car15]. $\square$

Example 4 The condition in theorem 7 is not necessary for systems with more than 2 inputs : the 3-input and 5-state system

$$\begin{array}{rcl} \dot{x}^1 &=& u^1 \\ \dot{x}^2 &=& u^2 \\ \dot{x}^3 &=& u^3 \\ \dot{x}^4 &=& x^2u^1 \\ \dot{x}^5 &=& x^3u^1. \end{array}$$

does not satisfy the condition ($\dim E_0 = 3$ and $\dim E_1 = 5$), but is nevertheless feedback linearizable : the submersion

$$\varphi:\ (y_0^1, y_1^1, y_0^2, y_1^2, y_0^3, y_1^3) \longmapsto \left(y_0^1, \frac{y_1^2}{y_1^1}, \frac{y_1^3}{y_1^1}, y_0^2, y_0^3 \right)$$

pulls back $\{dx^4 - x^2dx^1, dx^5 - x^3dx^1\} = (E_0)^\perp$ into $C^3_{1,1,1}$, i.e., into $\{dy_0^1 - y_1^1dt, dy_0^2 - y_1^2dt, dy_0^3 - y_1^3dt\}$. $\square$

5 Concluding remarks

By way of conclusion, we would like to draw the reader's attention on some important points.

The proof of theorem 7 shows that a Pfaffian system satisfying the flag condition is, locally on a dense open subset, diffeomorphic to a so-called Goursat system

$$I := \{dx^2 - x^3dx^1, \ldots, dx^{n-m+1} - x^{n-m+2}dx^1\}.$$

In other words, it means that a driftless system satisfying the flag condition can, locally on a dense open subset, be transformed into the chained system [MS93]

$$\begin{array}{rcl} \dot{z}^1 &=& v^1 \\ \dot{z}^2 &=& z^3v^1 \\ & \vdots & \\ \dot{z}^{n-m+1} &=& z^{n-m+2}v^1 \\ \dot{z}^{n-m+2} &=& v^2 \\ \dot{\zeta}^1 &=& v^3 \\ & \vdots & \\ \dot{\zeta}^{m-2} &=& v^m \end{array}$$

by *static* (and invertible) feedback (though it is not linearizable by invertible static feedback). This chained system is feedback linearizable, as follows from the proof of theorem 7 (construction of the submersion φ ; alternatively, notice that the system augmented by $n-m+1$ integrators on v^1 is now linearizable by static feedback). Moreover, if we set

$$y := (z^1, z^{n-m+2}, \zeta^1, \ldots, \zeta^{m-2}),$$

it is not difficult to see that z, ζ and v can be expressed in terms of y and its derivatives :

$$(z, \zeta, v) = a(y, \dot{y}, \ddot{y}, \ldots, y_{(n-m)}).$$

This means that a system satisfying the flag condition is *flat* [FLMR92b, FLMR95, Mar92, FLMR, NRM94], with y as a *flat output*. Flatness corresponds to a notion of dynamic equivalence to a linear system, and a flat system can be linearized by a special type of feedback, called *endogenous* (cf. [Mar92, Mar93]), which is in particular invertible.

Hence, by theorem 10 a linearizable Pfaffian system of dimension $n-2$ in n variables is, on a dense open subset, locally diffeomorphic to a Goursat system, and a 2-input driftless system which is linearizable by a *dynamic* (and possibly not invertible) feedback is in fact flat and may be put, around every point of a dense open subset, into chained form by a *static* (and invertible) feedback. In this sense, nothing is fundamentally gained by using dynamic feedback. This is analogous in spirit to the picture for single-input general systems, where linearization by dynamic feedback turns out to be no more general than linearization by static feedback [CLM89]. This is no longer true for driftless systems with more than 2 inputs (see e.g., [MR]).

The chained form is interesting because it may be used as a local model for a linearizable driftless system around a "singular" point, such as an equilibrium point, where the linear approximation is not controllable ; of course such a "singular" point can not be too "degenerate". From a cursory look at the proof of theorem 7, one might erroneously conclude it is sufficient for that to require the $I^k(x)$'s or $E_k(x)$'s to have constant dimension around the point of interest. This rather subtle fact, apparently overlooked by Cartan (in the context of Pfaffian systems), was pointed out in [GKR78] (see also [KR82] for a more detailed study). A recent work of Murray [Mur92] shows which extra regularity conditions must be met. Murray's result may be interpreted as a necessary and sufficient condition for putting a 2-input driftless system into chained form around a *given* point (x_0, u_0) by *static* feedback.

One may also wonder about the possibility of finding a result similar to theorem 10 for driftless systems with more than two inputs. Since theorem 10 is equivalent, in some loose sense, to Cartan's result [Car15] on Pfaffian system of codimension two, it is interesting to notice that Cartan tried, but apparently did not succeed in generalizing his result to Pfaffian systems of higher codimension (see [Car14], last paragraph of the introduction). It is thus not unwise to think that this is indeed a difficult task, at least within the classical framework of differential forms and Pfaffian systems.

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FIG. 1 – the rolling hoop.

Fig. 2 – *the general* 1-*trailer system.*