A real-time synchronization feedback for single-atom frequency standards [1]

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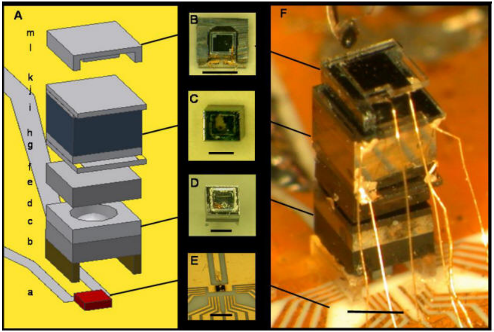
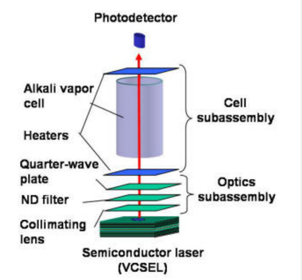
Outline

The NIST MicroClock[2]

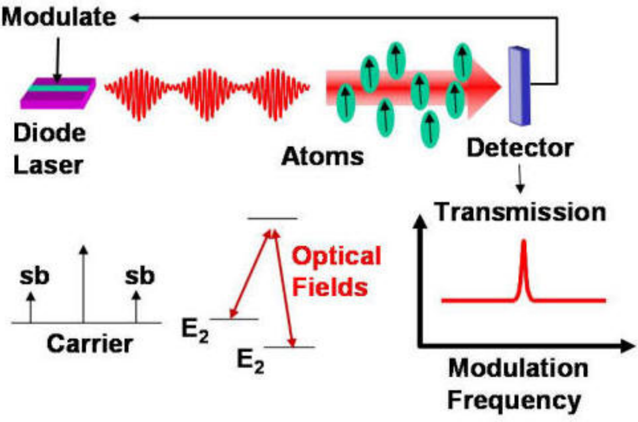




- Quartz crystal clocks: 1 second over few days.
- NIST chip-scale atomic clock: 1 second over 300 years
- High-Perf. atomic clocks: 1 second over 100 million years.

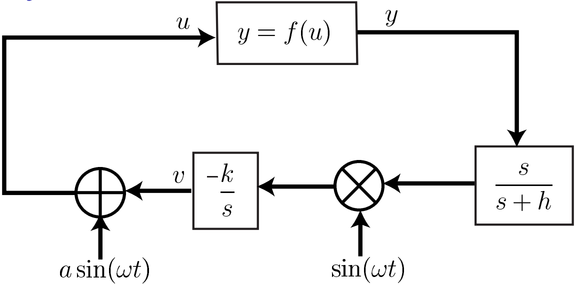
[2]NIST: National Institute of Standards and Technology, web-site: `http://tf.nist.gov/timefreq/index.html`.

The principle: Coherent Population Trapping[3]



[3] From the web-site: `http://tf.nist.gov/timefreq/index.html`.

The synchronization via extremum seeking



Here $u = \omega_{diode}$ and $y = f(\omega_{diode})$ where f admits a sharp maximum at the unknown value $\bar{u} = \omega_{atom}$.

$s = \frac{d}{dt}$, constant > 0 parameters (h, k, a, ω).

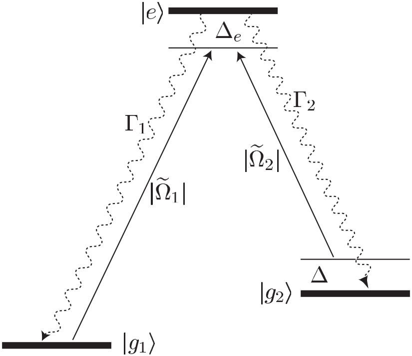
Extremum seeking via feedback: $u(t) = v(t) + a\sin(\omega t)$ where $v(t) \approx \omega_{atom}$ is adjusted via a dynamic time-varying output feedback (with $\omega, a, h, \sqrt{k} \ll \omega_{atom}$):

$$\frac{d}{dt}v(t) = -k\sin(\omega t)(y(t) - \xi(t)) \quad \text{with} \quad \frac{d}{dt}\xi = h(y(t) - \xi(t))$$

Our contribution[4]: a real-time synchronization scheme when the atomic cloud is replaced by a single atom.

[4] Mirrahimi-R, 2008, arxiv:0806.1392v1

The system and its synchronization scheme



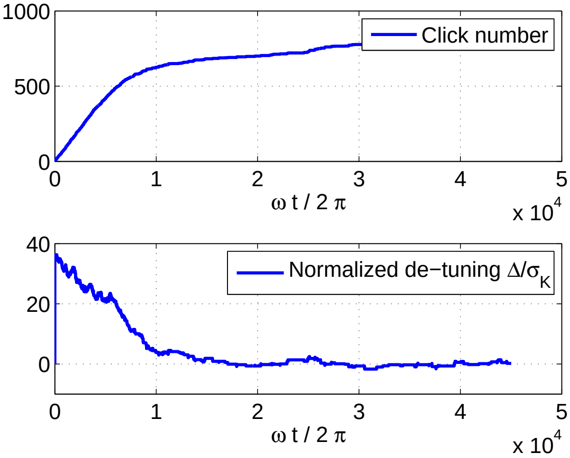
Input: $\tilde{\Omega}_1, \tilde{\Omega}_2 \in \mathbb{C}$ and $u = \frac{d}{dt}\Delta$.
Output: photo-detector click times corresponding to stochastic jumps from $|e\rangle$ to $|g_1\rangle$ or $|g_2\rangle$.
Synchronization goal: stabilize the unknown detuning Δ to 0.
Two time-scales: $|\tilde{\Omega}_1|, |\tilde{\Omega}_2|, |\Delta_e|, |\Delta| \ll \Gamma_1, \Gamma_2$

Modulation of Rabi complex amplitudes $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$:
$\tilde{\Omega}_1(t) = \Omega_1 - \imath\varepsilon\Omega_2 \cos(\omega t)$, $\tilde{\Omega}_2(t) = \imath\varepsilon\Omega_1 \cos(\omega t) + \Omega_2$,
with $\Omega_1, \Omega_2 > 0$ constant, $\omega \ll \Gamma_1, \Gamma_2$ and $0 < \varepsilon \ll 1$.

Detuning update $$\Delta_{N+1} = \Delta_N - K \frac{2\Omega_1\Omega_2}{\Omega_1^2 + \Omega_2^2} \cos(\omega t_N)$$

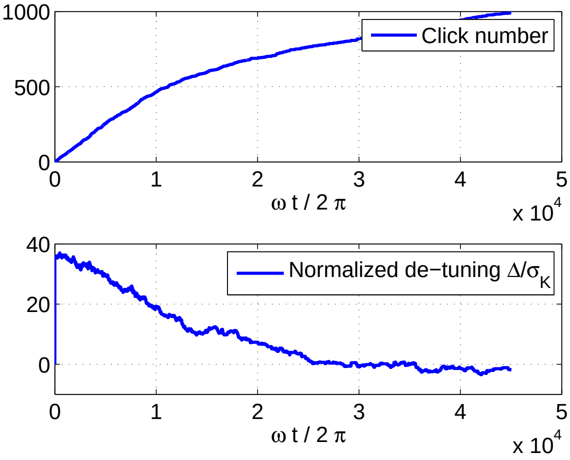
at each detected jump-time t_N. The gain $K > 0$ fixes the standard deviation σ_K: $4\sigma_K^2 = \varepsilon K \frac{\Omega_1^2 + \Omega_2^2}{\Gamma_1 + \Gamma_2}$.

Closed-loop quantum trajectory (matlab M-file: `QCCQI08PR.m`)



Λ-system parameters: $\Gamma_1 = \Gamma_2 = 10$, $\Delta_e = 2.0$
Modulation parameters: $\Omega_1 = \Omega_2 = 1.0$, $\omega = 2.8$, $\varepsilon = 0.14$
Feedback gain $K = 0.0023$ leading to a standard deviation $\sigma_K = 0.0057$

Robustness



Detector efficiency of 50%, wrong jump detection of 50%, feedback-loop delay of τ with $\omega\tau = \pi/4$.

The slow/fast master equation

Master equation of the Λ-system

$$\frac{d}{dt}\rho = -\frac{\imath}{\hbar}[\tilde{H},\rho] + \frac{1}{2}\sum_{j=1}^{2}(2Q_j\rho Q_j^\dagger - Q_j^\dagger Q_j\rho - \rho Q_j^\dagger Q_j),$$

with jump operators $Q_j = \sqrt{\Gamma_j}\,|g_j\rangle\langle e|$ and Hamiltonian

$$\frac{\tilde{H}}{\hbar} = \frac{\Delta}{2}(|g_2\rangle\langle g_2| - |g_1\rangle\langle g_1|) - \left(\Delta_e + \frac{\Delta}{2}\right)|e\rangle\langle e| + \widetilde{\Omega}_1|g_1\rangle\langle e| + \widetilde{\Omega}_1^*|e\rangle\langle g_1| + \widetilde{\Omega}_2|g_2\rangle\langle e| + \widetilde{\Omega}_2^*|e\rangle\langle g_2|.$$

Since $|\tilde{\Omega}_1|, |\tilde{\Omega}_2|, |\Delta_e|, |\Delta| \ll \Gamma_1, \Gamma_2$ we have two time-scales: a fast exponential decay for "$|e\rangle$" and a slow evolution for "$(|g_1\rangle, |g_2\rangle)$".

The slow master equation

Geometric reduction via center manifold techniques[5] leads to a reduced master equation that is still of Lindblad type with a slow Hamiltonian H and slow jump operators L_j:

$$\frac{d}{dt}\rho = -\frac{\imath}{\hbar}[H,\rho] + \frac{1}{2}\sum_{j=1}^{2}(2L_j\rho L_j^\dagger - L_j^\dagger L_j \rho - \rho L_j^\dagger L_j),$$

with $H = \frac{\Delta}{2}\sigma_z = \frac{\Delta(|g_2\rangle\langle g_2| - |g_1\rangle\langle g_1|)}{2}$ and $L_j = \sqrt{\tilde{\gamma}_j}\,|g_j\rangle\langle b_{\tilde{\Omega}}|$ and where $\tilde{\gamma}_j = 4\frac{|\widetilde{\Omega}_1|^2+|\widetilde{\Omega}_2|^2}{(\Gamma_1+\Gamma_2)^2}\Gamma_j$ and $|b_{\tilde{\Omega}}\rangle$ is the bright state:

$$|b_{\tilde{\Omega}}\rangle = \frac{\widetilde{\Omega}_1}{\sqrt{|\widetilde{\Omega}_1|^2 + |\widetilde{\Omega}_2|^2}}|g_1\rangle + \frac{\widetilde{\Omega}_2}{\sqrt{|\widetilde{\Omega}_1|^2 + |\widetilde{\Omega}_2|^2}}|g_2\rangle$$

For $\Delta = 0$, ρ converges towards the dark state $|d_{\tilde{\Omega}}\rangle\langle d_{\tilde{\Omega}}|$:

$$|d_{\tilde{\Omega}}\rangle = -\frac{\widetilde{\Omega}_2^*}{\sqrt{|\widetilde{\Omega}_1|^2 + |\widetilde{\Omega}_2|^2}}|g_1\rangle + \frac{\widetilde{\Omega}_1^*}{\sqrt{|\widetilde{\Omega}_1|^2 + |\widetilde{\Omega}_2|^2}}|g_2\rangle.$$

[5] Mirrahimi-R 2008, arXiv:0801.1602v1, accepted in IEEE-AC.

Quantum trajectories for the slow approximation

In the absence of the quantum jumps, ρ evolves on the Bloch sphere according to ($\tilde{\gamma} = 4\frac{|\widetilde{\Omega}_1|^2+|\widetilde{\Omega}_2|^2}{\Gamma_1+\Gamma_2}$)

$$\frac{1}{\tilde{\gamma}}\frac{d}{dt}\rho = -\imath\frac{\Delta}{2\tilde{\gamma}}[\sigma_z,\rho] - \frac{\left|b_{\widetilde{\Omega}}\right\rangle\left\langle b_{\widetilde{\Omega}}\right|\rho + \rho\left|b_{\widetilde{\Omega}}\right\rangle\left\langle b_{\widetilde{\Omega}}\right|}{2} + \left\langle b_{\widetilde{\Omega}}\right|\rho\left|b_{\widetilde{\Omega}}\right\rangle\rho.$$

At each time step dt, ρ may jump towards the state $|g_1\rangle\langle g_1|$ or $|g_2\rangle\langle g_2|$ with a jump probability given by:

$$dt\ \tilde{\gamma}\ \left\langle b_{\widetilde{\Omega}}\right|\rho\left|b_{\widetilde{\Omega}}\right\rangle$$

Since $\tilde{\Omega}_1(t) = \Omega_1 - \imath\varepsilon\Omega_2\cos(\omega t)$ and $\tilde{\Omega}_2(t) = \imath\varepsilon\Omega_1\cos(\omega t) + \Omega_2$,

$$\tilde{\gamma}\left|b_{\widetilde{\Omega}}\right\rangle\left\langle b_{\widetilde{\Omega}}\right| = \gamma(|b\rangle + \imath\varepsilon\cos(\omega t)|d\rangle)(\langle b| - \imath\varepsilon\cos(\omega t)\langle d|)$$

with $\gamma = 4\frac{|\Omega_1|^2+|\Omega_2|^2}{\Gamma_1+\Gamma_2}$, $|b\rangle = \frac{\Omega_1|g_1\rangle+\Omega_2|g_2\rangle}{\sqrt{\Omega_1^2+\Omega_2^2}}$ and $|d\rangle = \frac{-\Omega_2|g_1\rangle+\Omega_1|g_2\rangle}{\sqrt{\Omega_1^2+\Omega_2^2}}$

Quantum trajectories in Bloch-sphere coordinates

With $\beta = 2\arg(\Omega_1 + \imath\Omega_2)$ and
$\rho = \frac{1+X(|b\rangle\langle d|+|d\rangle\langle b|)+Y(\imath|b\rangle\langle d|-\imath|d\rangle\langle b|)+Z(|d\rangle\langle d|-|b\rangle\langle b|)}{2}$:

$$\frac{d}{dt}X = -\Delta\cos\beta\, Y - \gamma\left(\varepsilon\cos(\omega t)\, Y + \frac{1-\varepsilon^2\cos^2(\omega t)}{2} Z\right) X$$

$$\frac{d}{dt}Y = \Delta\cos\beta\, X - \Delta\sin\beta\, Z + \gamma\varepsilon\cos(\omega t)$$
$$- \gamma\left(\varepsilon\cos(\omega t)\, Y + \frac{1-\varepsilon^2\cos^2(\omega t)}{2} Z\right) Y$$

$$\frac{d}{dt}Z = \Delta\sin\beta\, Y + \gamma\left(\frac{1-\varepsilon^2\cos^2(\omega t)}{2}\right)$$
$$- \gamma\left(\varepsilon\cos(\omega t)\, Y + \frac{1-\varepsilon^2\cos^2(\omega t)}{2} Z\right) Z$$

The jump probability per unit of time is

$$P_{jump} = \frac{\gamma}{2}(1 - Z - 2\varepsilon\cos(\omega t)\, Y + \varepsilon^2\cos^2(\omega t)(1+Z)).$$

Just after a jump (X, Y, Z) is reset to $\pm(\sin\beta, 0, \cos\beta)$.

Convergence of the no-jump dynamics

$$\frac{d}{dt}X = -\Delta\cos\beta\, Y - \gamma\left(\varepsilon\cos(\omega t)Y + \frac{1-\varepsilon^2\cos^2(\omega t)}{2}Z\right)X$$
$$\frac{d}{dt}Y = \Delta\cos\beta\, X - \Delta\sin\beta\, Z + \gamma\varepsilon\cos(\omega t) - \gamma\left(\varepsilon\cos(\omega t)Y + \frac{1-\varepsilon^2\cos^2(\omega t)}{2}Z\right)Y$$
$$\frac{d}{dt}Z = \Delta\sin\beta\, Y + \gamma\left(\frac{1-\varepsilon^2\cos^2(\omega t)}{2}\right) - \gamma\left(\varepsilon\cos(\omega t)Y + \frac{1-\varepsilon^2\cos^2(\omega t)}{2}Z\right)Z$$

For $|\Delta| < \frac{\gamma}{2}$ and $0 < \varepsilon \ll 1$, the above time-periodic nonlinear system admits a quasi-global asymptotically stable periodic orbit (proof: Poincaré-Bendixon and perturbation). It reads

$$(X, Y, Z) = \left(0 \quad , \quad -\sin\beta\frac{\Delta}{\gamma} + \frac{\gamma^2\cos(\omega t) + \gamma\omega\sin(\omega t)}{\omega^2+\gamma^2}\varepsilon \quad , \quad 1\right)$$

up to second order terms in ε and $\frac{\Delta}{\gamma}$.

When $\omega \gg \gamma$, $P_{jump} \approx \gamma\left(\varepsilon\cos(\omega t) + \frac{\Delta\sin\beta}{2\gamma}\right)^2$ if the last jump occurs more that few $-\log\varepsilon/\gamma$ second(s) ago.[6].

[6] Replace Z by $1 - \frac{X^2+Y^2}{2}$ in previous formula giving P_{jump}.

Detuning update as a discrete-time stochastic process

Our analysis neglects the transient just after a jump.
When a jump occurs at t_N, we have

$$\Delta_{N+1} = \Delta_N - K \sin\beta \cos(\omega t_N)$$

and its probability was proportional to $\left(\varepsilon \cos(\omega t_N) + \frac{\Delta_N \sin\beta}{2\gamma}\right)^2$.
The phase $\varpi = \omega t_N$ can be seen as a stochastic variable in $[0, 2\pi]$ with the following probability density $P_{\Delta_N}(\varpi)$ on $[0, 2\pi]$:

$$P_{\Delta_N}(\varpi) = \frac{\left(\varepsilon \cos(\varpi) + \frac{\Delta_N \sin\beta}{2\gamma}\right)^2}{2\pi \left(\frac{\varepsilon^2}{2} + \frac{\Delta_N^2 \sin^2\beta}{4\gamma^2}\right)}$$

The de-tuning update is thus a discrete-time stochastic process

$$\Delta_{N+1} = \Delta_N - K \sin\beta \cos\varpi$$

where the probability of $\varpi \in [0, 2\pi]$ depends on Δ_N.

Convergence proof

We assume here $|\Delta| \ll \varepsilon\gamma$ (remember $\gamma \ll \omega \ll \Gamma_1 + \Gamma_2$):

$$\Delta_{N+1} = \Delta_N - K \sin\beta \cos\varpi$$

with ϖ of probability density $P_{\Delta_N}(\varpi) \approx \frac{1}{2\pi} + \frac{\Delta_N \sin\beta}{\pi\varepsilon\gamma} \cos\varpi$.

Simple computations yield to[7]

$$E(\Delta_{N+1}|\Delta_N) = \left(1 - \frac{K\sin^2\beta}{\varepsilon\gamma}\right)\Delta_N$$

For $0 < K \leq \frac{\varepsilon\gamma}{\sin^2\beta}$, $E(\Delta_N)$ tends to zero.

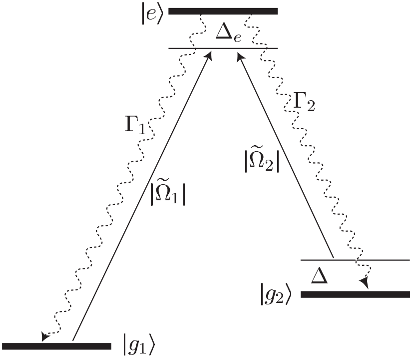
Similarly, we have

$$E(\Delta_{N+1}^2|\Delta_N) = \left(1 - \frac{2K\sin^2\beta}{\varepsilon\gamma}\right)\Delta_N^2 + \frac{K^2\sin^2\beta}{2}$$

For $0 < K \leq \frac{\varepsilon\gamma}{2\sin^2\beta}$, $E(\Delta_N^2)$ converges to $\sigma_K^2 = \frac{\varepsilon\gamma K}{4}$.

[7] $E(\Delta_{N+1}|\Delta_N)$ stands for the conditional expectation-value of Δ_{N+1} knowing Δ_N.

Summary: scales and feedback-gain design



Rabi frequency modulations:

$\tilde{\Omega}_1(t) = \Omega_1 - \imath\varepsilon\Omega_2\cos(\omega t)$

$\tilde{\Omega}_2(t) = \imath\varepsilon\Omega_1\cos(\omega t) + \Omega_2$

with $\Omega_1, \Omega_2 \ll \Gamma = \Gamma_1 + \Gamma_2$,

$0 < \varepsilon \ll 1$ and

$\frac{\Omega_1^2+\Omega_2^2}{\Gamma_1+\Gamma_2} = \gamma \ll \omega \ll \Gamma$

Detuning update

$\Delta_{N+1} = \Delta_N - K\sin\beta\cos(\omega t_N)$

with $K > 0$, $\beta = 2\arg(\Omega_1 + \imath\Omega_2)$.

A discrete-time stochastic process where the gain $K > 0$ drives

- the convergence speed with a contraction of $\left(1 - \frac{K\sin^2\beta}{\varepsilon\gamma}\right)$ for $E(\Delta_N)$ at each iteration
- the precision via the asymptotic root-mean-square $\sigma_K = \frac{\sqrt{\varepsilon\gamma K}}{2}$.

Concluding remarks

- For a nonlinear convergence proof with $\Delta < \gamma/2$, ε small enough and well tuned gain K, see Mirrahimi-R 2008, arxiv:0806.1392v1. Sensitivity analysis to wrong jump detection and noise remains to be done.
- Such simple feedback can be also developed for other single quantum systems such as the 3-level system illustrating the Dehmelt's electron shelving scheme[8]
- Such feedback scheme could be a preliminary guide for inventing the "quantum regulator", a quantum analogue of the classical PID regulator.

[8] C. Cohen-Tannoudji, J. Dalibard: Single atom Laser spectroscopy: looking for dark periods in fluorescent light. Europhys. Lett. 1 (9), pp:441-448, 1986.