Stabilisation par feedback de systèmes quantiques ouverts
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Feedback for quantum systems: the back-action of the measure.

A typical stabilizing feedback-loop for a classical system

Two kinds of stabilizing feedbacks for quantum systems

1. **Measurement-based feedback:** measurement back-action on $S$ is stochastic (collapse of the wave-packet); controller is classical; the control input $u$ is a classical variable appearing in some controlled Schrödinger equation; $u$ depends on the past measures.

2. **Coherent-autonomous feedback and reservoir engineering:** the system $S$ is coupled to another quantum system (the controller); the composite system, $\mathcal{H}_S \otimes \mathcal{H}_{controller}$, is an open-quantum system relaxing to some target (separable) state or decoherence free subspace.
Outline

Feedback stabilization of photons: the LKB photon box
- The closed-loop experiment (2011)
- Quantum stochastic model
- QND measurement and the quantum-state feedback

Dynamical models of open quantum systems
- Discrete-time: Markov process/Kraus maps
- Continuous-time: stochastic/Lindblad master equations

Stabilization of "Schrödinger cats" by reservoir engineering
- The principle
- Discrete-time example: the LKB photon box
- Continuous-time examples and Fokker-Planck equations

Conclusion

Appendix
- Design of a strict control Lyapunov function
- State estimation and stability of quantum filtering
- Schrödinger cats and Wigner functions
- Reservoir engineering stabilization: complements
- Books on open quantum systems
The first experimental realization of a quantum state feedback

The LKB photon box: group of S.Haroche and J.M.Raimond

Stabilization of a quantum state with exactly $n$ photon(s) ($n = 0, 1, 2, 3, \ldots$).


1 Courtesy of I. Dotsenko. Sampling period 80 $\mu$s.
Three quantum features emphasized by the LKB photon box

1. Schrödinger equation: wave function $|\psi\rangle \in \mathcal{H}$, density operator $\rho$

$$\frac{d}{dt} |\psi\rangle = -\frac{i}{\hbar} H |\psi\rangle, \quad \frac{d}{dt} \rho = -\frac{i}{\hbar} [H, \rho], \quad H = H_0 + uH_1$$

2. Origin of dissipation: collapse of the wave packet induced by the measure of observable $O$ with spectral decomposition $\sum_{\mu} \lambda_{\mu} P_{\mu}$:
   - measure outcome $\mu$ with proba. $p_{\mu} = \langle \psi | P_{\mu} |\psi\rangle = \text{Tr}(\rho P_{\mu})$ depending on $|\psi\rangle$, $\rho$ just before the measurement
   - measure back-action if outcome $\mu$:

$$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{P_{\mu}|\psi\rangle}{\sqrt{\langle \psi | P_{\mu} |\psi\rangle}} , \quad \rho \mapsto \rho_+ = \frac{P_{\mu}\rho P_{\mu}}{\text{Tr}(\rho P_{\mu})}$$

3. Tensor product for the description of composite systems $(S, M)$:
   - Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M$
   - Hamiltonian $H = H_S \otimes I_M + H_{\text{int}} + I_S \otimes H_M$
   - observable on sub-system $M$ only: $O = I_S \otimes O_M$.

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The composite system

- **System** $S$ corresponds to a quantized mode in $C$:

  \[ \mathcal{H}_S = \left\{ \sum_{n=0}^{\infty} \psi_n |n\rangle \mid (\psi_n)_{n=0}^{\infty} \in l^2(\mathbb{C}) \right\}, \]

  where $|n\rangle$ represents the Fock state associated to exactly $n$ photons inside the cavity.

- **Meter** $M$ is associated to atoms: $\mathcal{H}_M = \mathbb{C}^2$, each atom admits two energy levels and is described by a wave function $c_g |g\rangle + c_e |e\rangle$ with $|c_g|^2 + |c_e|^2 = 1$; atoms leaving $B$ are all in state $|g\rangle$.

- When atom comes out $B$, the state $|\Psi\rangle_B \in \mathcal{H}_S \otimes \mathcal{H}_M$ of the composite system atom/field is separable:

  \[ |\Psi\rangle_B = |\psi\rangle \otimes |g\rangle. \]
\( S: \) quantum harmonic oscillator

- **Hilbert space:**
  \[ \mathcal{H}_S = \{ \sum_{n \geq 0} \psi_n |n\rangle, \ (\psi_n)_{n \geq 0} \in l^2(\mathbb{C}) \}. \]

- **Quantum state space:**
  \[ \mathcal{D} = \{ \rho \in \mathcal{L}(\mathcal{H}_S), \rho^\dagger = \rho, \ \text{Tr}(\rho) = 1, \rho \geq 0 \} . \]

- **Operators and commutations:**
  \[ a |n\rangle = \sqrt{n} |n-1\rangle, \ a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle; \]
  \[ N = a^\dagger a, \ N |n\rangle = n |n\rangle; \]
  \[ [a, a^\dagger] = I, \ af(N) = f(N + I)a; \]
  \[ D_\alpha = e^{\alpha a^\dagger - \alpha^\dagger a}. \]
  \[ a = X + iP = \frac{1}{\sqrt{2}} (x + \frac{\partial}{\partial x}), \ [X, P] = iI/2. \]

- **Hamiltonian:**
  \[ H_{S/\hbar} = \omega_c a^\dagger a + u_c (a + a^\dagger). \]
  (associated classical dynamics:
  \[ \frac{dx}{dt} = \omega_c p, \ \frac{dp}{dt} = -\omega_c x - \sqrt{2}u_c. \]

- **Coherent state of amplitude**
  \[ \alpha \in \mathbb{C}: \ |\alpha\rangle = \sum_{n \geq 0} \left( e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \right) |n\rangle; \ |\alpha\rangle \equiv \frac{1}{\pi^{1/4}} e^{\sqrt{2}x \Im \alpha} e^{-(x - \sqrt{2} \Re \alpha)^2/2} \]
  \[ a |\alpha\rangle = \alpha |\alpha\rangle, \ D_\alpha |0\rangle = |\alpha\rangle. \]
**M: 2-level system, i.e. a qubit**

- **Hilbert space:**
  \[ \mathcal{H}_M = \mathbb{C}^2 = \{ c_g |g\rangle + c_e |e\rangle, \ c_g, c_e \in \mathbb{C} \}. \]

- **Quantum state space:**
  \[ \mathcal{D} = \{ \rho \in \mathcal{L}(\mathcal{H}_M), \rho^\dagger = \rho, \ \text{Tr}(\rho) = 1, \rho \geq 0 \}. \]

- **Operators and commutations:**
  \[
  \begin{align*}
  \sigma_+ &= |g\rangle \langle e|,
  \sigma_- &= \sigma_+^\dagger = |e\rangle \langle g|,
  \\
  \sigma_x &= \sigma_- + \sigma_+ = |g\rangle \langle e| + |e\rangle \langle g|,
  \\
  \sigma_y &= i \sigma_- - i \sigma_+ = i |g\rangle \langle e| - i |e\rangle \langle g|,
  \\
  \sigma_z &= \sigma_+ \sigma_- - \sigma_- \sigma_+ = |e\rangle \langle e| - |g\rangle \langle g|,
  \\
  \sigma_x^2 &= \mathbb{I},
  \sigma_x \sigma_y = i \sigma_z,
  [\sigma_x, \sigma_y] = 2i \sigma_z,
  \ldots
  \end{align*}
  \]

- **Hamiltonian:**
  \[ \frac{\mathcal{H}_M}{\hbar} = \omega_q \sigma_z / 2 + u_q \sigma_x. \]

- **Bloch sphere representation:**
  \[ \mathcal{D} = \left\{ \frac{1}{2} (\mathbb{I} + x \sigma_x + y \sigma_y + z \sigma_z) \mid (x, y, z) \in \mathbb{R}^3, \ x^2 + y^2 + z^2 \leq 1 \right\} \]
When atom comes out \( B \): \( |\psi\rangle_B = |\psi\rangle \otimes |g\rangle \).

Just before the measurement in \( D \), the state is in general entangled (not separable):

\[
|\psi\rangle_{R_2} = U_{SM}(|\psi\rangle \otimes |g\rangle) = (M_g|\psi\rangle) \otimes |g\rangle + (M_e|\psi\rangle) \otimes |e\rangle
\]

where \( U_{SM} \) is the total unitary transformation (Schrödinger propagator) defining the linear measurement operators \( M_g \) and \( M_e \) on \( \mathcal{H}_S \). Since \( U_{SM} \) is unitary, \( M_g^\dagger M_g + M_e^\dagger M_e = I \).
The unitary propagator $U_{SM}$ is derived from Jaynes-Cummings Hamiltonian $H_{SM}$ in the interaction frame. Two kind of qubit/cavity Hamiltonians:

**resonant**, $H_{SM}/\hbar = i(\Omega(vt)/2) \ (a^\dagger \otimes \sigma_- - a \otimes \sigma_+)$,

**dispersive**, $H_{SM}/\hbar = (\Omega^2(vt)/(2\delta)) \ N \otimes \sigma_z$,

where $\Omega(x) = \Omega_0 e^{-x^2/w^2}$, $x = vt$ with $v$ atom velocity, $\Omega_0$ vacuum Rabi pulsation, $w$ radial mode-width and where $\delta = \omega_q - \omega_c$ is the detuning between qubit pulsation $\omega_q$ and cavity pulsation $\omega_c$ ($|\delta| \ll \Omega_0$).
The Markov model (3)

Just before the measurement in $D$, the atom/field state is:

$$M_g |\psi\rangle \otimes |g\rangle + M_e |\psi\rangle \otimes |e\rangle$$

Denote by $\mu \in \{g, e\}$ the measurement outcome in detector $D$: with probability $p_\mu = \langle \psi | M_\mu^\dagger M_\mu | \psi \rangle$ we get $\mu$. Just after the measurement outcome $\mu$, the state becomes separable:

$$|\psi\rangle_D = \frac{1}{\sqrt{p_\mu}} (M_\mu |\psi\rangle) \otimes |\mu\rangle = \left( \frac{M_\mu}{\sqrt{\langle \psi | M_\mu^\dagger M_\mu | \psi \rangle}} |\psi\rangle \right) \otimes |\mu\rangle.$$

Markov process (density matrix formulation $\rho \sim |\psi\rangle \langle \psi|$)

$$\rho_+ = \begin{cases} 
M_g(\rho) = \frac{M_g \rho M_g^\dagger}{\text{Tr}(M_g \rho M_g^\dagger)}, & \text{with probability } p_g = \text{Tr} \left( M_g \rho M_g^\dagger \right); \\
M_e(\rho) = \frac{M_e \rho M_e^\dagger}{\text{Tr}(M_e \rho M_e^\dagger)}, & \text{with probability } p_e = \text{Tr} \left( M_e \rho M_e^\dagger \right).
\end{cases}$$

Kraus map: $\mathcal{E}(\rho_+ / \rho) = K(\rho) = M_g \rho M_g^\dagger + M_e \rho M_e^\dagger$. 
A controlled Markov process (input $u$, state $\rho$, output $y$)

**Input** $u$: classical amplitude of a coherent micro-wave pulse.

**State** $\rho$: the density operator of the photon(s) trapped in the cavity.

**Output** $y$: quantum projective measure of the probe atom.

The ideal model reads

\[
\rho_{k+1} = \begin{cases} 
\frac{D_{u_k} M_g \rho_k M_g^\dagger D_{u_k}^\dagger}{\text{Tr} \left( M_g \rho_k M_g^\dagger \right)} & y_k = g \text{ with probability } p_{g,k} = \text{Tr} \left( M_g \rho_k M_g^\dagger \right) \\
\frac{D_{u_k} M_e \rho_k M_e^\dagger D_{u_k}^\dagger}{\text{Tr} \left( M_e \rho_k M_e^\dagger \right)} & y_k = e \text{ with probability } p_{e,k} = \text{Tr} \left( M_e \rho_k M_e^\dagger \right) 
\end{cases}
\]

- **Displacement unitary operator** ($u \in \mathbb{R}$): $D_u = e^{u a^\dagger - u a}$ with $a = \text{upper diag}(\sqrt{1}, \sqrt{2}, \ldots)$ the photon annihilation operator.

- **Measurement Kraus operators in the linear dispersive case**

  $M_g = \cos \left( \frac{\phi_0 N + \phi_R}{2} \right)$ and $M_e = \sin \left( \frac{\phi_0 N + \phi_R}{2} \right)$: $M_g^\dagger M_g + M_e^\dagger M_e = I$

  with $N = a^\dagger a = \text{diag}(0, 1, 2, \ldots)$ the photon number operator.
\( u = 0: \text{Quantum Non Demolition (QND) measure of photons.} \)

\[
\rho_{k+1} = \begin{cases} 
\cos\left(\frac{\phi_0 N + \phi_R}{2}\right) \rho_k \cos\left(\frac{\phi_0 N + \phi_R}{2}\right) \\
\quad \text{with prob. } \text{Tr} \left( \cos^2 \left(\frac{\phi_0 N + \phi_R}{2}\right) \rho_k \right) \\
\sin\left(\frac{\phi_0 N + \phi_R}{2}\right) \rho_k \sin\left(\frac{\phi_0 N + \phi_R}{2}\right) \\
\quad \text{with prob. } \text{Tr} \left( \sin^2 \left(\frac{\phi_0 N + \phi_R}{2}\right) \rho_k \right)
\end{cases}
\]

Steady state: any Fock state \( \rho = |\tilde{n}\rangle \langle \tilde{n}| \) (\( \tilde{n} \in \mathbb{N} \)) is a steady-state (no other steady state when \( (\phi_R, \phi_0, \pi) \) are \( \mathbb{Q} \)-independent)

Martingales: for any real function \( g \), \( V_g(\rho) = \text{Tr} \left( g(N) \rho \right) \) is a martingale:

\[
\mathbb{E} \left( V_{g_k}(\rho_{k+1}) / \rho_k \right) = V_{g_k}(\rho_k).
\]

Convergence to a Fock state when \( (\phi_R, \phi_0, \pi) \) are \( \mathbb{Q} \)-independent:

\( V(\rho) = -\frac{1}{2} \sum_n \langle n | \rho | n \rangle^2 \) is a super-martingale with

\[
\mathbb{E} \left( V(\rho_{k+1}) / \rho_k \right) = V(\rho_k) - Q(\rho_k)
\]

where \( Q(\rho) \geq 0 \) and \( Q(\rho) = 0 \) iff, \( \rho \) is a Fock state.

For a realization starting from \( \rho_0 \), the probability to converge towards the Fock state \( |\tilde{n}\rangle \langle \tilde{n}| \) is equal to \( \text{Tr} \left( |\tilde{n}\rangle \langle \tilde{n}| \rho_0 \right) = \langle \tilde{n} | \rho_0 | \tilde{n} \rangle \).
Structure of the stabilizing quantum feedback scheme

With a sampling time of 80 $\mu$s, the controller is classical here

- Goal: stabilization of the steady-state $|\bar{n}\rangle\langle\bar{n}|$ (controller set-point).
- At each time step $k$:
  1. read $y_k$ the measurement outcome for probe atom $k$.
  2. update the quantum state estimation $\rho_{k-1}^{\text{est}}$ to $\rho_k^{\text{est}}$ from $y_k$
  3. compute $u_k$ as a function of $\rho_k^{\text{est}}$ (state feedback).
  4. apply the micro-wave pulse of amplitude $u_k$.

An observer/controller structure:

1. real-time state estimation based on asymptotic observer: here quantum filtering techniques;
2. state feedback stabilization towards a stationary regime: here control Lyapunov techniques based on open-loop martingales $\text{Tr} (g(N)\rho)$.

It takes into account imperfections, delays (5 sampling) and cavity decoherence.

In finite dimension (truncation to $n^{\text{max}}$ photons), all the mathematical details and convergence proof are given in the Automatica 2013 paper.
With pure state $\rho = |\psi\rangle\langle\psi|$, we have

$$\rho_+ = |\psi_+\rangle\langle\psi_+| = \frac{1}{\text{Tr} \left( M_\mu \rho M_\mu^\dagger \right)} M_\mu \rho M_\mu^\dagger$$

when the atom collapses in $\mu = g, e$ with proba. $\text{Tr} \left( M_\mu \rho M_\mu^\dagger \right)$.

Detection error rates: $P(y = e/\mu = g) = \eta_g \in [0, 1]$ the probability of erroneous assignation to $e$ when the atom collapses in $g$; $P(y = g/\mu = e) = \eta_e \in [0, 1]$ (given by the contrast of the Ramsey fringes).

Bayes law: expectation $\rho_+$ of $|\psi_+\rangle\langle\psi_+|$ knowing $\rho$ and the imperfect detection $y$.

$$\rho_+ = \begin{cases} 
\frac{(1-\eta_g)M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger}{\text{Tr} \left( (1-\eta_g)M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger \right)} & \text{if } y = g, \text{ prob. } \text{Tr} \left( (1-\eta_g)M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger \right); \\
\frac{\eta_g M_g \rho M_g^\dagger + (1-\eta_e) M_e \rho M_e^\dagger}{\text{Tr} \left( \eta_g M_g \rho M_g^\dagger + (1-\eta_e) M_e \rho M_e^\dagger \right)} & \text{if } y = e, \text{ prob. } \text{Tr} \left( \eta_g M_g \rho M_g^\dagger + (1-\eta_e) M_e \rho M_e^\dagger \right). 
\end{cases}$$

$\rho_+$ does not remain pure: the quantum state $\rho_+$ becomes a mixed state; $|\psi_+\rangle$ becomes physically irrelevant (not numerically).
LKB photon-box: Markov process with detection errors (2)

We get

\[ \rho_+ = \begin{cases} 
\frac{(1-\eta_g)M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger}{\text{Tr}((1-\eta_g)M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger)}, & \text{with prob. } \text{Tr}\left((1-\eta_g)M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger\right); \\
\frac{\eta_g M_g \rho M_g^\dagger + (1-\eta_e) M_e \rho M_e^\dagger}{\text{Tr}(\eta_g M_g \rho M_g^\dagger + (1-\eta_e) M_e \rho M_e^\dagger)}, & \text{with prob. } \text{Tr}\left(\eta_g M_g \rho M_g^\dagger + (1-\eta_e) M_e \rho M_e^\dagger\right). 
\end{cases} \]

Key point:

\[ \text{Tr}\left((1-\eta_g)M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger\right) \text{ and } \text{Tr}\left(\eta_g M_g \rho M_g^\dagger + (1-\eta_e) M_e \rho M_e^\dagger\right) \]

are the probabilities to detect \( y = g \) and \( e \), knowing \( \rho \).

**Generalization:** with \( (\eta_{\mu',\mu}) \) a left stochastic matrix \( \eta_{\mu',\mu} \geq 0 \) and \( \sum_{\mu'} \eta_{\mu',\mu} = 1 \), we have

\[ \rho_+ = \frac{\sum_{\mu} \eta_{\mu',\mu} M_{\mu} \rho M_{\mu}^\dagger}{\text{Tr}\left(\sum_{\mu} \eta_{\mu',\mu} M_{\mu} \rho M_{\mu}^\dagger\right)} \]

when we detect \( y = \mu' \).

The probability to detect \( y = \mu' \) knowing \( \rho \) is \( \text{Tr}\left(\sum_{\mu} \eta_{\mu',\mu} M_{\mu} \rho M_{\mu}^\dagger\right) \).
Continuous/discrete-time Stochastic Master Equation (SME)

**Discrete-time models** are Markov processes

\[
\rho_{k+1} = \frac{\sum_{\mu} \eta_{\mu', \mu} M_{\mu} \rho_k M_{\mu}^\dagger}{\text{Tr}(\sum_{\mu} \eta_{\mu', \mu} M_{\mu} \rho_k M_{\mu}^\dagger)},
\]

with proba.

\[
\rho_{\mu'}(\rho_k) = \sum_{\mu} \eta_{\mu', \mu} \text{Tr}(M_{\mu} \rho_k M_{\mu}^\dagger)
\]

associated to Kraus maps (ensemble average, quantum channel)

\[
\mathbb{E}(\rho_{k+1}|\rho_k) = K(\rho_k) = \sum_{\mu} M_{\mu} \rho_k M_{\mu}^\dagger
\]

with \(\sum_{\mu} M_{\mu}^\dagger M_{\mu} = I\)

**Continuous-time models** are stochastic differential systems

\[
d\rho_t = \left( -\frac{i}{\hbar} [H, \rho_t] + \sum_{\nu} L_{\nu} \rho_t L_{\nu}^\dagger - \frac{1}{2}(L_{\nu}^\dagger L_{\nu} \rho_t + \rho_t L_{\nu}^\dagger L_{\nu}) \right) dt
\]

\[
+ \sum_{\nu} \sqrt{\eta_{\nu}} \left( L_{\nu} \rho_t + \rho_t L_{\nu}^\dagger - \text{Tr} \left( (L_{\nu} + L_{\nu}^\dagger) \rho_t \right) \rho_t \right) dW_{\nu, t}
\]

driven by Wiener process

\[
dW_{\nu, t} = dy_{\nu, t} - \sqrt{\eta_{\nu}} \text{Tr} \left( (L_{\nu} + L_{\nu}^\dagger) \rho_t \right) dt
\]

with measures \(y_{\nu, t}\), detection efficiencies \(\eta_{\nu} \in [0, 1]\) and

Lindblad-Kossakowski master equations (\(\eta_{\nu} \equiv 0\)):

\[
\frac{d}{dt} \rho = -\frac{i}{\hbar} [H, \rho] + \sum_{\nu} L_{\nu} \rho_t L_{\nu}^\dagger - \frac{1}{2}(L_{\nu}^\dagger L_{\nu} \rho_t + \rho_t L_{\nu}^\dagger L_{\nu})
\]
Continuous/discrete-time diffusive SME

With a single imperfect measure \( dy_t = \sqrt{\eta} \operatorname{Tr} \left( (L + L^\dagger) \rho_t \right) \, dt + dW_t \)
and detection efficiency \( \eta \in [0, 1] \), the quantum state \( \rho_t \) is usually mixed and obeys to

\[
d\rho_t = \left( -\frac{i}{\hbar} [H, \rho_t] + L\rho_t L^\dagger - \frac{1}{2} (L^\dagger L\rho_t + \rho_t L^\dagger L) \right) dt
+ \sqrt{\eta} \left( L\rho_t + \rho_t L^\dagger - \operatorname{Tr} \left( (L + L^\dagger) \rho_t \right) \rho_t \right) dW_t
\]

driven by the Wiener process \( dW_t \)

With Itô rules, it can be written as the following "discrete-time" Markov model

\[
\rho_{t+dt} = \frac{M_{dy_t} \rho_t M_{dy_t}^\dagger + (1 - \eta) L\rho_t L^\dagger \, dt}{\operatorname{Tr} \left( M_{dy_t} \rho_t M_{dy_t}^\dagger + (1 - \eta) L\rho_t L^\dagger \, dt \right)}
\]

with \( M_{dy_t} = I + \left( -\frac{i}{\hbar} H - \frac{1}{2} \left( L^\dagger L \right) \right) dt + \sqrt{\eta} dy_t L. \)
A key physical example in circuit QED

Superconducting qubit dispersively coupled to a cavity traversed by a microwave signal (input/output theory). The back-action on the qubit state of a single measurement of both output field quadratures $I_t$ and $Q_t$ is described by a simple SME for the qubit density operator.

$$d \rho_t = \left( [u^* \sigma_ - u \sigma_+, \rho_t] + \gamma_t (\sigma_z \rho \sigma_z - \rho) \right) dt$$
$$+ \sqrt{\eta \gamma_t/2} (\sigma_z \rho_t + \rho_t \sigma_z - 2 \text{Tr} (\sigma_z \rho_t) \rho_t) dW_t^I$$
$$+ i \sqrt{\eta \gamma_t/2} [\sigma_z, \rho_t] dW_t^Q$$

with $l_t$ and $Q_t$ given by $dl_t = \sqrt{\eta \gamma_t/2} \text{Tr} (2\sigma_z \rho_t) dt + dW_t^I$ and $dQ_t = dW_t^Q$, where $\gamma_t \geq 0$ is related by the read-out pulse shape and $\eta \in [0, 1]$ is the detection efficiency.

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Watt regulator: classical analogue of quantum coherent feedback.  

The first variations of speed $\delta \omega$ and governor angle $\delta \theta$ obey to

\[
\begin{align*}
\frac{d}{dt} \delta \omega &= -a \delta \theta \\
\frac{d^2}{dt^2} \delta \theta &= -\Lambda \frac{d}{dt} \delta \theta - \Omega^2 (\delta \theta - b \delta \omega)
\end{align*}
\]

with $(a, b, \Lambda, \Omega)$ positive parameters.

Third order system

\[
\frac{d^3}{dt^3} \delta \omega + \Lambda \frac{d^2}{dt^2} \delta \omega + \Omega^2 \frac{d}{dt} \delta \omega + ab\Omega^2 \delta \omega = 0.
\]

Characteristic polynomial $P(s) = s^3 + \Lambda s^2 + \Omega^2 s + ab\Omega^2$ with roots having negative real parts iff $\Lambda > ab$: governor damping must be strong enough to ensure asymptotic stability.

Key issues: asymptotic stability and convergence rates.

Reservoir Engineering and coherent feedback\textsuperscript{5}

\[ H = H_{\text{res}} + H_{\text{int}} + H_{\text{syst}} \]

If \( \rho \xrightarrow{t \to \infty} \rho_{\text{res}} \otimes |\bar{\psi}\rangle\langle\bar{\psi}| \) exponentially on a time scale of \( \tau \approx 1/\kappa \) then . . . . . .

\textsuperscript{5}See, e.g., the lectures of H. Mabuchi delivered at the "Ecole de physique des Houches", July 2011.
Reservoir Engineering and coherent feedback

\[ H = H_{\text{res}} + H_{\text{int}} + H_{\text{syst}} \]

\[
\rho \xrightarrow{t \to \infty} \rho_{\text{res}} \otimes |\bar{\psi}\rangle \langle \bar{\psi}| + \Delta, \text{ if } \kappa \gg \gamma \text{ then } \|\Delta\| \ll 1
\]

See, e.g., the lectures of H. Mabuchi delivered at the "Ecole de physique des Houches", July 2011.
Reservoir stabilizing "Schrödinger cats" $|c_\alpha\rangle = (|\alpha\rangle + i|-\alpha\rangle)/\sqrt{2}$. 

Jaynes-Cumming Hamiltonian

$$H(t)/\hbar = \omega_c a^\dagger a \otimes I_M + \omega_q(t) I_S \otimes \sigma_z/2 + i\Omega(t)(a^\dagger \otimes \sigma_- - a \otimes \sigma_+)/2$$

with the open-loop control $t \mapsto \omega_q(t)$ combining dispersive $\omega_q \neq \omega_c$ and resonant $\omega_q = \omega_c$ interactions.

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Composite interaction with $\delta(t) = \omega_q(t) - \omega_c$

$U = U_{\text{off-resonant-1}}$

$U = X(\xi_N) Z(\phi_N)$,
Composite interaction with $\delta(t) = \omega_q(t) - \omega_c$

\[ U = U_{\text{resonant}} U_{\text{off-resonant-1}} \]
\[ U = Y(\theta_N) \text{ } X(\xi_N) \text{ } Z(\phi_N), \]
Composite interaction with $\delta(t) = \omega_q(t) - \omega_c$

$$U = U_{\text{off-resonant-2}} U_{\text{resonant}} U_{\text{off-resonant-1}}$$

$$U = Z(-\phi_N) X(\xi_N) Y(\theta_N) X(\xi_N) Z(\phi_N)$$
Composite interaction with $\delta(t) = \omega_q(t) - \omega_c$

\[
U \approx e^{-i\phi^Kerr N^2} U_{\text{resonant}} e^{i\phi^Kerr N^2}
\]
Convergence of $K$ iterates towards $(|\alpha_\infty\rangle + i|\!\!-\alpha_\infty\rangle) / \sqrt{2}$

Iterations $\rho_{k+1} = K(\rho_k) = M_g \rho_k M_g^\dagger + M_e \rho_k M_e^\dagger$ in the Kerr frame

\[ \rho = e^{-ih_N^{Kerr}} \rho^{Kerr} e^{ih_N^{Kerr}} \]

yields

\[ \rho^{Kerr}_{k+1} = K^{Kerr} (\rho^{Kerr}_k) = M_g^{Kerr} \rho^{Kerr}_k (M_g^{Kerr})^\dagger + M_e^{Kerr} \rho^{Kerr}_k (M_e^{Kerr})^\dagger. \]

with $M_g^{Kerr} = \cos(u/2) \cos(\theta_N/2) + \sin(u/2) \frac{\sin(\theta_N/2)}{\sqrt{N}} a^\dagger$ and $M_e^{Kerr} = \sin(u/2) \cos(\theta_{N+1}/2) - \cos(u/2) a \frac{\sin(\theta_N/2)}{\sqrt{N}}$.

Assume $|u| \leq \pi/2$, $\theta_0 = 0$, $\theta_n \in ]0, \pi[$ for $n > 0$ and $\lim_{n \to +\infty} \theta_n = \pi/2$, then (Zaki Leghtas, PhD thesis (2012))

- exists a unique common eigen-state $|\psi^{Kerr}\rangle$ of $M_g^{Kerr}$ and $M_e^{Kerr}$:

  \[ \rho^{Kerr}_\infty = |\psi^{Kerr}\rangle \langle \psi^{Kerr}| \] fixed point of $K^{Kerr}$.

- if, moreover $n \mapsto \theta_n$ is increasing, $\lim_{k \to +\infty} \rho^{Kerr}_k = \rho^{Kerr}_\infty$.

For well chosen experimental parameters, $\rho^{Kerr}_\infty \approx |\alpha_\infty\rangle \langle \alpha_\infty|$ and $h_N^{Kerr} \approx \pi N^2/2$. Since $e^{-i\pi N^2/2} |\alpha_\infty\rangle = \frac{e^{-i\pi/4}}{\sqrt{2}} (|\alpha_\infty\rangle + i|\!\!-\alpha_\infty\rangle)$:

\[ \lim_{k \to +\infty} \rho_k = \frac{1}{2} \left( |\alpha_\infty\rangle + i|\!\!-\alpha_\infty\rangle \right) \left( \langle \alpha_\infty | + i\langle \!\!-\alpha_\infty | \right) \]

\[ \neq \frac{1}{2} |\alpha_\infty\rangle \langle \alpha_\infty | + \frac{1}{2} |\!\!-\alpha_\infty\rangle \langle \!\!-\alpha_\infty |. \]
Wigner function $W^\rho$ for different values of the density operator $\rho$

$$W^\rho : \mathbb{C} \ni \xi \rightarrow \frac{2}{\pi} \text{Tr} \left( e^{i\pi N} D_{-\xi} \rho D_{\xi} \right) \in [-2/\pi, 2/\pi]$$
An approximation by a continuous-time model\textsuperscript{7}

\begin{center}
\begin{tikzpicture}
  \node (atom) at (0,0) {atom (reservoir)};
  \node (system) at (2,0) {Cavity mode (system)};
  \node (box) at (-2,0) {Box of atoms};
  \node (ENS) at (0,-2) {ENS experiment};
  \node (DC) at (1,-1) {DC field: (controls atom frequency)};
  \node (interaction) at (2,1) {Aim: engineer atom-mode interaction, to stabilize $|\alpha\rangle + |\alpha\rangle$};
  \draw[->] (atom) -- (system);
  \draw[->] (box) -- (system);
  \draw[->] (ENS) -- (system);
  \draw[->] (DC) -- (system);
\end{tikzpicture}
\end{center}

In the Kerr frame $\rho = e^{-i\pi/2} N^2 \rho_{\text{Kerr}} e^{i\pi/2} N^2$:

$$\frac{d}{dt} \rho_{\text{Kerr}} = u[a^\dagger - a, \rho_{\text{Kerr}}] + \kappa (a^\dagger \rho_{\text{Kerr}} a - (N \rho_{\text{Kerr}} + \rho_{\text{Kerr}} N)/2)$$

Identical to the Lindbald master equation of a damped harmonic oscillator ($\kappa > 0$) driven by a coherent input field of amplitude $u$. Simulations: convergence from vacuum in ideal and realistic cases.

Convergence of the quantum damped harmonic oscillator

**Lemma:** the solutions of

\[
\frac{d}{dt} \rho = u[a^\dagger - a, \rho] + \kappa (a \rho a^\dagger - (N \rho + \rho N)/2)
\]

converge exponentially towards \(|\alpha_\infty\rangle\langle\alpha_\infty|\) with \(\alpha_\infty = 2u/\kappa\).  

**Elementary proof:** under the unitary change of frame

\[
\rho = e^{(\alpha_\infty a^\dagger - \alpha_\infty a)} \xi e^{-(\alpha_\infty a^\dagger - \alpha_\infty a)}
\]

the new density operator \(\xi\) is governed by

\[
\frac{d}{dt} \xi = \kappa (a \xi a^\dagger - (N \xi + \xi N)/2)
\]

its energy \(E = \text{Tr}(N\xi) = \text{Tr}(a^\dagger a\xi)\) converges exponentially to 0 since it obeys to \(\frac{d}{dt} E = -\kappa E\); thus \(\xi\) converges exponentially to \(|0\rangle\langle0|\).  

**Computation only based on commutation relations:**

\[
[a, a^\dagger] = 1, \quad a f(N) = f(N + I) a, \quad e^{-(\alpha a^\dagger + \alpha^* a)} a e^{(\alpha a^\dagger - \alpha^* a)} = a + \alpha.
\]
Quantum Fokker-Planck equation: damped harmonic oscillator

\[
\frac{d}{dt} \rho = u[a^\dagger - a, \rho] + \kappa (a\rho a^\dagger - (N\rho + \rho N)/2)
\]

\(\rho\) can be represented by its Wigner function \(W^\rho\) defined by

\[
\mathbb{C} \ni \xi = x + ip \mapsto W^\rho(\xi) = \frac{2}{\pi} \text{Tr} \left( e^{i\pi N} e^{-\xi a^\dagger + \xi^* a} \rho e^{\xi a^\dagger - \xi^* a} \right)
\]

With the correspondences

\[
\frac{\partial}{\partial \xi} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial p} \right), \quad \frac{\partial}{\partial \xi^*} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial p} \right)
\]

\[
W^{\rho a} = \left( \xi - \frac{1}{2} \frac{\partial}{\partial \xi^*} \right) W^\rho, \quad W^{a\rho} = \left( \xi + \frac{1}{2} \frac{\partial}{\partial \xi^*} \right) W^\rho
\]

\[
W^{\rho a^\dagger} = \left( \xi^* + \frac{1}{2} \frac{\partial}{\partial \xi} \right) W^\rho, \quad W^{a^\dagger \rho} = \left( \xi^* - \frac{1}{2} \frac{\partial}{\partial \xi} \right) W^\rho
\]

we get the following PDE for \(W^\rho\) (\(\alpha_\infty = 2u/\kappa\)):

\[
\frac{\partial W^\rho}{\partial t} = \frac{\kappa}{2} \left( \frac{\partial}{\partial x} \left( (x - \alpha_\infty) W^\rho \right) + \frac{\partial}{\partial p} \left( p W^\rho \right) + \frac{1}{4} \Delta W^\rho \right)
\]

converging toward the Gaussian \(W^{\rho_\infty}(x, p) = \frac{2}{\pi} e^{-2(x - \alpha_\infty)^2 - 2p^2}\).
Reservoir with the cavity relaxation \((1/\kappa_c)\) photon life-time\(^8\)

In the Kerr representation frame \(\rho = e^{-i\pi/2} N^2 \rho_{\text{Kerr}} e^{i\pi/2} N^2:\)

\[
\frac{d}{dt} \rho_{\text{Kerr}} = \underbrace{u[a^\dagger - a, \rho_{\text{Kerr}}]}_{\text{reservoir relaxation}} + \kappa \Big( a \rho_{\text{Kerr}} a^\dagger - (N \rho_{\text{Kerr}} + \rho_{\text{Kerr}} N)/2 \Big)
+ \kappa_c \Big( e^{i\pi N} a \rho_{\text{Kerr}} a^\dagger e^{-i\pi N} - (N \rho_{\text{Kerr}} + \rho_{\text{Kerr}} N)/2 \Big).\]

\(\kappa_c\) cavity decoherence

The steady-state for $\kappa_c > 0$

The steady state $\rho_{\infty}^{\text{Kerr}}$ in the Kerr frame

$$0 = u[\hat{a}^\dagger - \hat{a}, \rho_{\infty}^{\text{Kerr}}] + \kappa(\hat{a}\rho_{\infty}^{\text{Kerr}} \hat{a}^\dagger - (N\rho_{\infty}^{\text{Kerr}} + \rho_{\infty}^{\text{Kerr}}N)/2)$$
$$+ \kappa_c(e^{i\pi N}\hat{a}\rho_{\infty}^{\text{Kerr}} \hat{a}^\dagger e^{-i\pi N} - (N\rho_{\infty}^{\text{Kerr}} + \rho_{\infty}^{\text{Kerr}}N)/2)$$

is unique

$$\rho_{\infty}^{\text{Kerr}} = \int_{-\alpha_{\infty}^c}^{\alpha_{\infty}^c} \mu(x)|x\rangle\langle x| \, dx.$$ 

The positive weight function $\mu$ (Glauber-Shudarshan $P$ distribution) is given by

$$\mu(x) = \mu_0 \frac{((\alpha_{\infty}^c)^2 - x^2)(\alpha_{\infty}^c)^2 e^{x^2}}{\alpha_{\infty}^c - x},$$

with $r_c = 2\kappa_c/(\kappa + \kappa_c)$ and $\alpha_{\infty}^c = 2u/(\kappa + \kappa_c)$. The normalization factor $\mu_0 > 0$ ensures that $\int_{-\alpha_{\infty}^c}^{\alpha_{\infty}^c} \mu(x)dx = 1$.

Conjecture: global (exponential) convergence towards $\rho_{\infty}^{\text{Kerr}}$ of $\rho_{\text{Kerr}}^{\text{Kerr}}(t)$ as $t \mapsto +\infty$. 
Robustness of the reservoir stabilizing the two-leg cat.

Since \( W e^{i\pi N} \rho^{Kerr} e^{-i\pi N} (\xi) = W \rho^{Kerr} (-\xi) \) the master Lindblad equation

\[
\frac{d}{dt} \rho^{Kerr} = u[a^\dagger - a, \rho^{Kerr}] + \kappa \left( a \rho^{Kerr} a^\dagger - (N \rho^{Kerr} + \rho^{Kerr} N)/2 \right) \\
+ \kappa_c \left( a e^{i\pi N} \rho^{Kerr} e^{-i\pi N} a^\dagger - (N \rho^{Kerr} + \rho^{Kerr} N)/2 \right).
\]

yields to the following non local diffusion PDE (quantum Fokker-Planck equation):

\[
\frac{\partial W^{\rho^{Kerr}}}{\partial t} \bigg|_{(x,p)} = \frac{\kappa + \kappa_c}{2} \left( \frac{\partial}{\partial x} \left((x - \alpha_\infty) W^{\rho^{Kerr}}\right) + \frac{\partial}{\partial p} \left(p W^{\rho^{Kerr}}\right) + \frac{1}{4} \Delta W^{\rho^{Kerr}} \right) \bigg|_{(x,p)} \\
+ \kappa_c \left( \left[x^2 + p^2 + \frac{1}{2}\right] \left(W^{\rho^{Kerr}} \bigg|_{(-x,-p)} - W^{\rho^{Kerr}} \bigg|_{(x,p)}\right) + \frac{1}{16} \left( \Delta W^{\rho^{Kerr}} \bigg|_{(-x,-p)} - \Delta W^{\rho^{Kerr}} \bigg|_{(x,p)}\right) \right) \\
- \kappa_c \left( \frac{x}{2} \left( \frac{\partial W^{\rho^{Kerr}}}{\partial x} \bigg|_{(-x,-p)} + \frac{\partial W^{\rho^{Kerr}}}{\partial x} \bigg|_{(x,p)}\right) + \frac{p}{2} \left( \frac{\partial W^{\rho^{Kerr}}}{\partial p} \bigg|_{(-x,-p)} + \frac{\partial W^{\rho^{Kerr}}}{\partial p} \bigg|_{(x,p)}\right) \right)
\]

Convergence towards \( W^{\rho^{\infty}} (x, p) = \int^{-\alpha_\infty^c}_{\alpha_\infty^c} \frac{2\mu(\alpha)}{\pi} e^{-2(x - \alpha)^2 - 2p^2} d\alpha \)
remains to be proved.
Quantum information processing with cat-qubits

It is possible with circuit QED to design an open quantum system governed by

\[ \frac{d}{dt} \rho = u[(a^2)\dagger - a^2, \rho] + \kappa \left( a^2 \rho (a^2)\dagger - ((a^2)\dagger a^2 \rho + \rho (a^2)\dagger a^2)/2 \right) \]

where \( a \) is replaced by \( a^2 \). The supports of all solutions \( \rho(t) \) converge to the decoherence free space spanned by the even and odd cat-state;

\[ |C_{\alpha_\infty}^+\rangle \propto |\alpha_\infty\rangle + |-\alpha_\infty\rangle, \quad |C_{\alpha_\infty}^-\rangle \propto |\alpha_\infty\rangle - |-\alpha_\infty\rangle \quad \text{with} \quad \alpha_\infty = \sqrt{2u/\kappa}. \]

The corresponding PDE for \( W^\rho \) is of order 4 in \( x \) and \( p \).

A similar system where \( a \) is replaced now with \( a^4 \) could be very interesting for quantum information processing where the logical qubit is encoded in the planes spanned by even and odd cat-states:

\[ \{ |C_{\alpha_\infty}^+\rangle, |C_{i\alpha_\infty}^\dagger\rangle \}, \quad \{ |C_{\alpha_\infty}^-\rangle, |C_{i\alpha_\infty}^-\rangle \}. \quad \text{with} \quad \alpha_\infty = \sqrt[4]{2u/\kappa}. \]

The corresponding PDE for \( W^\rho \) is of order 8 in \( x \) and \( p \).

Conclusion: convergence issues of open-quantum systems

Discrete time model (Kraus maps):
\[ \rho_{k+1} = K(\rho_k) = \sum_{\nu} M_{\nu} \rho_k M_{\nu}^\dagger \quad \text{with} \quad \sum_{\nu} M_{\nu}^\dagger M_{\nu} = I \]

Continuous-time model (Lindbald, Fokker-Planck eq.):
\[ \frac{d}{dt}\rho = -\frac{i}{\hbar}[H,\rho] + \sum_{\nu} \left( L_{\nu}\rho L_{\nu}^\dagger - (L_{\nu}^\dagger L_{\nu}\rho + \rho L_{\nu}^\dagger L_{\nu})/2 \right), \]

Stability induces by contraction for a lot of metrics (nuclear norm \( \text{Tr}(|\rho - \sigma|) \), fidelity \( \text{Tr}(\sqrt{\sqrt{\rho} \sigma \sqrt{\rho}}) \), see the work of D. Petz).

Open issues motivated by robust quantum information processing:

1. characterization of the \( \Omega \)-limit support of \( \rho \): decoherence free spaces are affine spaces where the dynamics are of Schrödinger types; they can be reduced to a point (pointer-state);
2. Estimation of convergence rate and robustness.
3. Reservoir engineering: design of realistic \( M_{\nu} \) and \( L_{\nu} \) to achieve rapid convergence towards prescribed affine spaces (protection against decoherence).
Collaborators and scientific environment

- **Former PhDs and PostDocs**: Hadis Amini, Hector Bessa Silveira, Zaki Leghtas, Alain Sarlette, Ram Somaraju.
- **Quantic project (INRIA/ENS/MINES)**: Benjamin Huard, François Mallet, Mazyar Mirrahimi, Landry Bretheau, Philippe Campagne, Joachim Cohen, Emmanuel Flurin, Ananda Roy, Pierre Six.
- **Mathematicians**: Karine Beauchard, Jean-Michel Coron, Thomas Chambion, Bernard Bonnard, Ugo Boscain, Sylvain Ervedoza, Stéphane Gaubert, Andrea Grigoriu, Claude Le Bris, Yvon Maday, Vahagn Nersesyan, Clément Pellégrini, Paulo Sergio Pereira da Silva, Jean-Pierre Puel, Lionel Rosier, Julien Salomon, Rodolphe Sepulchre, Mario Sigalotti, Gabriel Turinici.
- **Centre Automatique et Systèmes**: François Chaplais, Florent Di Méglio, Jean Lévine, Philippe Martin, Nicolas Petit, Laurent Praly.
- **And also**: Lectures at Collège de France, ANR projects CQUID and EMAQS, UPS-COFECUB, . . .
Quantum state feedback to stabilize the set-point $|\bar{n}\rangle\langle\bar{n}|$

The Lyapunov feedback scheme is based on a strict control Lyapunov function:

$$V_\epsilon(\rho) = \sum_n \left( -\epsilon \langle n | \rho | n \rangle^2 + \sigma_n \langle n | \rho | n \rangle \right)$$

where $\epsilon > 0$ is small enough and

$$\sigma_n = \begin{cases} 
\frac{1}{4} + \sum_{\nu=1}^{\bar{n}} \frac{1}{\nu} - \frac{1}{\nu^2}, & \text{if } n = 0; \\
\sum_{\nu=n+1}^{\bar{n}} \frac{1}{\nu} - \frac{1}{\nu^2}, & \text{if } n \in [1, \bar{n} - 1]; \\
0, & \text{if } n = \bar{n}; \\
\sum_{\nu=\bar{n}+1}^{\infty} \frac{1}{\nu} + \frac{1}{\nu^2}, & \text{if } n \in [\bar{n} + 1, +\infty].
\end{cases}$$

Feedback law: $u = f(\rho) =: \text{Argmin}_{\nu \in [-\bar{u}, \bar{u}]} V_\epsilon \left( D\nu \left( M_g \rho M_g^\dagger + M_e \rho M_e^\dagger \right) D_{\nu}^\dagger \right)$.

Achieve global stabilization since the decrease is strict

$$\forall \rho \neq |\bar{n}\rangle\langle\bar{n}|, \quad V_\epsilon \left( D_{f(\rho)} \left( M_g \rho M_g^\dagger + M_e \rho M_e^\dagger \right) D_{f(\rho)}^\dagger \right) < V_\epsilon \left( \rho \right).$$
The control Lyapunov function used for experiment.

Coefficients $\sigma_n$ of the control Lyapunov function

\[
V_\epsilon(\rho) = \sum_n \left( -\epsilon \langle n | \rho | n \rangle^2 + \sigma_n \langle n | \rho | n \rangle \right)
\] for $\bar{n} = 3$.

$\sigma_n \sim \log(n)$: key issue to avoid trajectories escaping to $n = +\infty$. 
The Markov process with imperfections: $|\psi_k\rangle$ and $\rho_k$

Take $|\psi_{k+1}\rangle\langle\psi_{k+1}| = \frac{1}{\text{Tr}(M_{\mu_k}|\psi_k\rangle\langle\psi_k|M_{\mu_k}^\dagger)} \left( M_{\mu_k} |\psi_k\rangle\langle\psi_k|M_{\mu_k}^\dagger \right)$ with measure imperfections and decoherence described by the left stochastic matrix $\eta$: $\eta_{\mu',\mu} \in [0, 1]$ is the probability of having the imperfect outcome $\mu' \in \{1, \ldots, m'\}$ knowing that the perfect one is $\mu \in \{1, \ldots, m\}$.

The optimal Belavkin filter: $\rho_k = \mathbb{E} \left( |\psi_k\rangle\langle\psi_k| \Big| |\psi_0\rangle, \mu'_0, \ldots, \mu'_{k-1} \right)$ can be computed efficiently via the following recurrence

$$
\rho_{k+1} = \frac{1}{\text{Tr} \left( \sum_{\mu=1}^{m} \eta_{\mu',\mu} M_{\mu} \rho_k M_{\mu}^\dagger \right)} \left( \sum_{\mu=1}^{m} \eta_{\mu',\mu} M_{\mu} \rho_k M_{\mu}^\dagger \right)
$$

where the detector outcome $\mu'_k$ takes values $\mu'$ in $\{1, \ldots, m'\}$ with probability $p_{\mu',\rho_k} = \text{Tr} \left( \sum_{\mu=1}^{m} \eta_{\mu',\mu} M_{\mu} \rho_k M_{\mu}^\dagger \right)$. 
The quantum state $\rho_k = \mathbb{E} \left( |\psi_k\rangle \langle \psi_k| \right | \psi_0\rangle, \mu'_0, \ldots, \mu'_{k-1})$ is given by the following optimal Belavkin filtering process

$$\rho_{k+1} = \frac{1}{\text{Tr} \left( \sum_{\mu=1}^{m} \eta_{\mu'_k, \mu} M_{\mu} \rho_k M_{\mu}^\dagger \right)} \left( \sum_{\mu=1}^{m} \eta_{\mu'_k, \mu} M_{\mu} \rho_k M_{\mu}^\dagger \right)$$

with the perfect initialization: $\rho_0 = |\psi_0\rangle \langle \psi_0|$. Its estimate $\rho^{\text{est}}$ follows the same recurrence

$$\rho_{k+1}^{\text{est}} = \frac{1}{\text{Tr} \left( \sum_{\mu=1}^{m} \eta_{\mu'_k, \mu} M_{\mu} \rho_k^{\text{est}} M_{\mu}^\dagger \right)} \left( \sum_{\mu=1}^{m} \eta_{\mu'_k, \mu} M_{\mu} \rho_k^{\text{est}} M_{\mu}^\dagger \right)$$

but with imperfect initialization $\rho_{0}^{\text{est}} \neq |\psi_0\rangle \langle \psi_0|$. A natural question: $\rho^{\text{est}}_k \rightarrow \rho_k$ when $k \rightarrow +\infty$?
Stability and convergence issues (2)

Markov process of state \((\rho_k, \rho_k^{\text{est}})\)

\[
\rho_{k+1} = \frac{\sum_{\mu=1}^{m} \eta_{\mu_k,\mu} M_{\mu} \rho_k M_{\mu}^\dagger}{\text{Tr}(\sum_{\mu=1}^{m} \eta_{\mu_k,\mu} M_{\mu} \rho_k M_{\mu}^\dagger)}, \quad \rho_k^{\text{est}} = \frac{\sum_{\mu=1}^{m} \eta_{\mu_k,\mu} M_{\mu} \rho_k^{\text{est}} M_{\mu}^\dagger}{\text{Tr}(\sum_{\mu=1}^{m} \eta_{\mu_k,\mu} M_{\mu} \rho_k^{\text{est}} M_{\mu}^\dagger)}
\]

Proba. to get \(\mu'_k\) at step \(k\), \(\text{Tr} \left( \sum_{\mu=1}^{m} \eta_{\mu_k,\mu} M_{\mu} \rho_k M_{\mu}^\dagger \right)\), depends on \(\rho_k\).

- Convergence of \(\rho_k^{\text{est}}\) towards \(\rho_k\) when \(k \rightarrow +\infty\) is an open problem.


- Stability\(^{10}\): the fidelity \(F(\rho_k, \rho_k^{\text{est}}) = \text{Tr}^2 \left( \sqrt{\sqrt{\rho_k} \rho_k^{\text{est}} \sqrt{\rho_k}} \right)\) is a sub-martingale for any \(\eta\) and \(M_{\mu}\):

  \[
  \mathbb{E} \left( F(\rho_{k+1}, \rho_k^{\text{est}}) / \rho_k \right) \geq F(\rho_k, \rho_k^{\text{est}}).
  \]

  Fidelity: \(0 \leq F(\rho, \rho^e) \leq 1\) and \(F(\rho, \rho^e) = 1\) iff \(\rho = \rho^e\).

The key inequality underlying $F(\rho, \rho^e)$ is sub-martingale\(^{11}\)

For

- any set of $m$ matrices $M_\mu$ with $\sum_{\mu=1}^{m} M_\mu \dagger M_\mu = 1$,
- any partition of $\{1, \ldots, m\}$ into $p \geq 1$ sub-sets $P_\nu$,
- any Hermitian non-negative matrices $\rho$ and $\sigma$ of trace one, the following inequality holds

$$\sum_{\nu=1}^{\nu=p} \text{Tr} \left( \sum_{\mu \in P_\nu} M_\mu \rho M_\mu \dagger \right) F \left( \frac{\sum_{\mu \in P_\nu} M_\mu \sigma M_\mu \dagger}{\text{Tr}(\sum_{\mu \in P_\nu} M_\mu \sigma M_\mu \dagger)} \right), \quad \frac{\sum_{\mu \in P_\nu} M_\mu \rho M_\mu \dagger}{\text{Tr}(\sum_{\mu \in P_\nu} M_\mu \rho M_\mu \dagger)} \geq F(\sigma, \rho)$$

where $F(\sigma, \rho) = \text{Tr}^2 \left( \sqrt{\sqrt{\sigma \rho} \sqrt{\sigma}} \right)$.

Proof combines on a lifting procedure with Ulhmann’s theorem.

\(^{11}\)PR. Fidelity is a Sub-Martingale for Discrete-Time Quantum Filters. IEEE Transactions on Automatic Control, 2011, 56, 2743-2747.
Continuous/discrete-time jump SME

With Poisson process \( N(t) \), \( \langle dN(t) \rangle = \left( \frac{\bar{\theta} + \bar{\eta}}{\bar{\theta} + \bar{\eta}} \text{Tr} \left( V \rho_t V^\dagger \right) \right) dt \), and detection imperfections modeled by \( \bar{\theta} \geq 0 \) and \( \bar{\eta} \in [0, 1] \), the quantum state \( \rho_t \) is usually mixed and obeys to

\[
d\rho_t = \left( -\frac{i}{\hbar} [H, \rho_t] + V \rho_t V^\dagger - \frac{1}{2} (V^\dagger V \rho_t + \rho_t V^\dagger V) \right) dt \\
+ \left( \frac{\bar{\theta} \rho_t + \bar{\eta} V \rho_t V^\dagger}{\bar{\theta} + \bar{\eta} \text{Tr} \left( V \rho_t V^\dagger \right)} - \rho_t \right) \left( dN(t) - \left( \frac{\bar{\theta} + \bar{\eta}}{\bar{\theta} + \bar{\eta}} \text{Tr} \left( V \rho_t V^\dagger \right) \right) dt \right)
\]

For \( N(t + dt) - N(t) = 1 \) we have \( \rho_{t+dt} = \frac{\bar{\theta} \rho_t + \bar{\eta} V \rho_t V^\dagger}{\bar{\theta} + \bar{\eta} \text{Tr} \left( V \rho_t V^\dagger \right)} \).

For \( dN(t) = 0 \) we have

\[
\rho_{t+dt} = \frac{M_0 \rho_t M_0^\dagger + (1 - \bar{\eta}) V \rho_t V^\dagger dt}{\text{Tr} \left( M_0 \rho_t M_0^\dagger + (1 - \bar{\eta}) V \rho_t V^\dagger dt \right)}
\]

with \( M_0 = I + \left( -\frac{i}{\hbar} H + \frac{1}{2} \left( \bar{\eta} \text{Tr} \left( V \rho_t V^\dagger \right) I - V^\dagger V \right) \right) dt \).
The quantum state $\rho_t$ is usually mixed and obeys to

$$
d\rho_t = \left( -\frac{i}{\hbar}[H, \rho_t] + L\rho_t L^\dagger - \frac{1}{2}(L^\dagger L\rho_t + \rho_t L^\dagger L) + V\rho_t V^\dagger - \frac{1}{2}(V^\dagger V\rho_t + \rho_t V^\dagger V) \right) dt $$

$$+ \sqrt{\eta} \left( L\rho_t + \rho_t L^\dagger - \text{Tr} \left( (L + L^\dagger)\rho_t \right) \rho_t \right) dW_t$$

$$+ \left( \frac{\bar{\theta}\rho_t + \bar{\eta}V\rho_t V^\dagger}{\bar{\theta} + \bar{\eta} \text{Tr} \left( V\rho_t V^\dagger \right)} - \rho_t \right) \left( dN(t) - \left( \bar{\theta} + \bar{\eta} \text{Tr} \left( V\rho_t V^\dagger \right) \right) dt \right)$$

For $N(t + dt) - N(t) = 1$ we have $\rho_{t+dt} = \frac{\bar{\theta}\rho_t + \bar{\eta}V\rho_t V^\dagger}{\bar{\theta} + \bar{\eta} \text{Tr} \left( V\rho_t V^\dagger \right)}$.

For $dN(t) = 0$ we have

$$\rho_{t+dt} = \frac{M_{dyt}\rho_t M^\dagger_{dyt} + (1 - \eta)L\rho_t L^\dagger dt + (1 - \bar{\eta})V\rho_t V^\dagger dt}{\text{Tr} \left( M_{dyt}\rho_t M^\dagger_{dyt} + (1 - \eta)L\rho_t L^\dagger dt + (1 - \bar{\eta})V\rho_t V^\dagger dt \right)}$$

with $M_{dyt} = I + \left( -\frac{i}{\hbar}H - \frac{1}{2}L^\dagger L + \frac{1}{2} (\bar{\eta} \text{Tr} \left( V\rho_t V^\dagger \right) I - V^\dagger V) \right) dt + \sqrt{\eta} dyt L$. 
Continuous/discrete-time general diffusive-jump SME

The quantum state $\rho_t$ is usually mixed and obeys to

$$
\frac{d\rho_t}{dt} = \left(-\frac{i}{\hbar}[H, \rho_t] + \sum_\nu L_\nu \rho_t L_\nu^\dagger - \frac{1}{2}(L_\nu^\dagger L_\nu \rho_t + \rho_t L_\nu^\dagger L_\nu) + \sum_\mu V_\mu \rho_t V_\mu^\dagger - \frac{1}{2}(V_\mu^\dagger V_\mu \rho_t + \rho_t V_\mu^\dagger V_\mu)\right) dt
$$

$$
+ \sum_\nu \sqrt{\eta_\nu} \left(L_\nu \rho_t + \rho_t L_\nu^\dagger - \text{Tr}\left((L_\nu + L_\nu^\dagger) \rho_t\right)\right) dW_\nu, t
$$

$$
+ \sum_\mu \left(\frac{\bar{\theta}_\mu \rho_t + \sum_\mu' \bar{\eta}_{\mu, \mu'} V_\mu \rho_t V_\mu^\dagger}{\bar{\theta}_\mu + \sum_\mu' \bar{\eta}_{\mu, \mu'} \text{Tr}\left(V_\mu' \rho_t V_\mu^\dagger\right)} - \rho_t\right) \left(dN_\mu(t) - \left(\bar{\theta}_\mu + \sum_\mu' \bar{\eta}_{\mu, \mu'} \text{Tr}\left(V_\mu' \rho_t V_\mu^\dagger\right)\right) dt\right)
$$

where $\eta_\nu \in [0, 1], \bar{\theta}_\mu, \bar{\eta}_{\mu, \mu'} \geq 0$ with $\bar{\eta}_{\mu, \mu'} = \sum_\mu \bar{\eta}_{\mu, \mu'} \leq 1$ are parameters modelling measurements imperfections.

If, for some $\mu$, $N_\mu(t + dt) - N_\mu(t) = 1$, we have $\rho_{t+dt} = \frac{\bar{\theta}_\mu \rho_t + \sum_\mu' \bar{\eta}_{\mu, \mu'} V_\mu \rho_t V_\mu^\dagger}{\bar{\theta}_\mu + \sum_\mu' \bar{\eta}_{\mu, \mu'} \text{Tr}\left(V_\mu' \rho_t V_\mu^\dagger\right)}$.

When $\forall \mu$, $dN_\mu(t) = 0$, we have

$$
\rho_{t+dt} = \frac{M_{dy_t} \rho_t M_{dy_t}^\dagger + \sum_\nu (1 - \eta_\nu) L_\nu \rho_t L_\nu^\dagger dt + \sum_\mu (1 - \bar{\eta}_\mu) V_\mu \rho_t V_\mu^\dagger dt}{\text{Tr}\left(M_{dy_t} \rho_t M_{dy_t}^\dagger + \sum_\nu (1 - \eta_\nu) L_\nu \rho_t L_\nu^\dagger dt + \sum_\mu (1 - \bar{\eta}_\mu) V_\mu \rho_t V_\mu^\dagger dt\right)}
$$

with $M_{dy_t} = I + \left(-\frac{i}{\hbar} H - \frac{1}{2} \sum_\nu L_\nu^\dagger L_\nu + \frac{1}{2} \sum_\mu \left(\bar{\eta}_\mu \text{Tr}\left(V_\mu \rho_t V_\mu^\dagger\right) I - V_\mu^\dagger V_\mu\right)\right) dt + \sum_\nu \sqrt{\eta_\nu} dy_\nu dt L_\nu$

and where $dy_\nu dt = \sqrt{\eta_\nu} \text{Tr}\left((L_\nu + L_\nu^\dagger) \rho_t\right) dt + dW_\nu, t$.

Could be used as a numerical integration scheme that preserves the positiveness of $\rho$. 
Continuous-time diffusive SME and quantum filtering

For clarity’s sake, take a single measure $y_t$ associated to operator $L$ and detection efficiency $\eta \in [0, 1]$. Then $\rho_t$ obeys to the following diffusive SME

$$
    d\rho_t = -\frac{i}{\hbar} [H, \rho_t] dt + \left( L\rho_t L^\dagger - \frac{1}{2} (L^\dagger L\rho_t + \rho_t L^\dagger L) \right) dt \\
    + \sqrt{\eta} \left( L\rho_t + \rho_t L^\dagger - \text{Tr} \left( (L + L^\dagger)\rho_t \right) \rho_t \right) dW_t
$$

driven by the Wiener processes $W_t$.

Since $dy_t = \sqrt{\eta} \text{Tr} \left( (L + L^\dagger)\rho_t \right) dt + dW_t$, the estimate $\rho_t^{\text{est}}$ is given by

$$
    d\rho_t^{\text{est}} = -\frac{i}{\hbar} [H, \rho_t^{\text{est}}] dt + \left( L\rho_t^{\text{est}} L^\dagger - \frac{1}{2} (L^\dagger L\rho_t^{\text{est}} + \rho_t^{\text{est}} L^\dagger L) \right) dt \\
    + \sqrt{\eta} \left( L\rho_t^{\text{est}} + \rho_t^{\text{est}} L^\dagger - \text{Tr} \left( (L + L^\dagger)\rho_t^{\text{est}} \right) \rho_t^{\text{est}} \right) \left( dy_t - \sqrt{\eta} \text{Tr} \left( (L + L^\dagger)\rho_t^{\text{est}} \right) dt \right)
$$

initialized to any density matrix $\rho_0^{\text{est}}$. 
Assume that \((\rho, \rho^\text{est})\) obey to

\[
d\rho_t = -\frac{i}{\hbar}[H, \rho_t] \, dt + \left(L\rho_t L^\dagger - \frac{1}{2}(L^\dagger L\rho_t + \rho_t L^\dagger L)\right) \, dt
\]

\[
+ \sqrt{\eta} \left(L\rho_t + \rho_t L^\dagger - \text{Tr} \left((L + L^\dagger)\rho_t\right) \rho_t\right) \, dW_t
\]

\[
d\rho^\text{est}_t = -\frac{i}{\hbar}[H, \rho^\text{est}_t] \, dt + \left(L\rho^\text{est}_t L^\dagger - \frac{1}{2}(L^\dagger L\rho^\text{est}_t + \rho^\text{est}_t L^\dagger L)\right) \, dt
\]

\[
+ \sqrt{\eta} \left(L\rho^\text{est}_t + \rho^\text{est}_t L^\dagger - \text{Tr} \left((L + L^\dagger)\rho^\text{est}_t\right) \rho^\text{est}_t\right) \, dW_t
\]

\[
+ \eta \left(L\rho^\text{est}_t + \rho^\text{est}_t L^\dagger - \text{Tr} \left((L + L^\dagger)\rho^\text{est}_t\right) \rho^\text{est}_t\right) \text{Tr} \left((L + L^\dagger)(\rho_t - \rho^\text{est}_t)\right) \, dt.
\]

\text{correction terms vanishing when } \rho_t = \rho^\text{est}_t

Then for any \(H, L\) and \(\eta \in [0, 1]\),

\[
F(\rho_t, \rho^\text{est}_t) = \text{Tr}^2 \left(\sqrt{\sqrt{\rho_t} \rho^\text{est}_t} \sqrt{\rho_t}\right)
\]

is a sub-martingale:

\[
t \mapsto \mathbb{E}\left(F(\rho_t, \rho^\text{est}_t)\right) \text{ is non-decreasing.}
\]

Cat states obtained via Kerr transformations of coherent states\textsuperscript{13}

Take a coherent state $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$ of complex amplitude $\alpha$. Depending on $\phi^Kerr$, the Kerr-propagated state

$$e^{-i\phi^Kerr N^2} |\alpha\rangle$$

can take a number of nonclassical forms:

1. squeezed states for $\phi^Kerr \ll \pi$;
2. states with ‘banana’-shaped Wigner function for slightly larger $\phi^Kerr$;
3. mesoscopic field state superpositions $|k\alpha\rangle$ with $k$ equally spaced components for $t_K \gamma_K = \pi/k$.
4. in particular, for $\phi^Kerr = \frac{\pi}{2}$, a superposition of two coherent states with opposite amplitudes:

$$|c\alpha\rangle = (|\alpha\rangle + i |\alpha\rangle) / \sqrt{2}.$$

Wigner functions of $e^{-i\phi_{Kerr} N^2} |\alpha\rangle$ for different values of $\phi_{Kerr}$.

(a): $\phi_{Kerr} = \pi/2$; (b): $\phi_{Kerr} = \pi/3$; (c): $\phi_{Kerr} = 0.28$; (d): $\phi_{Kerr} = 0.08$

(e-h): similar states stabilized, despite decoherence, by the atomic reservoir onto which we focus in this talk.
Reservoir engineering stabilization for discrete-time systems

Data: $\mathcal{H}_S$ with Hamiltonian $H_S$, a pure goal state $\tilde{\rho}_S = |\tilde{\psi}_S\rangle\langle\tilde{\psi}_S|$. **Find a "realistic" meter system** of Hilbert space $\mathcal{H}_M$ with initial state $|\theta_M\rangle$, with Hamiltonian $H_M$ and interaction Hamiltonian $H_{int}$ such that

1. the propagator $U_{S,M} = U(T)$ between 0 and time $T$
   \[
   \left( \frac{d}{dt} U = -\frac{i}{\hbar}(H_S + H_M + H_{int}) U, \ U(0) = I \right) \]
   reads:
   \[
   \forall |\psi_S\rangle \in \mathcal{H}_S, \quad U_{S,M}(|\psi_S\rangle \otimes |\theta_M\rangle) = \sum_{\mu} (M_\mu |\psi_S\rangle) \otimes |\lambda_\mu\rangle
   \]
   where $|\lambda_\mu\rangle$ is an ortho-normal basis of $\mathcal{H}_M$.

2. the resulting measurement operators $M_\mu$ admit $|\tilde{\psi}_S\rangle$ as common eigen-vector, i.e., $\tilde{\rho}_S$ is a fixed point of the Kraus map $K(\rho) = \sum_\mu M_\mu \rho M_\mu^\dagger$: $K(\tilde{\rho}_S) = \tilde{\rho}_S$.

3. iterates of $K$ converge to $\tilde{\rho}_S$ for any initial condition $\rho_0$:
   \[
   \lim_{k \to +\infty} \rho_k = \tilde{\rho}_S \text{ where } \rho_k = K(\rho_{k-1}) \quad (\text{asymptotic stability}).
   \]

Here the reservoir is made of the infinite set of identical meter systems with initial state $|\theta_M\rangle$ at $t = (k - 1)T$ and interacting with $\mathcal{H}_S$ during $[(k - 1)T, kT]$, $k = 1, 2, \ldots$. 
\[ U = Z(-\phi_N) X(\xi_N) \ Y(\theta^r_N) \ X(\xi_N) Z(\phi_N), \]

Generalized rotations around Bloch spheres labeled with \( n \):

\[
X(f_N) = \cos \left( \frac{f_N}{2} \right) \otimes |g\rangle \langle g| + \cos \left( \frac{f_{N+1}}{2} \right) \otimes |e\rangle \langle e|
- i a \frac{\sin \left( \frac{f_N}{2} \right)}{\sqrt{N}} \otimes |e\rangle \langle g| - i \frac{\sin \left( \frac{f_N}{2} \right)}{\sqrt{N}} a^\dagger \otimes |g\rangle \langle e|
\]

\[
Y(f_N) = \cos \left( \frac{f_N}{2} \right) \otimes |g\rangle \langle g| + \cos \left( \frac{f_{N+1}}{2} \right) \otimes |e\rangle \langle e|
- a \frac{\sin \left( \frac{f_N}{2} \right)}{\sqrt{N}} \otimes |e\rangle \langle g| + \frac{\sin \left( \frac{f_N}{2} \right)}{\sqrt{N}} a^\dagger \otimes |g\rangle \langle e|
\]

\[
Z(f_N) = e^{if_N/2} \otimes |g\rangle \langle g| + e^{-if_{N+1}/2} \otimes |e\rangle \langle e|
.\]

The different angles depending on the photon-numbers:

\[
\theta^r_n = \sqrt{n} \int_{-T/2}^{T/2} \Omega(vt) \, dt, \quad \phi_n = \delta_0 \int_{-T/2}^{T/2} \sqrt{1 + n(\Omega(vt)/\delta_0)^2} \, dt
\]

\[
\tan \xi_n = \frac{\Omega(vT/2)}{\delta_0} \sqrt{n} \quad \text{with} \quad \xi_n \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right).
\]

1. With $\theta_n \in [0, 2\pi)$ defined by $\cos(\theta_n/2) = \cos(\theta_N^r/2) \cos \xi_n$ and $\phi^c_N = \phi_N + \text{angle}[\sin(\theta_N^r/2) - i \cos(\theta_N^r/2) \sin \xi_N]$:

$$U = \cos(\theta_N/2) \otimes |g\rangle \langle g| + \cos(\theta_N^r/2) \otimes |e\rangle \langle e|$$

$$- a \frac{\sin(\theta_N/2)}{\sqrt{N}} e^{i\phi^c_N} \otimes |e\rangle \langle g| + \frac{\sin(\theta_N^r/2)}{\sqrt{N}} e^{-i\phi^c_N} a^\dagger \otimes |g\rangle \langle e|.$$

2. Using $a f(N) \equiv f(N + 1) a$ we get

$$U = e^{-i h^\text{Kerr}_N} Y(\theta^c_N) e^{i h^\text{Kerr}_N}$$

with $h^\text{Kerr}_{n+1} - h^\text{Kerr}_n = \phi^c_{n+1}$ defining "Kerr Hamiltonian" $h^\text{Kerr}_N$.

3. With $|u_{at}\rangle = \cos(u/2)|g\rangle + \sin(u/2)|e\rangle$,

$$U (|\psi\rangle \otimes |u_{at}\rangle) = M_g |\psi\rangle \otimes |g\rangle + M_e |\psi\rangle \otimes |e\rangle$$

where

$$M_g = e^{-i h^\text{Kerr}_N} M^g e^{i h^\text{Kerr}_N}, \quad M_e = e^{-i h^\text{Kerr}_N} M^e e^{i h^\text{Kerr}_N}.$$
Convergence issues in \( \{ \sum_{n \geq 0} \psi_n |n\rangle, (\psi_n)_{n \geq 0} \in l^2(\mathbb{C}) \} \)

The two measurement operators in the Kerr frame

\[
M^\text{Kerr}_g = \cos\left(\frac{u}{2}\right) \cos(\theta_N/2) + \sin\left(\frac{u}{2}\right) \frac{\sin(\theta_N/2)}{\sqrt{N}} \ a^\dagger
\]

\[
M^\text{Kerr}_e = \sin\left(\frac{u}{2}\right) \cos(\theta_{N+1}/2) - \cos\left(\frac{u}{2}\right) \ a \frac{\sin(\theta_N/2)}{\sqrt{N}}
\]

and the Kraus map:

\[
\rho^{\text{Kerr}}_{k+1} = K^{\text{Kerr}}(\rho_k) = M^\text{Kerr}_g \rho_k (M^\text{Kerr}_g)^\dagger + M^\text{Kerr}_e \rho_k (M^\text{Kerr}_e)^\dagger.
\]

When \( |u| < \pi/2, \theta_0 = 0, \theta_n \in ]0, \pi[ \) for \( n > 0 \) and \( \lim_{n \to +\infty} \theta_n = \pi/2 \):

- exists a unique common eigen-state \( |\psi^{Kerr}\rangle \) of \( M^\text{Kerr}_g \) and \( M^\text{Kerr}_e \):
  \[
  \rho^{\text{Kerr}}_{\infty} = |\psi^{Kerr}\rangle \langle \psi^{Kerr}| \text{ fixed point of } K^{\text{Kerr}}.
  \]

- if \( n \mapsto \theta_n \) is increasing, Zaki Leghtas has proved in his PhD thesis (2012) global convergence (Lyapunov function \( \text{Tr} (\rho^{\text{Kerr}}_{\infty} \rho^{\text{Kerr}}_k) \), precompacity of the trajectories, Lassalle invariance principle, . . .).

**Conjecture:** global convergence without \( n \mapsto \theta_n \) increasing.
Books on open quantum systems