Euler-Lagrange models with complex currents of three-phase electrical machines and observability issues

Pierre Rouchon, in collaboration with Duro Basic, François Malrait from Schneider Electric, STIE.

Mines ParisTech
Centre Automatique et Systèmes
Mathématiques et Systèmes
pierre.rouchon@mines-paristech.fr
http://cas.ensmp.fr/~rouchon/index.html

Electrical and Mechatronical Systems Workshop
Bernoulli Center, Lausanne
18-20 February 2009

1Preprint arXiv:0806.0387v3 [math.OC]
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PM machines: usual models

In the \((\alpha, \beta)\) frame the dynamic equations read\(^2\):

\[
\begin{align*}
\frac{d}{dt} \left( J\dot{\theta} \right) &= n_p \Re \left( \left( \bar{\phi} e^{jnp\theta} \right)^* \cdot s \right) - \tau_L \\
\frac{d}{dt} \left( \lambda \cdot s + \bar{\phi} e^{jnp\theta} \right) &= u_s - R_s \cdot s
\end{align*}
\]

where

- \(^*\) stands for complex-conjugation, \(j = \sqrt{-1}\) and \(n_p\) is the number of pairs of poles.
- \(\theta\) is the rotor mechanical angle, \(J\) and \(\tau_L\) are the inertia and load torque, respectively.
- \(\cdot s = \cdot s_{\alpha} + j \cdot s_{\beta}\) (resp \(u_s = u_{s\alpha} + j u_{s\beta}\)) is the stator current (resp. voltage): complex quantities.
- \(\lambda = (L_d + L_q)/2\) with inductances \(L_d = L_q > 0\) (no saliency here).
- The stator flux is \(\phi_s = \lambda \cdot s + \bar{\phi} e^{jnp\theta}\) with the constant \(\bar{\phi} > 0\) representing to the rotor flux due to permanent magnets.

PM machines: Euler-Lagrange setting

Lagrangian: sum of kinetic and magnetic Lagrangian $\mathcal{L}_c + \mathcal{L}_m$:

$$
\mathcal{L}_c = \frac{J}{2} \dot{\theta}^2, \quad \mathcal{L}_m = \frac{\lambda}{2} |\iota_s + \bar{i} e^{in_p\theta}|^2
$$

where $\bar{i} = \bar{\phi} / \lambda > 0$ is the permanent magnetizing current.

Euler-Lagrange setting: with additional variable $q_s \in \mathbb{C}$ defined by $\frac{d}{dt} q_s = \iota_s$, take the Lagrangian $\mathcal{L} = \mathcal{L}_c + \mathcal{L}_m$ as a real function of $q = (\theta, q_{s\alpha}, q_{s\beta})$ and $\dot{q} = (\dot{\theta}, \iota_{s\alpha}, \iota_{s\beta})$:

$$
\mathcal{L}(q, \dot{q}) = \frac{J}{2} \dot{\theta}^2 + \frac{\lambda}{2} \left( (\iota_{s\alpha} + \bar{i} \cos n_p \theta)^2 + (\iota_{s\beta} + \bar{i} \sin n_p \theta)^2 \right)
$$

Then the dynamics (3 real ODE) read:

$$
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = -\tau_L
$$

$$
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial q_{s\alpha}} \right) - \frac{\partial \mathcal{L}}{\partial q_{s\alpha}} = u_{s\alpha} - R_s \iota_{s\alpha}, \quad \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial q_{s\beta}} \right) - \frac{\partial \mathcal{L}}{\partial q_{s\beta}} = u_{s\beta} - R_s \iota_{s\beta}
$$

---

Euler-Lagrange equation with complex variables

Two generalized coordinates $q_1$ and $q_2$ correspond to a point $q = q_1 + jq_2$ in the complex plane ($j = \sqrt{-1}$). The Lagrangian $\mathcal{L}(q_1, q_2, \dot{q}_1, \dot{q}_2)$ is a real function and the Euler-Lagrange equations are

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_1} \right) - \frac{\partial \mathcal{L}}{\partial q_1} = 0, \quad \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_2} \right) - \frac{\partial \mathcal{L}}{\partial q_2} = 0.$$ 

Using the complex notation $q$

$$\tilde{\mathcal{L}}(q, q^*, \dot{q}, \dot{q}^*) \equiv \mathcal{L} \left( \frac{q + q^*}{2}, \frac{q - q^*}{2j}, \frac{\dot{q} + \dot{q}^*}{2}, \frac{\dot{q} - \dot{q}^*}{2j} \right).$$

Since $2 \frac{\partial \tilde{\mathcal{L}}}{\partial q} = \frac{\partial \mathcal{L}}{\partial q_1} - j \frac{\partial \mathcal{L}}{\partial q_2}$, $2 \frac{\partial \tilde{\mathcal{L}}}{\partial q^*} = \frac{\partial \mathcal{L}}{\partial q_1} + j \frac{\partial \mathcal{L}}{\partial q_2}$ we get

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_1} + j \frac{\partial \mathcal{L}}{\partial \dot{q}_2} \right) = \frac{\partial \mathcal{L}}{\partial q_1} + j \frac{\partial \mathcal{L}}{\partial q_2}$$

that reads

$$\frac{d}{dt} \left( 2 \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{q}^*} \right) - 2 \frac{\partial \tilde{\mathcal{L}}}{\partial q^*} = 0.$$
PM machines: Lagrangian with complex stator currents

With \( \frac{d}{dt} q_s = \nu_s \) (\( q_s \) complex cyclic variables) and the Lagrangian

\[
\mathcal{L}(\theta, \dot{\theta}, \nu_s, \nu_s^*) = \frac{J}{2} \dot{\theta}^2 + \frac{\lambda}{2} \left( \nu_s + i e^{jn_p\theta} \right) \left( \nu_s^* + i e^{-jn_p\theta} \right)
\]

the usual equations

\[
\frac{d}{dt} \left( J \dot{\theta} \right) = n_p \Im \left( \left( \lambda i e^{jn_p\theta} \right)^* \nu_s \right) - \tau_L, \quad \frac{d}{dt} \left( \lambda (\nu_s + i e^{jn_p\theta}) \right) = u_s - R_s \nu_s
\]

read

\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) = \frac{\partial \mathcal{L}}{\partial \theta} - \tau_L, \quad 2 \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \nu_s^*} \right) = u_s - R_s \nu_s
\]

since \( \frac{\partial \mathcal{L}}{\partial q_s^*} = 0 \) and \( \frac{\partial \mathcal{L}}{\partial q_s} = \frac{\partial \mathcal{L}}{\partial \nu_s} \).
PM machines: structure of any dynamical models

More generally, the magnetic Lagrangian $\mathcal{L}_m$ is a real value function of $\theta$, $i_s$ and $i_s^\ast$ that is $\frac{2\pi}{n_p}$ periodic versus $\theta$. Thus any Lagrangian $\mathcal{L}_{PM}$ representing a 3-phases permanent magnet machine admits the following form

$$\mathcal{L}_{PM} = \frac{J}{2} \dot{\theta}^2 + \mathcal{L}_m(\theta, i_s, i_s^\ast)$$

Consequently, any model (with saliency, magnetic-saturation, space-harmonics, ...) of permanent magnet machines admits the following structure ($J$ independent of $\theta$ here):

$$\frac{d}{dt} \left( J \dot{\theta} \right) = \frac{\partial \mathcal{L}_m}{\partial \theta} - \tau_L, \quad \frac{d}{dt} \left( 2 \frac{\partial \mathcal{L}_m}{\partial i_s^\ast} \right) = u_s - R_s i_s$$

with $\phi_s = 2 \frac{\partial \mathcal{L}_m}{\partial i_s^\ast}$ as stator flux.
PM machines: saliency effects

With a positive magnetic Lagrangian of the form

\[
\mathcal{L}_m = \frac{\lambda}{2} \left( \dot{\lambda} + \bar{\lambda} e^{jnp\theta} \right) \left( \dot{\lambda}^* + \bar{\lambda} e^{-jnp\theta} \right) - \frac{\mu}{4} \left( (\dot{\lambda} e^{jnp\theta})^2 + (\lambda e^{-jnp\theta})^2 \right)
\]

where \( \lambda = (L_d + L_q)/2 \) and \( \mu = (L_q - L_d)/2 \) (inductances \( L_d > 0 \) and \( L_q > 0 \)), we recover the usual model with saliency:

\[
\begin{align*}
\frac{d}{dt} \left( J \dot{\theta} \right) &= n_{p} \mathcal{S} \left( \left( \lambda \dot{\lambda}^* + \lambda \bar{\lambda} e^{-jnp\theta} - \mu \lambda \bar{\lambda} e^{-2jnp\theta} \right) \lambda \right) - \tau_L \\
\frac{d}{dt} \left( \lambda \lambda + \lambda \bar{\lambda} e^{jnp\theta} - \mu \lambda^* e^{2jnp\theta} \right) &= u_s - R_s \lambda.
\end{align*}
\]
PM machines: magnetic-saturation and saliency effects

Inductances depend on the currents as, e.g.,

\[
\lambda = \lambda(|i_s + \bar{i}e^{jnp\theta}|) = \lambda \left( \sqrt{(i_s + \bar{i}e^{jnp\theta})(i_s^* + \bar{i}e^{-jnp\theta})} \right)
\]

where \(i_s + \bar{i}e^{jnp\theta}\) stands for total magnetizing current. With magnetic Lagrangien

\[
L_m = \frac{\lambda(|i_s + \bar{i}e^{jnp\theta}|)}{2} |i_s + \bar{i}e^{jnp\theta}|^2 - \frac{\mu}{4} \left( (i_s^*e^{jnp\theta})^2 + (i_se^{-jnp\theta})^2 \right)
\]

the dynamics read (\(\Lambda = \lambda + \frac{|i_s+\bar{i}e^{jnp\theta}|}{2}\lambda'\)):

\[
\begin{align*}
\frac{d}{dt} (J\dot{\theta}) &= n_p \mathcal{S} \left( \left( \Lambda \left( i_s^* + \bar{i}e^{-jnp\theta} \right) - \mu i_s e^{-2jnp\theta} \right) i_s \right) - \tau_L \\
\frac{d}{dt} \left( \Lambda \left( i_s + \bar{i}e^{jnp\theta} \right) - \mu i_s^* e^{2jnp\theta} \right) &= u_s - R_s i_s
\end{align*}
\]

Similarly \(\mu\) could also depend on \(|i_s + \bar{i}e^{jnp\theta}|\).
Induction machines: usual models

Dynamics with complex stator and rotor currents:

\[
\begin{align*}
\frac{d}{dt} \left( J_\dot{\theta} \right) &= n_p \mathcal{S} \left( L_m \dot{i}_r e^{-jnp\theta} i_s \right) - \tau_L \\
\frac{d}{dt} \left( L_r \dot{i}_r + L_m i_s e^{-jnp\theta} \right) &= -R_r \dot{i}_r \\
\frac{d}{dt} \left( L_s \dot{i}_s + L_m \dot{i}_r e^{jnp\theta} \right) &= u_s - R_s \dot{i}_s
\end{align*}
\]

where

- \( i_r \in \mathbb{C} \) (resp. \( i_s \in \mathbb{C} \)) is the rotor (resp. stator) current;
- \( u_s \in \mathbb{C} \) is the stator voltage
- \( R_s > 0 \) and \( R_r > 0 \) are stator and rotor resistances.
- \( L_s > 0 \), \( L_r > 0 \) and \( L_m \) are the inductances satisfying \( L_s L_r > L_m^2 \) for physical reasons (positive magnetic Lagrangien). They are constant here.
- the stator (resp. rotor) flux is \( \phi_s = L_s \dot{i}_s + L_m \dot{i}_r e^{jnp\theta} \) (resp. \( \phi_r = L_r \dot{i}_r + L_m \dot{i}_s e^{-jnp\theta} \)).
Induction machines: Lagrangian with complex currents

The Lagrangian of the usual model is

\[ \mathcal{L}_m = \frac{J}{2} \dot{\theta}^2 + \frac{L_m}{2} \left| \nu_s + \nu_r e^{jnp\theta} \right|^2 + \frac{L_{fr}}{2} |\nu_r|^2 + \frac{L_{fs}}{2} |\nu_s|^2 \]

where \( L_s = L_m + L_{fs} \) and \( L_r = L_m + L_{fr} \) with \( L_m > 0 \) and \( 0 < L_{fr}, L_{fs} \ll L_m \). More generally, a physically consistent model should be obtained with a Lagrangian of the form

\[ \mathcal{L}_{\text{im}} = \frac{J}{2} \dot{\theta}^2 + \mathcal{L}_m (\theta, \nu_r, \nu_r^*, \nu_s, \nu_s^*) \]

where \( \mathcal{L}_m \) is the magnetic Lagrangian expressed with the rotor angle and currents. It is \( \frac{2\pi}{np} \) periodic versus \( \theta \). Any physically admissible model reads (\( J \) independent of \( \theta \))

\[ \frac{d}{dt} \left( J \dot{\theta} \right) = \frac{\partial \mathcal{L}_m}{\partial \theta} - \tau_L, \quad \frac{d}{dt} \phi_r = -R_r \nu_r, \quad \frac{d}{dt} \phi_s = u_s - R_s \nu_s, \]

where the rotor and stator fluxes are given by

\[ \phi_r = 2 \frac{\partial \mathcal{L}_m}{\partial \nu_r^*}, \quad \phi_s = 2 \frac{\partial \mathcal{L}_m}{\partial \nu_s^*}. \]
Induction machines: magnetic-saturation

With positive inductances of the form

\[ L_m = L_m \left( \left| i_s + i_r e^{jnp} \right| \right), \quad L_s = L_m + L_{fs}, \quad L_r = L_m + L_{fr} \]

the magnetic Lagrangien remains positive

\[ \mathcal{L}_m = \frac{L_m \left( \left| i_s + i_r e^{jnp} \right| \right)}{2} \left| i_s + i_r e^{jnp} \right|^2 + \frac{L_{fr}}{2} i_r^* i_r + \frac{L_{fs}}{2} i_s^* i_s \]

and the saturation model reads

\[
\begin{cases}
\frac{d}{dt} \left( J \dot{\theta} \right) = n_p \mathcal{S} \left( \Lambda_m i_r^* e^{-jnp} i_s \right) - \tau_L \\
\frac{d}{dt} \left( \Lambda_m \left( i_r + i_s e^{-jnp} \right) + L_{fr} i_r \right) = -R_r i_r \\
\frac{d}{dt} \left( \Lambda_m \left( i_s + i_r e^{jnp} \right) + L_{fs} i_s \right) = u_s - R_s i_s
\end{cases}
\]

with \( \Lambda_m = L_m + \frac{\left| i_s + i_r e^{jnp} \right|}{2} L_m' \) function of \( \left| i_s + i_r e^{jnp} \right| \).
Induction machines: space-harmonics and magnetic-saturation.

Add contribution of space harmonics to magnetic Lagrangian:

\[
L_m \left( \frac{|i_s + i_r e^{jn_p\theta}|}{2} \right)^2 + \frac{L_{fr}}{2} i_r^* + \frac{L_{fs}}{2} i_s i_s^* \\
+ \frac{L_\nu}{2} \left( i_s i_r^* e^{-j\sigma_\nu n_p\theta} + i_s^* i_r e^{j\sigma_\nu n_p\theta} \right)
\]

with \( L_\nu > 0 \) a small parameter (\(|L_\nu| \ll L_m\)) and \( \sigma_\nu = \pm 1 \) depending on arithmetic conditions\(^5\). The dynamical model is changed as follows:

\[
\frac{d}{dt} \left( J \dot{\theta} \right) = n_p \otimes \left( (\Lambda_m e^{-jn_p\theta} + L_\nu \sigma_\nu v e^{-j\sigma_\nu n_p\theta}) i_r^* i_s \right) - \tau_L \\
\frac{d}{dt} \left( \Lambda_m (i_r + i_s e^{-jn_p\theta}) + L_{fr} i_r + L_\nu i_s e^{-j\sigma_\nu n_p\theta} \right) = -R_r i_r \\
\frac{d}{dt} \left( \Lambda_m (i_s + i_r e^{jn_p\theta}) + L_{fs} i_s + L_\nu i_r e^{j\sigma_\nu n_p\theta} \right) = u_s - R_s i_s
\]

Sensorless control of PM machines

Sensorless control: a load torque $\tau_L$ constant but unknown, control inputs $u_s$ and measured outputs $i_s$.\(^6\)

Physical models including saliency and magnetic saturation associated to Lagrangian $L_{\text{PM}} = \frac{1}{2} J \dot{\theta}^2 + L_m(\theta, i_s, i_s^*)$,

$$\frac{d}{dt} \left( J \dot{\theta} \right) = \frac{\partial L_m}{\partial \theta} - \tau_L, \quad \frac{d}{dt} \left( 2 \frac{\partial L_m}{\partial i_s^*} \right) = u_s - R_s i_s$$

can be always written in state-space form

$$\frac{d}{dt} X = f(X, U), \quad Y = h(X)$$

where $X = (\tau_L, \theta, \dot{\theta}, \Re(i_s), \Im(i_s))$ with $U = (\Re(u_s), \Im(u_s))$, $Y = (\Re(i_s), \Im(i_s))$ and $\frac{d}{dt} \tau_L = 0$.

Sensorless control around zero stator frequency

A stationary regime at zero stator frequency corresponds then to a steady state \((\bar{X}, \bar{U}, \bar{Y})\) satisfying \(f(\bar{X}, \bar{U}) = 0, \bar{Y} = h(\bar{X})\). For a PM machines we get

\[
\frac{\partial L_m}{\partial \theta}(\theta, i_s, i_s^*) - \tau_L = 0, \quad i_s = \bar{i}_s
\]

to recover \((\theta, i_s, \tau_L)\) from the stationary values \(\bar{u}_s\) and \(\bar{i}_s\). This implies severe observability difficulties:

- to any constant input and output \(\bar{u}_s\) and \(\bar{i}_s\) satisfying \(\bar{u}_s = R_s \bar{i}_s\) correspond a one dimensional family of steady states parameterized by the scalar variable \(\xi\) with

  \[
  \tau_L = \frac{\partial L_m}{\partial \theta}(\xi, \bar{i}_s, \bar{i}_s^*), \quad \theta = \xi, \quad i_s = \bar{i}_s.
  \]

- the linear tangent systems around such steady-states are not observable;

The situation is similar for induction machines: including space-harmonic and magnetic-saturation does not canceled such lack of observability.
Concluding remarks

- Extensions to network of machines and generators connected via long lines can also be developed with similar variational principles and Euler-Lagrange equations with complex currents and voltages (ODE or PDE).

- Observability issues at zero stator frequency: a strong motivation for theoretical works on the following specific stabilization problem involving an unknown constant parameter $p$: take $\frac{d}{dt} x = f(x, u, p)$, $y = h(x)$ a nonlinear system where $\{(x, p) \mid f(x, \bar{u}, p) = 0, h(x) = \bar{y}\}$ is a smooth curve; take any $(\bar{x}, \bar{p})$ on this equilibrium curve; under which conditions is it possible to construct (without knowing $p$) a (dynamic) output feedback stabilizing $x$ around $\bar{x}$ in a robust way.