Design and Stability of Quantum Filters with Measurement Imperfections: discrete-time and continuous-time cases

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Quantum filtering: some references . . .


The first experimental realization of a quantum state feedback

The LKB photon box: sampling time (\(\sim 100 \, \mu s\)) long enough to estimate in real-time the quantum-state \(\rho\) and to compute the control \(u = Ae^{i\Phi}\) as a function of \(\rho\) (quantum state feedback).

\(^1\)Courtesy of Igor Dotsenko
\(^2\)C. Sayrin et al., Nature, 1-September 2011
Experimental data

An open-loop trajectory starting from coherent state with an average of 3 photons relaxes towards vacuum (decoherence due to finite photon life time around 70 ms).

Detection efficiency 40%
Detection error rate 10%
Delay 4 sampling periods.

The quantum filter takes into account cavity decoherence, measure imperfections and delays (Bayes law).

Truncation to 9 photons.
Several "quantum states": $|\psi_k\rangle$, $\hat{\rho}_k$ and $\hat{\rho}_{k}^{\text{est}}$.

The state estimation $\hat{\rho}_{k}^{\text{est}}$ used in the feedback law takes into account, measure imperfections, delays and cavity decoherence:

- Derived from Bayes law: depends on past detector outcomes between 0 and $k$; computed recursively from an initial value $\hat{\rho}_{0}^{\text{est}}$;
- Stable and tends to converge towards $\hat{\rho}_k$, the expectation value of $\rho_k = |\psi_k\rangle\langle\psi_k|$ knowing its initial value $\rho_0 = \hat{\rho}_0$ and the past detector outcomes between 0 and $k$. 

Outline

Quantum filter of the LKB Photon Box
  Markov chain in the ideal case
  The Markov chain with detection errors
  The Markov chain with cavity decoherence
  Structure of the complete quantum filter

Discrete-time quantum systems
  Markov chains in the ideal case
  Markov chains with imperfections and decoherence
  Stability and convergence issues
  Bayesian parameter estimations

Continuous-time quantum filters
  From discrete-time to continuous-time models
  SDE driven by Poisson and Wiener processes
  SDE with imperfections and decoherence

Conclusion
Markov chain in the ideal case (1)

- **System** $S$ corresponds to a quantized cavity mode:

$$\mathcal{H}_S = \left\{ \sum_{n=0}^{\infty} \psi^n |n\rangle \mid (\psi^n)_{n=0}^{\infty} \in l^2(\mathbb{C}) \right\},$$

where $|n\rangle$ represents the Fock state associated to exactly $n$ photons inside the cavity.

- **Meter** $M$ is associated to atoms: $\mathcal{H}_M = \mathbb{C}^2$, each atom admits two energy levels and is described by a wave function $c_g |g\rangle + c_e |e\rangle$ with $|c_g|^2 + |c_e|^2 = 1$; atoms leaving $B$ are all in state $|g\rangle$.

- When an atom comes out $B$, the state $|\Psi\rangle_B \in \mathcal{H}_S \otimes \mathcal{H}_M$ of the composite system atom/field is separable:

$$|\Psi\rangle_B = |\psi\rangle \otimes |g\rangle.$$
Markov chain in the ideal case (2)

When an atom comes out $B$: $|\psi\rangle_B = |\psi\rangle \otimes |g\rangle$.

Just before the measurement in $D$, the state is in general entangled (not separable):

$$|\psi\rangle_{R_2} = U_{SM}(|\psi\rangle \otimes |g\rangle) = (M_g|\psi\rangle) \otimes |g\rangle + (M_e|\psi\rangle) \otimes |e\rangle$$

where $U_{SM}$ is the total unitary transformation (Schrödinger propagator) defining the linear measurement operators $M_g$ and $M_e$ on $\mathcal{H}_S$. Since $U_{SM}$ is unitary, $M_g^\dagger M_g + M_e^\dagger M_e = I$. 
Markov chain in the ideal case (3)

Just before the measurement in $D$, the atom/field state is:

$$M_g |\psi\rangle \otimes |g\rangle + M_e |\psi\rangle \otimes |e\rangle$$

Denote by $\mu \in \{g, e\}$ the measurement outcome in detector $D$: with probability $p_\mu = \langle \psi | M_\mu^\dagger M_\mu |\psi\rangle$ we get $\mu$. Just after the measurement outcome $\mu$, the state becomes separable:

$$|\Psi\rangle_D = \frac{1}{\sqrt{p_\mu}} (M_\mu |\psi\rangle) \otimes |\mu\rangle = \frac{(M_\mu |\psi\rangle) \otimes |\mu\rangle}{\sqrt{\langle \psi | M_\mu^\dagger M_\mu |\psi\rangle}}.$$

Markov process (density matrix formulation $\rho \sim |\psi\rangle \langle \psi|$)

$$\rho_+ = \begin{cases} 
M_g(\rho) = \frac{M_g \rho M_g^\dagger}{\text{Tr}(M_g \rho M_g^\dagger)}, & \text{with probability } p_g = \text{Tr} \left( M_g \rho M_g^\dagger \right); \\
M_e(\rho) = \frac{M_e \rho M_e^\dagger}{\text{Tr}(M_e \rho M_e^\dagger)}, & \text{with probability } p_e = \text{Tr} \left( M_e \rho M_e^\dagger \right). 
\end{cases}$$

Kraus map: $E(\rho_+ / \rho) = K(\rho) = M_g \rho M_g^\dagger + M_e \rho M_e^\dagger.$
Markov chain with detection errors (1)

- \( \rho_+ = \frac{1}{\text{Tr}(M_\mu \rho M_\mu^\dagger)} M_\mu \rho M_\mu^\dagger \) when the atom collapses in \( \mu = g, e \).
  
  This happens with probability \( \text{Tr}(M_\mu \rho M_\mu^\dagger) \).

- Detection error rates: \( P(y = e/\mu = g) = \eta_g \in [0,1] \) the probability of erroneous assignation to \( e \) when the atom collapses in \( g \); \( P(y = g/\mu = e) = \eta_e \in [0,1] \) (given by the contrast of the Ramsey fringes).

Bayes law gives the probability that the atom collapses in \( \mu = g \) knowing the detector outcome \( y = g \):

\[
P(\mu = g/y = g) = \frac{(1 - \eta_g) \text{Tr}(M_g \rho M_g^\dagger)}{(1 - \eta_g) \text{Tr}(M_g \rho M_g^\dagger) + \eta_e \text{Tr}(M_e \rho M_e^\dagger)}
\]

since \( P(y = g/\mu = g) = (1 - \eta_g) \) and \( P(y = g/\mu = e) = \eta_e \).
Markov chain with detection errors (2)

The expectation value $\hat{\rho}_+$ of $\rho_+$ knowing $\rho$ and the imperfect detection $y = g$ is given by

$$\hat{\rho}_+ = P(\mu = g / y = g) \frac{M_g \rho M_g^\dagger}{\text{Tr}(M_g \rho M_g^\dagger)} + P(\mu = e / y = g) \frac{M_e \rho M_e^\dagger}{\text{Tr}(M_e \rho M_e^\dagger)}$$

Since

$$P(\mu = g / y = g) = \frac{(1 - \eta_g) \text{Tr}(M_g \rho M_g^\dagger)}{(1 - \eta_g) \text{Tr}(M_g \rho M_g^\dagger) + \eta_e \text{Tr}(M_e \rho M_e^\dagger)}$$

and $P(\mu = e / y = g) = 1 - P(\mu = g / y = g)$, this expectation value $\hat{\rho}_+$ is given by

$$\hat{\rho}_+ = \frac{1}{\text{Tr}((1 - \eta_g) M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger)} \left( (1 - \eta_g) M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger \right)$$

Similarly when $y = e$, the expectation value $\hat{\rho}_+$ is given by

$$\hat{\rho}_+ = \frac{1}{\text{Tr}((1 - \eta_e) M_e \rho M_e^\dagger + \eta_g M_g \rho M_g^\dagger)} \left( (1 - \eta_e) M_e \rho M_e^\dagger + \eta_g M_g \rho M_g^\dagger \right)$$
Markov chain with detection errors (3)

We get

\[
\hat{\rho}_+ = \begin{cases} 
\frac{(1-\eta_g)M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger}{\text{Tr}\left((1-\eta_g)M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger\right)}, & \text{with prob. } \text{Tr}\left((1-\eta_g)M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger\right); \\
\frac{\eta_g M_g \rho M_g^\dagger + (1-\eta_e) M_e \rho M_e^\dagger}{\text{Tr}\left(\eta_g M_g \rho M_g^\dagger + (1-\eta_e) M_e \rho M_e^\dagger\right)}, & \text{with prob. } \text{Tr}\left(\eta_g M_g \rho M_g^\dagger + (1-\eta_e) M_e \rho M_e^\dagger\right). 
\end{cases}
\]

Key point:

\[
\text{Tr}\left((1-\eta_g)M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger\right) \quad \text{and} \quad \text{Tr}\left(\eta_g M_g \rho M_g^\dagger + (1-\eta_e) M_e \rho M_e^\dagger\right)
\]

are the probabilities to detect \(y = g\) and \(e\), knowing \(\rho\).

With \(\eta_{\mu',\mu}\) being the probability to detect \(y = \mu'\) knowing that the atom collapses in \(\mu\), we have

\[
\hat{\rho}_+ = \frac{\sum_{\mu} \eta_{\mu',\mu} M_{\mu} \rho M_{\mu}^\dagger}{\text{Tr}\left(\sum_{\mu} \eta_{\mu',\mu} M_{\mu} \rho M_{\mu}^\dagger\right)} \quad \text{when we detect } y = \mu'.
\]

The probability to detect \(y = \mu'\) knowing \(\rho\) is \(\text{Tr}\left(\sum_{\mu} \eta_{\mu',\mu} M_{\mu} \rho M_{\mu}^\dagger\right)\).
The Markov chain with cavity decoherence

When the sampling time $\Delta T$ is much smaller than the photon life time $T_{cav}$, cavity decoherence (at zero temperature) can be described approximatively by the Kraus map

$$\rho \mapsto M_0 \rho M_0^\dagger + M_- \rho M_-^\dagger$$

with $M_0 = (1 - \frac{\Delta T}{2T_{cav}})I - \frac{\Delta T}{2T_{cav}} \mathbf{N}$ and $M_- = \sqrt{\frac{\Delta T}{T_{cav}}} \mathbf{a}$

$M_0$ and $M_-$ can be seen as "measurement" operators corresponding to information caught by the "environment", information unknown in the real life but known in "Matlab/Simulink world":

- $M_0$ corresponds to no photon destruction during the sampling interval $\Delta T$; probability $\text{Tr} \left( M_0 \rho M_0^\dagger \right)$.

- $M_-$ corresponds to one photon destruction during the sampling interval $\Delta T$; probability $\text{Tr} \left( M_- \rho M_-^\dagger \right)$.

The fact that we do not have access to this information can be interpreted as a detection error of 50\% for $M_0$ and $M_-$. We get

$$\hat{\rho}_+ = M_0 \rho M_0^\dagger + M_- \rho M_-^\dagger.$$
Photon-box quantum filter parameterized by left stochastic matrix $\eta_{\mu',\mu}$

$$\hat{\rho}_{k+1}^{\text{est}} = \frac{1}{\text{Tr}\left(\sum_{\mu} \eta_{\mu',\mu} M_{\mu} \hat{\rho}_k^{\text{est}} M_{\mu}^{\dagger}\right)} \left(\sum_{\mu} \eta_{\mu',\mu} M_{\mu} \hat{\rho}_k^{\text{est}} M_{\mu}^{\dagger}\right)$$

we have a total of $m = 3 \times 7 = 21$ Kraus operators $M_{\mu}$. The "jumps" are labeled by $\mu = (\mu^a, \mu^c)$ with $\mu^a \in \{\text{no}, g, e, gg, ge, eg, ee\}$ labeling atom related jumps and $\mu^c \in \{\text{o}, +, -\}$ cavity decoherence jumps.

we have only $m' = 6$ real detection possibilities $\mu' \in \{\text{no}, g, e, gg, ge, ee\}$ corresponding respectively to no detection, a single detection in $g$, a single detection in $e$, a double detection both in $g$, a double detection one in $g$ and the other in $e$, and a double detection both in $e$.

<table>
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<tr>
<th>$\mu' \setminus \mu$</th>
<th>(no, $\mu^c$)</th>
<th>(g, $\mu^c$)</th>
<th>(e, $\mu^c$)</th>
<th>(gg, $\mu^c$)</th>
<th>(ee, $\mu^c$)</th>
<th>(ge, $\mu^c$) or (eg, $\mu^c$)</th>
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<td>$1 - \epsilon_d$</td>
<td>$1 - \epsilon_e$</td>
<td>$(1 - \epsilon_d)^2$</td>
<td>$(1 - \epsilon_d)^2$</td>
<td>$(1 - \epsilon_d)^2$</td>
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<td>$\epsilon_d(1 - \eta_g)$</td>
<td>$\epsilon_d \eta_e$</td>
<td>$2 \epsilon_d(1 - \epsilon_d)(1 - \eta_g)$</td>
<td>$2 \epsilon_d(1 - \epsilon_d)\eta_e$</td>
<td>$\epsilon_d(1 - \epsilon_d)(1 - \eta_g)$</td>
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<tr>
<td>e</td>
<td>0</td>
<td>$\epsilon_d \eta_g$</td>
<td>$\epsilon_d(1 - \eta_e)$</td>
<td>$2 \epsilon_d(1 - \epsilon_d)\eta_g$</td>
<td>$2 \epsilon_d(1 - \epsilon_d)(1 - \eta_e)$</td>
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<tr>
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<td>$\epsilon_d^2(1 - \eta_g)^2$</td>
<td>$\epsilon_d^2 \eta_e^2$</td>
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<tr>
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<td>0</td>
<td>$2 \epsilon_d^2 \eta_g(1 - \eta_g)$</td>
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<td>$\epsilon_d^2((1 - \eta_g)(1 - \eta_e)$</td>
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<tr>
<td>ee</td>
<td>0</td>
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<td>0</td>
<td>$\epsilon_d^2 \eta_g^2$</td>
<td>$\epsilon_d^2 \eta_e^2$</td>
<td>$\epsilon_d^2 \eta_g(1 - \eta_e)$</td>
</tr>
</tbody>
</table>

Markov chain in ideal life (e.g. Matlab/Simulink world): pure state $\rho_k$

$$
\rho_{k+1} = \mathbb{M}_{\mu_k}(\rho_k) =: \frac{M_{\mu_k}\rho_k M_{\mu_k}^\dagger}{\text{Tr} \left( M_{\mu_k}\rho_k M_{\mu_k}^\dagger \right)}
$$

- To each measurement outcome $\mu$ is attached the Kraus operator $M_{\mu}$ depending on $\mu$ and also on time (not explicitly recalled here, $M_{\mu} = M_{\mu,k}$ could depend on $k$).

- $\mu_k$ is a random variable taking values $\mu$ in $\{1, \cdots, m\}$ with probability $p_{\mu,\rho_k} = \text{Tr} \left( M_{\mu}\rho_k M_{\mu}^\dagger \right)$. Conservation of probability ($\sum_{\mu} p_{\mu,\rho} = 1$ for all $\rho$) is guaranteed by $\sum_{\mu=1}^{m} M_{\mu}^\dagger M_{\mu} = I$.

- The Kraus map $\mathcal{K}(\rho) = \sum_{\mu=1}^{m} M_{\mu}\rho M_{\mu}^\dagger$ provides

$$
\mathbb{E} (\rho_{k+1}/\rho_k) = \mathcal{K}(\rho_k)
$$
The Markov chain in real life: mixed states, $\hat{\rho}_k$ and $\hat{\rho}_k^{\text{est}}$ (1) ⁴

Take $\rho_{k+1} = \frac{1}{\text{Tr}(M_{\mu_k} \rho_k M_{\mu_k}^\dagger)} \left( M_{\mu_k} \rho_k M_{\mu_k}^\dagger \right)$ with measure imperfections and decoherence described by the left stochastic matrix $\eta$:

$\eta_{\mu',\mu} \in [0, 1]$ is the probability of having the imperfect outcome $\mu' \in \{1, \ldots, m'\}$ knowing that the perfect one is $\mu \in \{1, \ldots, m\}$.

$\hat{\rho}_k = \mathbb{E} (\rho_k | \rho_0, \mu'_0, \ldots, \mu'_{k-1})$ can be computed efficiently via the following recurrence

$$
\hat{\rho}_{k+1} = \frac{1}{\text{Tr} \left( \sum_{\mu=1}^{m} \eta_{\mu',\mu} M_{\mu} \hat{\rho}_k M_{\mu}^\dagger \right)} \left( \sum_{\mu=1}^{m} \eta_{\mu',\mu} M_{\mu} \hat{\rho}_k M_{\mu}^\dagger \right)
$$

where the detector outcome $\mu'_k$ takes values $\mu'$ in $\{1, \ldots, m'\}$ with probability $p_{\mu', \hat{\rho}_k} = \text{Tr} \left( \sum_{\mu=1}^{m} \eta_{\mu',\mu} M_{\mu} \hat{\rho}_k M_{\mu}^\dagger \right)$.

Thus $\mathbb{E} (\hat{\rho}_{k+1} | \hat{\rho}_k) = \mathcal{K}(\hat{\rho}_k) = \sum_{\mu=1}^{m} M_{\mu} \hat{\rho}_k M_{\mu}^\dagger$.

The Markov chain in real life: mixed states, $\hat{\rho}_k$ and $\hat{\rho}^{est}_k$ (2)

$\hat{\rho}_k = \mathbb{E} (\rho_k | \rho_0, \mu'_0, \ldots, \mu'_{k-1})$ is given by

$$
\hat{\rho}_{k+1} = \frac{1}{\text{Tr} \left( \sum_{\mu=1}^{m} \eta_{\mu'_k, \mu} M_{\mu} \hat{\rho}_k M_{\mu}^\dagger \right)} \left( \sum_{\mu=1}^{m} \eta_{\mu'_k, \mu} M_{\mu} \hat{\rho}^{est}_k M_{\mu}^\dagger \right)
$$

with the perfect initialization: $\hat{\rho}_0 = \rho_0$.

$\hat{\rho}^{est}_{k+1} = \frac{1}{\text{Tr} \left( \sum_{\mu=1}^{m} \eta_{\mu'_k, \mu} M_{\mu} \hat{\rho}^{est}_k M_{\mu}^\dagger \right)} \left( \sum_{\mu=1}^{m} \eta_{\mu'_k, \mu} M_{\mu} \hat{\rho}^{est}_k M_{\mu}^\dagger \right)$ but with imperfect initialization $\hat{\rho}^{est}_0 \neq \rho_0$.

This filtering process is stable$^5$: the fidelity $F(\hat{\rho}_k, \hat{\rho}^{est}_k)$ is a sub-martingale for any $\eta$ and $M_{\mu}$:

$$
\mathbb{E} \left( F(\hat{\rho}_{k+1}, \hat{\rho}^{est}_{k+1}) / \hat{\rho}_k \right) \geq F(\hat{\rho}_k, \hat{\rho}^{est}_k)
$$

Convergence of $\hat{\rho}^{est}_k$ towards $\hat{\rho}_k$ when $k \mapsto +\infty$ is an open problem.$^6$

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$^5$PR. Fidelity is a Sub-Martingale for Discrete-Time Quantum Filters IEEE Transactions on Automatic Control, 2011, 56, 2743-2747.

Bayesian parameter estimations

Consider detector outcomes $\mu'_k$ corresponding to a parameter value $\bar{p}$ poorly known. Assume to simplify that either $\bar{p} = a$ or $\bar{p} = b$, with $a \neq b$. We can discriminate between $a$ and $b$ and recover $\bar{p}$ via the following Bayesian scheme using information contained in the $\mu'_k$'s:

$$
\hat{\rho}_{a,k+1}^{\text{est}} = \frac{\sum_{\mu} \eta_{\mu'_k,\mu}^a M_{\mu}^{a} \hat{\rho}_{a,k}^{\text{est}} M_{\mu}^{a \dagger}}{\text{Tr} \left( \sum_{p} \sum_{\mu} \eta_{\mu'_k,\mu}^p M_{\mu}^{p} \hat{\rho}_{p,k}^{\text{est}} M_{\mu}^{p \dagger} \right)}, \quad \hat{\rho}_{b,k+1}^{\text{est}} = \frac{\sum_{\mu} \eta_{\mu'_k,\mu}^b M_{\mu}^{b} \hat{\rho}_{b,k}^{\text{est}} M_{\mu}^{b \dagger}}{\text{Tr} \left( \sum_{p} \sum_{\mu} \eta_{\mu'_k,\mu}^p M_{\mu}^{p} \hat{\rho}_{p,k}^{\text{est}} M_{\mu}^{p \dagger} \right)}
$$

with initialization $\hat{\rho}_{a,k+1}^{\text{est}} = \hat{\rho}_{b,k+1}^{\text{est}} = \hat{\rho}_{0}^{\text{est}}/2$ where $\hat{\rho}_{0}^{\text{est}}$ is some guess of $\hat{\rho}_0$ assuming initial probability of $\frac{1}{2}$ to have $\bar{p} = a$ and $\bar{p} = b$.

This estimation/filtering process is also stable:

- $F(\hat{\rho}_k, \hat{\rho}_{a,k}^{\text{est}}) + F(\hat{\rho}_k, \hat{\rho}_{b,k}^{\text{est}})$ is a sub-martingale
- $\text{Tr} \left( \hat{\rho}_{a,k}^{\text{est}} \right), \text{Tr} \left( \hat{\rho}_{b,k}^{\text{est}} \right)$ estim. of proba. to have $\bar{p} = a$, $\bar{p} = b$.

Direct generalization to a continuum of choices for $\bar{p} \in [p_{\text{min}}, p_{\text{max}}]$ (see \(^7\) for a first experimental use)

Dynamical models with a precise structure

Discrete-time models are Markov chains

\[ \rho_{k+1} = \frac{1}{\rho_\mu(\rho_k)} M_\mu \rho_k M_\mu^\dagger \]  

with proba. \[ \rho_\mu(\rho_k) = \text{Tr} \left( M_\mu \rho_k M_\mu^\dagger \right) \]

associated to Kraus maps (ensemble average, open quantum channels)

\[ \mathbb{E} \left( \rho_{k+1} / \rho_k \right) = K(\rho_k) = \sum_\mu M_\mu \rho_k M_\mu^\dagger \quad \text{with} \quad \sum_\mu M_\mu^\dagger M_\mu = \mathbb{I} \]

Continuous-time models are stochastic differential systems

\[ d\rho = \left( -i[H, \rho] + L \rho L^\dagger - \frac{1}{2} (L^\dagger L \rho + \rho L^\dagger L) \right) dt \]

\[ + \left( L \rho + \rho L^\dagger - \text{Tr} \left( (L + L^\dagger) \rho \right) \rho \right) dw \]

driven by Wiener processes\(^8\)

\[ dw = dy - \text{Tr} \left( (L + L^\dagger) \rho \right) dt \]

with measure \( y \) and associated to Lindbald master equations:

\[ \frac{d}{dt} \rho = -i \hbar [H, \rho] + L \rho L^\dagger - \frac{1}{2} (L^\dagger L \rho + \rho L^\dagger L) \]

\(^8\)Another possibility: SDE driven by Poisson processes.
From discrete-time to continuous-time: heuristic connection

For Monte-Carlo simulations of

\[ d\rho = \left( -i[H, \rho] + L\rho L^\dagger - \frac{1}{2}(L^\dagger L\rho + \rho L^\dagger L) \right) dt \]

\[ + \left( L\rho + \rho L^\dagger - \text{Tr}\left( (L + L^\dagger)\rho \right) \rho \right) dw \]

take a small sampling time \( dt \), generate a random number \( dw_t \) according to a Gaussian law of standard deviation \( \sqrt{dt} \), and set \( \rho_{t+dt} = M_{dy_t}(\rho_t) \) where the jump operator \( M_{dy_t} \) is labelled by the measurement outcome \( dy_t = \text{Tr}\left( (L + L^\dagger)\rho_t \right) dt + dw_t \):

\[ M_{dy_t}(\rho_t) = \frac{\left( I + (-iH - \frac{1}{2} L^\dagger L) dt + dy_t L \right) \rho_t \left( I + (iH - \frac{1}{2} L^\dagger L) dt + dy_t L^\dagger \right)}{\text{Tr}\left( (I + (-iH - \frac{1}{2} L^\dagger L) dt + dy_t L) \rho_t \left( I + (iH - \frac{1}{2} L^\dagger L) dt + dy_t L^\dagger \right) \right)}. \]

Then \( \rho_{t+dt} \) remains always a density operator and using the Itô rules (\( dw \) of order \( \sqrt{dt} \) and \( dw^2 \equiv dt \)) we get the good

\[ d\rho = \rho_{t+dt} - \rho_t \text{ up to } O((dt)^{3/2}) \text{ terms.} \]
For the Lindblad equation

\[
\frac{d}{dt} \rho = -\frac{i}{\hbar} [H, \rho] + L\rho L^\dagger - \frac{1}{2} (L^\dagger L \rho + \rho L^\dagger L)
\]

take a small sampling time \( dt \) and set

\[
\rho_{t+dt} = \frac{(I + dt(-iH - \frac{1}{2} L^\dagger L)) \rho_t (I + dt(iH - \frac{1}{2} L^\dagger L)) + dt L \rho_t L^\dagger}{\text{Tr}((I + dt(-iH - \frac{1}{2} L^\dagger L)) \rho_t (I + dt(iH - \frac{1}{2} L^\dagger L)) + dt L \rho_t L^\dagger)}.\]

Then \( \rho_{t+dt} \) remains always a density operator and

\[
\frac{d}{dt} \rho = (\rho_{t+dt} - \rho_t)/dt \]

up to \( O(dt) \) terms.
SDE driven by Poisson and/or Wiener processes

\[ d\rho_t = \mathcal{L}(\rho_t) \, dt + \sum_{\nu=1}^{m_w} \Lambda_{\nu}(\rho_t) \, dw_\nu^t + \sum_{\mu=1}^{m_P} \Gamma_{\mu}(\rho_t) \left( dN^\mu_t - \text{Tr} \left( C_\mu \rho_t C_\mu^\dagger \right) \, dt \right) \]

where

\[ \mathcal{L}(\rho_t) := -i [H, \rho_t] + \sum_{\mu=1}^{m_P} \mathcal{L}^P_\mu(\rho_t) + \sum_{\nu=1}^{m_w} \mathcal{L}^W_\nu(\rho_t), \]
\[ \mathcal{L}^P_\mu(\rho) := -\frac{1}{2} \{ C_\mu^\dagger C_\mu, \rho \} + C_\mu \rho C_\mu^\dagger, \quad \mathcal{L}^W_\nu(\rho) := -\frac{1}{2} \{ L_\nu^\dagger L_\nu, \rho \} + L_\nu \rho L_\nu^\dagger; \]
\[ \Gamma_{\mu}(\rho) := \frac{C_\mu \rho C_\mu^\dagger}{\text{Tr}(C_\mu \rho C_\mu^\dagger)} - \rho, \quad \Lambda_{\nu}(\rho) := L_\nu \rho + \rho L_\nu^\dagger - \text{Tr} \left( (L_\nu + L_\nu^\dagger) \rho \right) \rho \]

- Detector click no \( \mu \) is related to the Poisson process
  \[ dN^\mu_t = N^\mu(t + dt) - N^\mu(t) = 1 \]
  and happens with probability \( \text{Tr} \left( C_\mu \rho_t C_\mu^\dagger \right) \, dt; \)

- Continuous detector \( y_\nu^t \) is related to the Wiener process
  \[ dw_\nu^t \text{ by } dy_\nu^t = dw_\nu^t + \text{Tr} \left( (L_\nu + L_\nu^\dagger) \rho_t \right) \, dt. \]
Quantum filter in the ideal case

\[ d \rho_t = \mathcal{L}(\rho_t) \, dt + \sum_{\nu=1}^{m_w} \Lambda_{\nu}(\rho_t) \, dw_{\nu}^{\nu} + \sum_{\mu=1}^{m_p} \gamma_{\mu}(\rho_t) \left( dN_{\mu}^{\mu} - \text{Tr} \left( C_{\mu} \rho_t C_{\mu}^\dagger \right) \, dt \right), \]

and the associated quantum filter

\[ d \hat{\rho}_{t}^{\text{est}} = \mathcal{L}(\hat{\rho}_{t}^{\text{est}}) \, dt + \sum_{\nu=1}^{m_w} \Lambda_{\nu}(\hat{\rho}_{t}^{\text{est}}) \left( dy_{\nu}^{\nu} - \text{Tr} \left( (L_{\nu} + L_{\nu}^\dagger) \hat{\rho}_t^{\text{est}} \right) \, dt \right) \]

\[ + \sum_{\mu=1}^{m_p} \gamma_{\mu}(\hat{\rho}_{t}^{\text{est}}) \left( dN_{\mu}^{\mu} - \text{Tr} \left( C_{\mu} \hat{\rho}_t^{\text{est}} C_{\mu}^\dagger \right) \, dt \right). \]

It can be rewritten as follows

\[ d \hat{\rho}_{t}^{\text{est}} = \mathcal{L}(\hat{\rho}_{t}^{\text{est}}) \, dt + \sum_{\nu=1}^{m_w} \Lambda_{\nu}(\hat{\rho}_{t}^{\text{est}}) \left( \text{Tr} \left( (L_{\nu} + L_{\nu}^\dagger) \rho_t \right) - \text{Tr} \left( (L_{\nu} + L_{\nu}^\dagger) \hat{\rho}_t^{\text{est}} \right) \right) \, dt \]

\[ + \sum_{\nu=1}^{m_w} \Lambda_{\nu}(\hat{\rho}_{t}^{\text{est}}) \, dw_{\nu}^{\nu} + \sum_{\mu=1}^{m_p} \gamma_{\mu}(\hat{\rho}_{t}^{\text{est}}) \left( dN_{\mu}^{\mu} - \text{Tr} \left( C_{\mu} \hat{\rho}_t^{\text{est}} C_{\mu}^\dagger \right) \, dt \right). \]
Quantum filters with imperfections and decoherence\(^9\) (1)

- Imperfection model for the Poisson processes \(dN_t^\mu\):
  - real outcomes \(\mu' \in \{0, 1, \ldots, m'_P\}\)
  - ideal outcomes \(\mu \in \{0, 1, \ldots, m_P\}\).
  - \((m'_P + 1) \times m_P\) left stochastic matrix
    \[\eta^P = (\eta_{\mu',\mu})_{0 \leq \mu' \leq m'_P, 1 \leq \mu \leq m_P}\]
  - positive vector \(\bar{\eta}^P = (\bar{\eta}_{\mu'})_{1 \leq \mu' \leq m'_P}\) in \(\mathbb{R}_{+}^{m'_P}\).

- Imperfection model for the diffusion processes \(dw_t^{\nu'}\):
  - \(m'_w\) real continuous signals \(y_t^{\nu'}\) with \(\nu' \in \{1, \ldots, m'_w\}\),
  - \(m_w\) ideal continuous signals \(y_t^{\nu}\) with \(\nu \in \{1, \ldots, m_w\}\)
  - correlation \(m'_w \times m_w\) matrix \(\eta^w = (\eta_{\nu',\nu})_{1 \leq \nu' \leq m'_w, 1 \leq \nu \leq m_w}\),
    with \(0 \leq \eta_{\nu',\nu} \leq 1\) and \(\sum_{\nu' = 1}^{m'_w} \eta_{\nu',\nu} \leq 1\).

Quantum filters with imperfections and decoherence (2)

\[ d\hat{\rho}_t = \mathcal{L}(\hat{\rho}_t)\ dt + \sum_{\nu' = 1}^{m'_w} \sqrt{\tilde{\eta}_{\nu',\nu}'^w} \hat{\Lambda}_{\nu'}(\hat{\rho}_t)\ d\hat{w}_{t}' \]

\[ + \sum_{\mu' = 1}^{m'_p} \hat{\Upsilon}_{\mu'}(\hat{\rho}_t) \left( d\hat{N}^\mu_\mu' - \tilde{\eta}_{\mu'}^P\ dt - \sum_{\mu = 1}^{m_P} \eta_{\mu',\mu}^P \ Tr\left( C_\mu \hat{\rho}_t C_\mu^\dagger \right) dt \right) \]

- \[ \tilde{\eta}_{\nu',\nu}'^w = \sum_{\nu = 1}^{m_w} \eta_{\nu',\nu}^w, \quad \hat{\Upsilon}_{\mu'}(\rho) := \frac{\tilde{\eta}_{\mu'}^P\rho + \sum_{\mu = 1}^{m_P} \eta_{\mu',\mu}^P C_\mu \rho C_\mu^\dagger}{\tilde{\eta}_{\mu'}^P + \sum_{\mu = 1}^{m_P} \eta_{\mu',\mu}^P \ Tr(C_\mu \rho C_\mu^\dagger)} - \rho, \]

- \[ \hat{\Lambda}_{\nu'}(\rho) = \hat{L}_{\nu'}\rho + \rho \hat{L}_{\nu'}^\dagger - \ Tr\left( (\hat{L}_{\nu'} + \hat{L}_{\nu'}^\dagger)\rho \right) \rho, \quad \hat{L}_{\nu'} := (\sum_{\nu = 1}^{m_w} \eta_{\nu',\nu}^w L_{\nu}^\mu) / \tilde{\eta}_{\nu'}^w; \]

- the jump detector \( \mu' \) corresponds to \( \hat{N}^\mu_\mu'(t) \):
  \[ d\hat{N}^\mu_\mu'(t) = \hat{N}^\mu_\mu'(t + dt) - \hat{N}^\mu_\mu'(t) = 1 \]
  happens with probability
  \[ \hat{P}_{\mu'}(\hat{\rho}_t) = \tilde{\eta}_{\mu'}^P\ dt + \sum_{\mu = 1}^{m_P} \eta_{\mu',\mu}^P \ Tr\left( C_\mu \hat{\rho}_t C_\mu^\dagger \right) dt ; \]

- the continuous detector \( \nu' \) refers to \( \hat{y}_t^{\nu'} \) and \( d\hat{w}_{t}' \):
  \[ d\hat{y}_t^{\nu'} = d\hat{w}_t^{\nu'} + \sqrt{\tilde{\eta}_{\nu'}^w} \ Tr\left( (\hat{L}_{\nu'} + \hat{L}_{\nu'}^\dagger)\hat{\rho}_t \right) dt. \]
Quantum filters with imperfections and decoherence (3)

\[
d\hat{\rho}_t = \mathcal{L}(\hat{\rho}_t)\,dt + \sum_{\nu'=1}^{m_w'} \sqrt{\bar{\eta}_{\nu'}^{w}} \hat{\Lambda}_{\nu'}(\hat{\rho}_t)\,d\hat{w}_{\nu}'
\]

\[
+ \sum_{\mu'=1}^{m_p'} \hat{\Upsilon}_{\mu'}(\hat{\rho}_t) \left( d\hat{N}_t^{\mu'} - \bar{\eta}_{\mu'} P \,dt - \sum_{\mu=1}^{m_p} \eta_{\mu',\mu} \operatorname{Tr}(C_{\mu} \hat{\rho}_t C_{\mu}^\dagger) \,dt \right)
\]

and the associated quantum filter

\[
d\hat{\rho}_{\text{est}} = \mathcal{L}(\hat{\rho}_{\text{est}})\,dt + \sum_{\nu'=1}^{m_w'} \sqrt{\bar{\eta}_{\nu'}^{w}} \hat{\Lambda}_{\nu'}(\hat{\rho}_{\text{est}})\left( d\hat{y}_t^{\nu'} - \sqrt{\bar{\eta}_{\nu'}^{w}} \operatorname{Tr}\left((\hat{L}_{\nu'} + \hat{L}_{\nu'}^\dagger)\hat{\rho}_{\text{est}}\right) \,dt \right)
\]

\[
+ \sum_{\mu'=1}^{m_p'} \hat{\Upsilon}_{\mu'}(\hat{\rho}_{\text{est}}) \left( d\hat{N}_t^{\mu'} - \bar{\eta}_{\mu'} P \,dt - \sum_{\mu=1}^{m_p} \eta_{\mu',\mu} \operatorname{Tr}(C_{\mu} \hat{\rho}_{\text{est}} C_{\mu}^\dagger) \,dt \right)
\]
Quantum filtering combines the following key points

1. **Bayes law:** $P(\mu'/\mu) = P(\mu/\mu') P(\mu') / (\sum_{\nu'} P(\mu/\nu') P(\nu'))$.

2. **Schrödinger equations** defining unitary transformations.

3. **Partial collapse of the wave packet:** irreversibility and convergence are induced by the measure of observables $\mathcal{O}$ with degenerate spectra, $\mathcal{O} = \sum_{\mu} \lambda_{\mu} P_{\mu}$:
   - measure outcome $\lambda_{\mu}$ with proba. $p_{\mu} = \langle \psi | P_{\mu} | \psi \rangle = \text{Tr} (\rho P_{\mu})$ depending $|\psi\rangle$, $\rho$ just before the measurement
   - measure back-action if outcome $\mu$:
     $$|\psi\rangle \mapsto |\psi\rangle^+ = \frac{P_{\mu} |\psi\rangle}{\sqrt{\langle \psi | P_{\mu} | \psi \rangle}}, \quad \rho \mapsto \rho^+ = \frac{P_{\mu} \rho P_{\mu}}{\text{Tr} (\rho P_{\mu})}$$

4. **Tensor product for the description of composite systems $(S, M)$:**
   - Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M$
   - Hamiltonian $H = H_S \otimes I_M + H_{\text{int}} + I_S \otimes H_M$
   - observable on sub-system $M$ only: $\mathcal{O} = I_S \otimes \mathcal{O}_M$. 