Singular perturbations and Lindblad-Kossakowski differential equations

Pierre Rouchon
In collaboration with Mazyar Mirrahimi

Mines ParisTech
Centre Automatique et Systèmes
Mathématiques et Systèmes
pierre.rouchon@mines-paristech.fr
http://cas.ensmp.fr/~rouchon/index.html

IMA, Minneapolis
March 2-6, 2009
Outline

The main result on Λ-systems

Optical pumping and coherence population trapping

Extension to V-systems

Proof of the main result

Concluding remarks
Open quantum systems

The Lindbald-Kossakowski equation

\[
\frac{d}{dt} \rho = -\frac{i}{\hbar} [H, \rho] + \sum_{k=1}^{N} \frac{1}{2} \left( 2L_k \rho L_k^\dagger - L_k^\dagger L_k \rho - \rho L_k^\dagger L_k \right)
\]

is the master equation associated to an ensemble average of quantum trajectories (stochastic jump dynamics of a single quantum system where the "environment is watching")\(^1\).

**Contribution:** when the Lindblad operators \( L_k \) are associated to highly unstable excited states, we propose a systematic method to eliminate the resulting fast and asymptotically stable dynamics. The obtained slow dynamics

\[
\frac{d}{dt} \rho_s = -\frac{i}{\hbar} [H_s, \rho_s] + \sum_{k=1}^{N} \frac{1}{2} \left( 2L_{s,k} \rho_s L_{s,k}^\dagger - L_{s,k}^\dagger L_{s,k} \rho_s - \rho_s L_{s,k}^\dagger L_{s,k} \right)
\]

is still of Lindbald-Kossakowski form ((\( \rho_s, H_s, L_{s,k} \)) = fnct(\( \rho, H, L_k \))).

Prototype of open quantum system: Λ-systems.

$N$ stable states $|g_k\rangle$, $k = 1, \ldots, N$.

Unstable state $|e\rangle$

Quasi resonant laser transition, $|g_k\rangle \leftrightarrow |e\rangle$ with de-tuning $\delta_k$ and Rabi pulsations $\Omega_k \in \mathbb{C}$.

Spontaneous emission rate $|e\rangle \leftrightarrow |g_k\rangle$: $\Gamma_k$.

Lindbald-Kossakowski master-equation for the density matrix $\rho$

$$\frac{d}{dt} \rho = -\frac{i}{\hbar} [H, \rho] + \sum_{k=1}^{N} \frac{1}{2} (2L_k \rho L_k^\dagger - L_k^\dagger L_k \rho - \rho L_k^\dagger L_k)$$

$$\frac{H}{\hbar} = \sum_{k=1}^{N} \delta_k |g_k\rangle \langle g_k| + \Omega_k |g_k\rangle \langle e| + \Omega_k^* |e\rangle \langle g_k|,$$

$L_k = \sqrt{\Gamma_k} |g_k\rangle \langle e|$. Photon flux (measure): $y = \sum_{k=1}^{N} \text{tr} \left( L_k^\dagger L_k \rho \right)$.

Two time-scales: $|\delta_k|, |\Omega_k| \ll \Gamma_k$. 
Main result: adiabatic elimination of the unstable state $|e\rangle$\(^2\)

The slow/fast dynamics

$$
\frac{d}{dt}\rho = -\frac{i}{\hbar}[H, \rho] + \sum_{k=1}^{N} \frac{1}{2} \left( 2L_k\rho L_k^\dagger - L_k^\dagger L_k\rho - \rho L_k^\dagger L_k \right)
$$

with $L_k = \sqrt{\Gamma_k} |g_k\rangle \langle e|$, $\Gamma = (\sum_k \Gamma_k)$ much larger than $\frac{H}{\hbar}$,

$y = \sum_k \text{tr} \left( L_k^\dagger L_k\rho \right)$, is approximated by the slow dynamics

$$
\frac{d}{dt}\rho_s = -\frac{i}{\hbar}[H_s, \rho_s] + \sum_{k=1}^{N} \frac{1}{2} \left( 2L_{s,k}\rho_s L_{s,k}^\dagger - L_{s,k}^\dagger L_{s,k}\rho_s - \rho_s L_{s,k}^\dagger L_{s,k} \right)
$$

with $\rho_s = (1 - P)\rho(1 - P)$ the slow density operator, $H_s = (1 - P)H(1 - P)$ the slow Hamiltonian and $L_{s,k} = 2\frac{L_k}{\Gamma} \frac{H}{\hbar} (1 - P)$ the slow jump operators ($P = |e\rangle \langle e|$). The slow approximation of $y$ is still given by the standard formula

$$
y_s = \sum_{k=1}^{n} \text{tr} \left( L_{s,k}^\dagger L_{s,k}\rho_s \right).
$$

Application to the 3-level system (coherence population trapping\(^3\))

Input: \(\Omega_1, \Omega_2 \in \mathbb{C}\) and \(\frac{d}{dt} \Delta\)

Output: photo-detector click times corresponding to jumps from \(|e\rangle\) to \(|g_1\rangle\) or \(|g_2\rangle\).

Two time-scales: \(|\Omega_1|, |\Omega_2|, |\Delta_e|, |\Delta| \ll \Gamma_1, \Gamma_2\)

The slow/fast master equation

Master equation of the $\Lambda$-system

$$\frac{d}{dt} \rho = -\frac{i}{\hbar} [H, \rho] + \frac{1}{2} \sum_{k=1}^{2} (2L_k \rho L_k^\dagger - L_k^\dagger L_k \rho - \rho L_k^\dagger L_k),$$

with jump operators $L_k = \sqrt{\Gamma_k} |g_k\rangle \langle e|$ and Hamiltonian

$$\frac{H}{\hbar} = \frac{\Delta}{2} (|g_2\rangle \langle g_2| - |g_1\rangle \langle g_1|) - \left( \Delta_e + \frac{\Delta}{2} \right) |e\rangle \langle e|$$

$$+ \Omega_1 |g_1\rangle \langle e| + \Omega_1^* |e\rangle \langle g_1| + \Omega_2 |g_2\rangle \langle e| + \Omega_2^* |e\rangle \langle g_2|.$$

Since $|\Omega_1|, |\Omega_2|, |\Delta_e|, |\Delta| \ll \Gamma_1, \Gamma_2$ we have two time-scales: a fast exponential decay for "$|e\rangle" and a slow evolution for "$(|g_1\rangle, |g_2\rangle)$".
The slow master equation with bright and dark states.

The above general result leads to a reduced master equation that is still of Lindblad type with a slow Hamiltonian $H_s$ and slow jump operators $L_{s,k}$:

$$\frac{d}{dt}\rho_s = -\frac{i}{\hbar}[H_s, \rho_s] + \frac{1}{2}\sum_{k=1}^{2}\left(2L_{s,k}\rho_s L_{s,k}^\dagger - L_{s,k}^\dagger L_{s,k}\rho_s - \rho_s L_{s,k}^\dagger L_{s,k}\right),$$

with $H_s = \frac{\Delta}{2}\sigma_z = \frac{\Delta}{2}(|g_2\rangle\langle g_2| - |g_1\rangle\langle g_1|)$, $L_{s,k} = \sqrt{\gamma_k}|g_k\rangle\langle b_\Omega|$ where

$$\gamma_k = 4\frac{|\Omega_1|^2 + |\Omega_2|^2}{(\Gamma_1 + \Gamma_2)^2}\Gamma_k$$

and $|b_\Omega\rangle$ is the bright state:

$$|b_\Omega\rangle = \frac{\Omega_1}{\sqrt{|\Omega_1|^2 + |\Omega_2|^2}}|g_1\rangle + \frac{\Omega_2}{\sqrt{|\Omega_1|^2 + |\Omega_2|^2}}|g_2\rangle.$$

For $\Delta = 0$, $\rho_s$ converges towards the dark state $|d_\Omega\rangle$:

$$|d_\Omega\rangle = -\frac{\Omega_2^*}{\sqrt{|\Omega_1|^2 + |\Omega_2|^2}}|g_1\rangle + \frac{\Omega_1^*}{\sqrt{|\Omega_1|^2 + |\Omega_2|^2}}|g_2\rangle.$$
Extension to V-systems: Dehmelt’s electron shelving scheme

A stable state $|g_1\rangle$. A quasi-stable state: $|g_2\rangle$ with a long life time $1/\gamma$.
An unstable state: $|e\rangle$ with a short life time $1/\Gamma$.

Quasi resonant transitions:

- $|g_1\rangle \leftrightarrow |e\rangle$ with de-tuning $\Delta$ and Rabi pulsation $\Omega \in \mathbb{C}$.
- $|g_1\rangle \leftrightarrow |g_2\rangle$ with de-tuning $\delta$ and Rabi pulsation $\omega \in \mathbb{C}$.

Lindbald-Kossakowski master-equation for the density matrix $\rho$

$$\frac{d}{dt} \rho = -\frac{i}{\hbar} [H, \rho] + \frac{1}{2} (2L\rho L^\dagger - L^\dagger L\rho - \rho L^\dagger L) + \frac{1}{2} (2l\rho l^\dagger - l^\dagger l\rho - \rho l^\dagger l)$$

$$H = \frac{\Delta}{\hbar} |e\rangle \langle e| + \Omega |g_1\rangle \langle e| + \Omega^* |e\rangle \langle g_1| + \delta |g_2\rangle \langle g_2| + \omega |g_1\rangle \langle g_2| + \omega^* |g_2\rangle \langle g_1|$$

$$L = \sqrt{\Gamma} |g_1\rangle \langle e|, \quad l = \sqrt{\gamma} |g_1\rangle \langle g_2|$$

Photon flux (measure): $y = \text{tr} (L^\dagger L\rho) + \text{tr} (l^\dagger l\rho)$. ($|\delta|, |\omega|, |\Omega|, \gamma \ll \Gamma$).

---

The slow master equation

The slow dynamics is still of Lindblad type with a slow Hamiltonian $H_s$, slow jump operators $L_s$ and $l_s = l$:

$$
\frac{d}{dt}\rho_s = -\frac{i}{\hbar}[H_s, \rho_s] + \frac{1}{2} (2L_s\rho_s L_s^\dagger - L_s^\dagger L_s \rho_s - \rho_s L_s L_s^\dagger) \\
+ \frac{1}{2} (2l_s \rho_s l_s^\dagger - l_s^\dagger l_s \rho_s - \rho_s l_s l_s^\dagger)
$$

$$
\frac{H_s}{\hbar} = \delta |g_2\rangle \langle g_2| + \omega |g_1\rangle \langle g_2| + \omega^* |g_2\rangle \langle g_1|
$$

$$
L_s = 2 \sqrt{\frac{|\Omega|^2}{\Gamma}} |g_1\rangle \langle g_1|, \quad l_s = l = \sqrt{\gamma} |g_1\rangle \langle g_2|
$$

Photon flux (measure): $y = \text{tr} \left( L_s^\dagger L_s \rho_s \right) + \text{tr} \left( l_s^\dagger l_s \rho_s \right)$. 
Slow/fast systems in Tikhonov normal form

\[ \begin{cases} \frac{dx}{dt} = f(x, z, \varepsilon) \\ \varepsilon \frac{dz}{dt} = g(x, z, \varepsilon) \end{cases} \]

with \( x \in \mathbb{R}^n, z \in \mathbb{R}^p, 0 < \varepsilon \ll 1 \) a small parameter, \( f \) and \( g \) regular functions.
Slow approximation (zero order in $\varepsilon$)

As soon as $g(x, z, 0) = 0$ admits a solution, $z = \rho(x)$, with $\rho$ smooth function of $x$ and $\frac{\partial g}{\partial z}(x, \rho(x), 0)$ is a stable matrix, the approximation of

$$
(\Sigma^\varepsilon) \begin{cases}
\frac{dx}{dt} = f(x, z, \varepsilon) \\
\frac{dz}{dt} = g(x, z, \varepsilon)
\end{cases}
$$

by

$$
(\Sigma^0) \begin{cases}
\frac{dx}{dt} = f(x, z, 0) \\
0 = g(x, z, 0)
\end{cases}
$$

is valid for time intervals of length 1. For longer intervals of length $1/\varepsilon$, correction terms of order 1 in $\varepsilon$ should be included in $\Sigma^0$. They can be computed via center manifold techniques and Carr’s approximation lemma\(^5\).

Proof based on matrix computations only \(^6\)

With \(Q_k = |g_k\rangle\langle e|\), \(\Gamma_k = \frac{\Gamma_k}{\varepsilon}\) and \(0 < \varepsilon \ll 1\) the slow/fast master equation reads

\[
\frac{d}{dt} \rho = -\frac{i}{\hbar} [H, \rho] + \sum_{k=1}^{N} \frac{\bar{\Gamma}_k}{2\varepsilon} (2Q_k \rho Q_k^\dagger - Q_k^\dagger Q_k \rho - \rho Q_k^\dagger Q_k).
\]

Change of variables \(\rho \mapsto (\rho_f, \rho_s)\) to put the system in Tikhonov normal form \((P = |e\rangle\langle e|)\): \(\rho_f = P \rho + \rho P - P \rho P\),

\[
\rho_s = (1 - P) \rho (1 - P) + \frac{1}{\left(\sum_{k=1}^{N} \bar{\Gamma}_k\right)} \sum_{k=1}^{N} \bar{\Gamma}_k Q_k \rho Q_k^\dagger,
\]

with inverse \(\rho = \rho_s + \rho_f - \frac{1}{\left(\sum_{k=1}^{N} \bar{\Gamma}_k\right)} \sum_{k=1}^{N} \bar{\Gamma}_k Q_k \rho_f Q_k^\dagger\).

The dynamics in \((\rho_s, \rho_f)\) "Tikhonov coordinates":

\[
\frac{d}{dt} \rho_s = (1 - P) \left[ -\frac{\varepsilon H}{\hbar}, \rho \right] (1 - P) + \frac{1}{\left(\sum_{k=1}^{N} \bar{\Gamma}_k\right)} \sum_{k=1}^{N} \bar{\Gamma}_k Q_k \left[ -\frac{\varepsilon H}{\hbar}, \rho \right] Q_k^\dagger
\]

\[
\varepsilon \frac{d}{dt} \rho_f = -\frac{1}{2} \left(\sum_{k=1}^{N} \bar{\Gamma}_k\right) (\rho_f + P \rho_f P) - \frac{\varepsilon}{\hbar} (P[H, \rho] + [H, \rho] P - P[H, \rho] P).
\]

Order zero approximation in $\varepsilon$

- Setting $\varepsilon$ to 0 in

\[
\frac{d}{dt} \rho_s = (1 - P) \left[ \frac{-iH}{\hbar}, \rho \right] (1 - P) + \frac{1}{\left( \sum_{k=1}^{N} \bar{\Gamma}_k \right)} \sum_{k=1}^{N} \bar{\Gamma}_k Q_k \left[ \frac{-iH}{\hbar}, \rho \right] Q_k^\dagger
\]

\[
\varepsilon \frac{d}{dt} \rho_f = -\frac{\left( \sum_{k=1}^{N} \bar{\Gamma}_k \right)}{2} (\rho_f + P \rho_f P) - \frac{\varepsilon i}{\hbar} (P[H, \rho] + [H, \rho]P - P[H, \rho]P).
\]

yields to the coherent dynamics

\[
\frac{i\hbar}{\hbar} \frac{d}{dt} \rho_s = \left[ (1 - P)H(1 - P), \rho_s \right]
\]

$\rho_f = 0$

with $y = 0$.

- Need for higher order corrections terms in $\varepsilon$
High order approximation via center manifold techniques

Consider the slow/fast system (\(f\) and \(g\) are regular functions)

\[
\frac{d}{dt} x = f(x, z), \quad \varepsilon \frac{d}{dt} z = -Az + \varepsilon g(x, z)
\]

where all the eigenvalues of the matrix \(A\) have strictly positive real parts, and \(0 < \varepsilon \ll 1\). The slow invariant attractive manifold admits for equation (boundary layer)

\[
z = \varepsilon A^{-1} g(x, 0) + O(\varepsilon^2)
\]

and the restriction of the dynamics on this slow invariant manifold reads

\[
\frac{d}{dt} x = f(x, \varepsilon A^{-1} g(x, 0)) + O(\varepsilon^2) = f(x, 0) + \varepsilon \left. \frac{\partial f}{\partial z} \right|_{(x,0)} A^{-1} g(x, 0) + O(\varepsilon^2).
\]

Center-manifold approximations yield to second order terms in the expansion of \(z\):

\[
z = \varepsilon A^{-1} g(x, 0) + \varepsilon^2 A^{-1} \left( \left. \frac{\partial g}{\partial z} \right|_{(x,0)} A^{-1} g(x, 0) - A^{-1} \left. \frac{\partial g}{\partial x} \right|_{(x,0)} f(x, 0) \right) + O(\varepsilon^3).
\]

Order one approximation in $\varepsilon$

Addition of first order correction terms in $\varepsilon$ are related to decoherence and thus to Lindblad terms:

$$\frac{d}{dt}\rho_s = -\frac{i}{\hbar}[H_s, \rho_s] + 2\varepsilon \sum_{k=1}^{N} \Gamma_k \left( 2Q_{s,k}\rho_s Q_{s,k}^\dagger - Q_{s,k}^\dagger Q_{s,k}\rho_s - \rho_s Q_{s,k}^\dagger Q_{s,k} \right)$$

where

$$H_s = (1-P)H(1-P) \quad \text{and} \quad Q_{s,k} = \frac{1}{\hbar \left( \sum_{l=1}^{N} \Gamma_l \right)}(1-P)Q_k H(1-P).$$

The boundary layer reads

$$\rho_f = \frac{-2i \varepsilon}{\hbar \left( \sum_{k=1}^{N} \Gamma_k \right)} (PH\rho_s - \rho_s HP) + O(\varepsilon^2).$$

and the output (measure)

$$y(t) = 4\varepsilon \left( \sum_{k=1}^{N} \Gamma_k \right) \text{tr} (P \rho_s) + O(\varepsilon^2),$$
Concluding remarks

- The proposed adiabatic reduction mixing non commutative computations with operators and dynamical systems theory (geometric singular perturbations theory, invariant manifold) preserves the "physics" (CPT slow dynamics).
- In the slow master equation, the decoherence terms depend on the control input $\Omega_k$: influence on controllability and optimal control?\(^8\)
- Straightforward extensions to: several unstable states $|e_r\rangle$ with fast relaxation to stable states $|g_k\rangle$; slow decoherence between the "stable" states $|g_k\rangle$.
- A method to approximate slow/fast quantum trajectories by slow quantum trajectories where the jumps from $|e\rangle$ to $|g_k\rangle$ are replaced by jumps inside the "slow space"\(^9\)

---

\(^8\)See, e.g., Altafini and Bonnard-Chyba-Sugny for the recent results on controllability and optimal control of such dissipative systems.

Quantum trajectories\textsuperscript{10} associated to the slow master equation

Set $\gamma_k = 4 \frac{|\Omega_1|^2 + |\Omega_2|^2}{(\Gamma_1 + \Gamma_2)^2} \Gamma_k$ for $k = 1, 2$.

At each infinitesimal time step $dt$,

- $\rho_s$ jumps
  - towards the state $|g_1\rangle \langle g_1|$ with a jump probability given by: $\langle b_\Omega | \rho_s | b_\Omega \rangle \gamma_1 dt$.
  - or towards the state $|g_2\rangle \langle g_2|$ with a jump probability given by: $\langle b_\Omega | \rho_s | b_\Omega \rangle \gamma_2 dt$.
  - or $\rho_s$ does not jump with probability

$$1 - \langle b_\Omega | \rho_s | b_\Omega \rangle (\gamma_1 + \gamma_2) dt$$

and then evolves on the Bloch sphere according to

$$\frac{1}{\gamma} \frac{d}{dt} \rho_s = -\frac{i}{2\gamma} [\sigma_z, \rho_s] - \frac{\langle b_\Omega | \rho_s + \rho_s | b_\Omega \rangle \langle b_\Omega \rangle}{2} + \langle b_\Omega | \rho_s | b_\Omega \rangle \rho_s.$$ 

with $\gamma = \gamma_1 + \gamma_2$.

Quantum trajectories in Bloch-sphere coordinates

Set $\beta = 2 \arg(\Omega_1 + i\Omega_2)$ and

$$\rho_s = \frac{1}{2} + X(|b\rangle\langle d| + |d\rangle\langle b|) + Y(i|b\rangle\langle d| - i|d\rangle\langle b|) + Z(|d\rangle\langle d| - |b\rangle\langle b|).$$

At each infinitesimal time step $dt$, the point $(X, Y, Z) \in S^2$,

- jumps
  - towards the state $(\sin \beta, 0, \cos \beta)$ with a jump probability given by: $\frac{(1-Z)^2}{2} \gamma_1 dt$.
  - or towards the state $(-\sin \beta, 0, -\cos \beta)$ with a jump probability given by: $\frac{(1-Z)^2}{2} \gamma_2 dt$.
  - or does not jump with probability $1 - \frac{(1-Z)^2}{2} (\gamma_1 + \gamma_2) dt$ and then evolves according to

$$\frac{d}{dt} X = -\Delta \cos \beta Y - \gamma \frac{XZ}{2}$$

$$\frac{d}{dt} Y = \Delta \cos \beta X - \Delta \sin \beta Z - \gamma \frac{YZ}{2}$$

$$\frac{d}{dt} Z = \Delta \sin \beta Y + \frac{\gamma(1 - Z^2)}{2}$$
Convergence of the no-jump dynamics

For $|\Delta| < \frac{\gamma}{2}$, the nonlinear system in $S^2$

\[
\begin{align*}
\frac{d}{dt} X &= -\Delta \cos \beta Y - \gamma \frac{XZ}{2} \\
\frac{d}{dt} Y &= \Delta \cos \beta X - \Delta \sin \beta Z - \gamma \frac{YZ}{2} \\
\frac{d}{dt} Z &= \Delta \sin \beta Y + \frac{\gamma(1 - Z^2)}{2}
\end{align*}
\]

admits a two equilibrium points, one is unstable and the other one is quasi-global asymptotically stable.

Proof: based on Poincaré-Bendixon on the sphere.\textsuperscript{11}

\textsuperscript{11}See, Mirrahimi-R, 2008, arxiv:0806.1392v1