Motion planing and tracking for differentially flat systems

Pierre Rouchon
Mines-ParisTech,
Centre Automatique et Systèmes
Mathématiques et Systèmes
pierre.rouchon@mines-paristech.fr

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Outline

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ODE: several definitions of flat-systems
  An elementary definition based on inversion
  The intrinsic definition with $D$-variety (diffiety)
  An extrinsic definition
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PDE: two kind of flat examples
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Conclusion for PDE

Conclusion for ODE: flatness characterization is an open problem
  Systems with only one control
  Driftless systems as Pfaffian system
  Ruled manifold criterion
  Symmetry preserving flat-output
Interest of flat systems

1. History: "integrability" for under-determined systems of differential equations (Monge, Hilbert, Cartan, ....).

2. Control theory: flat systems admit simple solutions to the motion planing and tracking problems (Fliess and coworkers 1991 and later).

3. Books on differentially flat systems:
Motion planning: controllability.

Difficult problem due to integration of

\[ \frac{d}{dt} x = f(x, u_r(t)), \quad x(0) = p. \]
Tracking for $\frac{d}{dt}x = f(x, u)$: stabilization.

Compute $\Delta u$, $u = u_r + \Delta u$, depending $\Delta x$ (feedback), such that $\Delta x = x - x_r$ tends to 0 (stabilization).
The simplest robot

- **Newton ODE):**
  \[
  \frac{d^2}{dt^2} \theta = -p \sin(\theta) + u
  \]
  Non linear oscillator with scalar input \( u \) and parameter \( p > 0 \).

- **Computed torque method:**
  \[
  u_r = \frac{d^2}{dt^2} \theta_r + p \sin \theta_r
  \]
  provides an explicit parameterization via \( KC^2 \) function: \( t \mapsto \theta_r(t) \), the flat output.

Motion planing and tracking \((\xi, \omega_0 > 0, \text{two feedback gains})\)

\[
 u \left( t, \theta, \frac{d}{dt} \theta \right) = \frac{d^2}{dt^2} \theta_r + p \sin \theta - 2 \xi \omega_0 \left( \frac{d}{dt} \theta - \frac{d}{dt} \theta_r \right) - (\omega_0)^2 \sin(\theta - \theta_r)
\]
where \( t \mapsto \theta_r(t) \) defines the reference trajectory (control goal).
The computed torque method for

\[
\frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{q}} \right] = \frac{\partial L}{\partial q} + M(q)u
\]

consists in setting \( t \mapsto q(t) \) to obtain \( u \) as a function of \( q \), \( \dot{q} \) and \( \ddot{q} \).

(Fully actuated: \( \dim q = \dim u \) and \( M(q) \) invertible).
Oscillators and linear systems

System with 2 ODEs and 3 unknowns \((x_1, x_2, u)\) \((a_1, a_2 > 0\) and \(a_1 \neq a_2)\)

\[
\frac{d^2}{dt^2} x_1 = -a_1(x_1 - u), \quad \frac{d^2}{dt^2} x_2 = -a_2(x_2 - u)
\]
defines a free module\(^1\) with basis \(y = \frac{a_2 x_1 - a_1 x_2}{a_2 - a_1}\):

\[
\begin{cases}
  x_1 = y + \frac{d^2}{dt^2} y / a_2, & \frac{d}{dt} x_1 = \frac{d}{dt} y + \frac{d^3}{dt^3} y / a_2 \\
  x_2 = y + \frac{d^2}{dt^2} y / a_1, & \frac{d}{dt} x_2 = \frac{d}{dt} y + \frac{d^3}{dt^3} y / a_1 \\
  u = y + \left(\frac{1}{a_1} + \frac{1}{a_2}\right) \frac{d^2}{dt^2} y + \left(\frac{1}{a_1 a_2}\right) \frac{d^4}{dt^4} y
\end{cases}
\]

Reference trajectory for equilibrium \(x_1 = x_2 = u = 0\) at \(t = 0\) to equilibrium \(x_1 = x_2 = u = D\) at \(t = T > 0\):

\[
y(t) = \begin{cases} 
  0 & \text{si } t \leq 0, \\
  \frac{(t)^4}{t^4 + (T - t)^4} D & \text{si } t \in [0, T], \\
  D & \text{si } t \geq T.
\end{cases}
\]

Generalization to \(n\) oscillators and any linear controllable system, \(\frac{d}{dt} X = AX + Bu\).

\(^1\)See the work of Alban Quadrat and co-workers....
$2k\pi$ juggling robot: prototype of implicit flat system

Isochronous punctual pendulum $H$ (Huygens):

$$ m\frac{d^2}{dt^2}H = \vec{T} + m\vec{g} $$

$$ \vec{T} \parallel \vec{HS} $$

$$ \|\vec{HS}\|^2 = l $$

| The suspension point $S \in \mathbb{R}^3$ stands for the control input |
| The oscillation center $H \in \mathbb{R}^3$ is the flat output: since $\frac{\vec{T}}{m} = \frac{d^2}{dt^2}H - \vec{g}$ et $\vec{T} \parallel \vec{HS}$, $S$ is solution of the algebraic system: |

$$ \vec{HS} \parallel \frac{d^2}{dt^2}H - \vec{g} \quad \text{and} \quad \|\vec{HS}\|^2 = l. $$
Return of the pendulum and smooth branch switch

In a vertical plane: $H$ of coordinates $(y_1, y_2)$ and $S$ of coordinates $(u_1, u_2)$ satisfy

$$(y_1 - u_1)^2 + (y_2 - u_2)^2 = l, \quad (y_1 - u_1) \left( \frac{d^2}{dt^2} y_2 + g \right) = (y_2 - u_2) \frac{d^2}{dt^2} y_1.$$ \hspace{1cm} (1)

Find $[0, T] \ni t \mapsto y(t) \in C^2$ such that $y(0) = (0, -l)$, $y(T) = (0, l)$ and $y^{(1, 2)}(0, T) = 0$, and such that exists also $[0, T] \ni t \mapsto u(t) \in C^0$ with $u(0) = u(T) = 0$ (switch between the stable and the unstable branches).
Planning the inversion trajectory

Any smooth trajectory connecting the stable to the unstable equilibrium is such that $\ddot{H}(t) = \vec{g}$ for at least one time $t$. During the motion there is a switch from the stable root to the unstable root (singularity crossing when $\ddot{H} = \vec{g}$)
Crossing smoothly the singularity $\ddot{H} = \vec{g}$

The geometric path followed by $H$ is a half-circle of radius $l$ of center $O$:

$$H(t) = 0 + l \begin{bmatrix} \sin \theta(s) \\ -\cos \theta(s) \end{bmatrix}$$

with $\theta(s) = \mu(s) \pi$, $s = t/T \in [0, 1]$

where $T$ is the transition time and $\mu(s)$ a sigmoid function of the form:

![Graph of sigmoid function](image)
Time scaling and dilatation of $\ddot{H} - \vec{g}$

Denote by $'$ derivation with respect to $s$. From

$$H(t) = 0 + l \begin{bmatrix} \sin \theta(s) \\ - \cos \theta(s) \end{bmatrix}, \quad \theta(s) = \mu(t/T)\pi$$

we have

$$\ddot{H} = H''/T^2.$$  

Changing $T$ to $\alpha T$ yields to a dilation of factor $1/\alpha^2$ of the closed geometric path described by $\ddot{H} - \vec{g}$ for $t \in [0, T]$ ($\ddot{H}(0) = \ddot{H}(T) = 0$), the dilation center being $-\vec{g}$. The inversion time is obtained when this closed path passes through 0. This construction holds true for generic $\mu$. 
The crane

\[ m\ddot{H} = \vec{T} + m\vec{g} \]
\[ \vec{T} // \vec{HD} \]
\[ HD = r \]
\[ \vec{OD} \cdot \vec{k} = 0 \]
The geometric construction for the crane

Singularity when $\ddot{H} - \vec{g}$ is horizontal.
Single car $^2$

\[
\begin{align*}
\frac{dx}{dt} &= v \cos \theta \\
\frac{dy}{dt} &= v \sin \theta \\
\frac{d\theta}{dt} &= \frac{v}{l} \tan \varphi = \omega
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
\cos \theta \\
\sin \theta
\end{bmatrix} &= \frac{d}{dt} \begin{bmatrix} P \end{bmatrix} \\
\tan \varphi &= \frac{l \det(\dot{P}, \dot{P})}{v \sqrt{|v|}}
\end{align*}
\]

The time scaling symmetry

For any $T \mapsto \sigma(T)$, the transformation

$$t = \sigma(T), \quad (x, y, \theta) = (X, Y, \Theta), \quad (v, \omega) = (V, \Omega)/\sigma'(t)$$

leave the equations

$$\frac{d}{dt} x = v \cos \theta, \quad \frac{d}{dt} y = v \sin \theta, \quad \frac{d}{dt} \theta = \omega$$

unchanged:

$$\frac{d}{dT} X = V \cos \Theta, \quad \frac{d}{dT} Y = V \sin \Theta, \quad \frac{d}{dT} \Theta = \Omega.$$
SE(2) invariance

For any \((a, b, \alpha)\), the transformation

\[
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} X \cos \alpha - Y \sin \alpha + a \\ X \sin \alpha + Y \cos \alpha + b \end{bmatrix}, \quad \theta = \Theta - \alpha, \quad (v, \omega) = (V, \Omega)
\]

leave the equations

\[
\frac{d}{dt} x = v \cos \theta, \quad \frac{d}{dt} y = v \sin \theta, \quad \frac{d}{dt} \theta = \omega
\]

unchanged:

\[
\frac{d}{dt} X = V \cos \Theta, \quad \frac{d}{dt} Y = V \sin \Theta, \quad \frac{d}{dt} \Theta = \Omega.
\]
Invariant tracking$^3$

Invariant tracking for the car: goal

Given the reference trajectory

\[ t \mapsto s_r \mapsto P_r(s_r), \quad \theta_r(s_r), \quad v_r = \dot{s}_r, \quad \omega_r = \dot{s}_r \kappa_r(s_r) \]

and the state \((P, \theta)\)

Find an invariant controller

\[ v = v_r + \ldots, \quad \omega = \omega_r + \ldots \]
Invariant tracking for the car: time-scaling

Set

\[ \nu = \bar{\nu} \dot{s}_r, \quad \omega = \bar{\omega} \dot{s}_r \]

and denote by \( \dot{\cdot} \) derivation versus \( s_r \).

Equations remain unchanged

\[ P' = \bar{\nu} \tau', \quad \tau' = \bar{\omega} \bar{\nu} \]

with \( P = (x, y), \quad \tau = (\cos \theta, \sin \theta) \) and \( \bar{\nu} = (- \sin \theta, \cos \theta) \).
Invariant errors

Construct the decoupling and/or linearizing controller with the two following invariant errors

\[ e_{\parallel} = (P - P_r) \cdot \vec{\tau}_r, \quad e_{\perp} = (P - P_r) \cdot \vec{\nu}_r. \]
Computations of $e_\parallel$ and $e_\bot$ derivatives

Since $e_\parallel = (P - P_r) \cdot \vec{\tau}_r$ and $e_\bot = (P - P_r) \cdot \vec{\nu}_r$ we have (remember that $' = d/ds_r$)

$$e'_\parallel = (P' - P'_r) \cdot \vec{\tau}_r + (P - P_r) \cdot \vec{\tau}'_r.$$  

But $P' = \vec{v}\vec{\tau}$, $P'_r = \vec{\tau}_r$ and $\vec{\tau}'_r = \kappa_r\vec{\nu}_r$, thus

$$e'_\parallel = \vec{v}\vec{\tau} \cdot \vec{\tau}_r - 1 + \kappa_r(P - P_r) \cdot \vec{\nu}_r.$$  

Similar computations for $e'_\bot$ yield:

$$e'_\parallel = \vec{v} \cos(\theta - \theta_r) - 1 + \kappa_r e_\bot, \quad e'_\bot = \vec{v} \sin(\theta - \theta_r) - \kappa_r e_\parallel.$$
Computations of $e_\parallel$ and $e_\perp$ second derivatives

Derivation of

$$e'_\parallel = \bar{v} \cos(\theta - \theta_r) - 1 + \kappa_r e_\perp, \quad e'_\perp = \bar{v} \sin(\theta - \theta_r) - \kappa_r e_\parallel$$

with respect to $s_r$ gives

$$e''_\parallel = \bar{v}' \cos(\theta - \theta_r) - \bar{\omega} \bar{v} \sin(\theta - \theta_r)$$
$$+ 2\kappa_r \bar{v} \sin(\theta - \theta_r) + \kappa'_r e_\perp - \kappa^2_r e_\parallel$$

$$e''_\perp = \bar{v}' \sin(\theta - \theta_r) + \bar{\omega} \bar{v} \cos(\theta - \theta_r)$$
$$- 2\kappa_r \bar{v} \cos(\theta - \theta_r) - \kappa'_r e_\parallel + \kappa_r + \kappa^2_r e_\parallel.$$
The dynamics feedback in $s_r$ time-scale

We have obtain

$$e''_\parallel = \bar{v}' \cos(\theta - \theta_r) - \bar{\omega} \bar{v} \sin(\theta - \theta_r)$$
$$+ 2\kappa_r \bar{v} \sin(\theta - \theta_r) + \kappa'_r e_\perp - \kappa^2_r e_\parallel$$

$$e''_\perp = \bar{v}' \sin(\theta - \theta_r) + \bar{\omega} \bar{v} \cos(\theta - \theta_r)$$
$$- 2\kappa_r \bar{v} \cos(\theta - \theta_r) - \kappa'_r e_\parallel + \kappa_r + \kappa^2_r e_\parallel.$$

Choose $\bar{v}'$ and $\bar{\omega}$ such that

$$e''_\parallel = - \left( \frac{1}{L^1_\parallel} + \frac{1}{L^2_\parallel} \right) e'_\parallel - \left( \frac{1}{L^1_\parallel L^2_\parallel} \right) e_\parallel$$

$$e''_\perp = - \left( \frac{1}{L^1_\perp} + \frac{1}{L^2_\perp} \right) e'_\perp - \left( \frac{1}{L^1_\perp L^2_\perp} \right) e_\perp$$

Possible around a large domain around the reference trajectory since the determinant of the decoupling matrix is $\bar{v} \approx 1.$
The dynamics feedback in physical time-scale

In the $s_r$ scale, we have the following dynamic feedback

$$
\bar{v}' = \Phi(\bar{v}, P, P_r, \theta, \theta_r, \kappa, \kappa')
$$

$$
\bar{\omega} = \Psi(\bar{v}, P, P_r, \theta, \theta_r, \kappa, \kappa')
$$

Since $' = d/ds_r = d/(\dot{s}_r dt)$ we have

$$
\frac{d\bar{v}}{dt} = \Phi(\bar{v}, P, P_r, \theta, \theta_r, \kappa, \kappa') \dot{s}_r(t)
$$

$$
\bar{\omega} = \Psi(\bar{v}, P, P_r, \theta, \theta_r, \kappa, \kappa')
$$

and the real control is

$$
v = \bar{v} \dot{s}_r(t), \quad \tan \phi = \frac{l \bar{\omega}}{\bar{v}}
$$

Nothing blows up when $\dot{s}_r(t)$ tends to 0: the controller is well defined around steady-state via a simple use of time-scaling symmetry
Conversion into chained form destroys $SE(2)$ invariance

The car model

\[
\frac{d}{dt} x = v \cos \theta, \quad \frac{d}{dt} y = v \sin \theta, \quad \frac{d}{dt} \theta = \frac{v}{l} \tan \varphi
\]

can be transformed into chained form

\[
\frac{d}{dt} x_1 = u_1, \quad \frac{d}{dt} x_2 = u_2, \quad \frac{d}{dt} x_3 = x_2 u_1
\]

via change of coordinates and static feedback

\[
x_1 = x, \quad x_2 = \frac{dy}{dx} = \tan \theta, \quad x_3 = y.
\]

But the symmetries are not preserved in such coordinates: one privileges axis $x$ versus axis $y$ without any good reason. The behavior of the system seems to depend on the origin you take to measure the angle (artificial singularity when $\theta = \pm \pi/2$).
The standard $n$-trailers system

\[ P_{n-1} = P_n + d_n \frac{dP_n}{ds_n} \]
Motion planning for the standard $n$ trailers system

initial state

final state

$P_n$

$(P_n, \frac{dP_n}{ds_n}, \ldots)$
The general 1-trailer system (CDC93)

Rolling without slipping conditions \((A = (x, y), \ u = (v, \varphi))\):

\[
\begin{align*}
\frac{dx}{dt} &= v \cos \alpha \\
\frac{dy}{dt} &= v \sin \alpha \\
\frac{d\alpha}{dt} &= \frac{v}{l} \tan \varphi \\
\frac{d\beta}{dt} &= \frac{v}{b} \left( \frac{a}{l} \tan \varphi \cos(\beta - \alpha) + \sin(\beta - \alpha) \right).
\end{align*}
\]
With \( \delta = \overrightarrow{BCA} \) we have

\[
D = P - L(\delta) \vec{\nu} \quad \text{with} \quad L(\delta) = ab \int_0^{\pi + \delta} \frac{-\cos \sigma}{\sqrt{a^2 + b^2 + 2ab \cos \sigma}} \, d\sigma
\]

Curvature is given by

\[
K(\delta) = \frac{\sin \delta}{\cos \delta \sqrt{a^2 + b^2 - 2ab \cos \delta} - L(\delta) \sin \delta}
\]
The geometric construction

Assume that \( s \mapsto P(s) \) is known. Let us show how to deduce \((A, B, \alpha, \beta)\) the system configuration.
We know thus \( P, \vec{\tau} = dP/ds \) and \( \kappa = d\theta/ds \) (\( \theta \) is the angle of \( \vec{\tau} \)):
From $\kappa$ we deduce $\delta = \widehat{BCA} = \widehat{BDA}$ by inverting $\kappa = K(\delta)$. $D$ is then known since $D = P - L(\delta)\vec{v}$. Finally $\vec{\tau}$ is parallel to $AB$ and $DB = a$ and $DA = b$. 

![Diagram](image-url)
The complete construction

One to one correspondence between $P$, $\vec{\tau}$ and $\kappa$ and $(A, \alpha, \beta)$. 
Differential forms

Eliminate $v$ from

$$\frac{d}{dt} x = v \cos \alpha, \quad \frac{d}{dt} y = v \sin \alpha, \quad \frac{d}{dt} \alpha = \frac{v}{l} \tan \varphi, \quad \frac{d}{dt} \beta = \ldots$$

to have 3 equations with 5 variables

$$\sin \alpha \frac{d}{dt} x - \cos \alpha \frac{d}{dt} y = 0$$
$$\frac{d}{dt} \alpha - \left( \frac{\tan \varphi \cos \alpha}{l} \right) \frac{d}{dt} x - \left( \frac{\tan \varphi \sin \alpha}{l} \right) \frac{d}{dt} y = 0$$
$$\frac{d}{dt} \beta \ldots$$

defining a module of differential forms, $I = \{\eta_1, \eta_2, \eta_3\}$

$$\eta_1 = \sin \alpha \, dx - \cos \alpha \, dy$$
$$\eta_2 = d\alpha - \left( \frac{\tan \varphi \cos \alpha}{l} \right) \, dx - \left( \frac{\tan \varphi \sin \alpha}{l} \right) \, dy$$
$$\eta_3 = d\beta - \ldots$$
Following 4, compute the sequence $I = I^{(0)} \supseteq I^{(1)} \supseteq I^{(2)} \ldots$

where

$$I^{(k+1)} = \{ \eta \in I^{(k)} \mid d\eta = 0 \mod (I^{(k)}) \}$$

and find that

$$\dim I^{(0)} = 3, \quad \dim I^{(1)} = 2, \quad \dim I^{(2)} = 1, \quad \dim I^{(3)} = 0.$$ 

The Cartesian coordinates $(X, Y)$ of $P$ are obtained via the Pfaff normal form of the differential form $\mu$ generating $I^{(2)}$

$$\mu = f(\alpha, \beta) \, dX + g(\alpha, \beta) \, dY.$$ 

$(X, Y)$ is not unique; $SE(2)$ invariance simplifies computations.

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Contact systems:

The driftless system \( \frac{d}{dt} x = f_1(x)u_1 + f_2(x)u_2 \) is also a Pfaffian system of codimension 2

\[
\omega_i \equiv \sum_{j=1}^{n} a_{ij}^i(x) \ dx_j = 0, \quad i = 1, \ldots, n - 2.
\]

Pfaffian systems equivalent via changes of \( x \)-coordinates to contact systems (related to chained-form, Murray-Sastry 1993)

\[
dx_2 - x_3 \ dx_1 = 0, \quad dx_3 - x_4 \ dx_1 = 0, \quad \ldots \ dx_{n-1} - x_n \ dx_1 = 0
\]

are mainly characterized by the derived flag (Weber(1898), Cartan(1916), Goursat (1923), Giaro-Kumpera-Ruiz(1978), Murray (1994), Pasillas-Respondek (2000), \ldots).
Interest of contact systems (chained form):

\[ dx_2 - x_3 dx_1 = 0, \quad dx_3 - x_4 dx_1 = 0, \quad \ldots \quad dx_{n-1} - x_n dx_1 = 0 \]

The general solution reads in terms of \( z \mapsto w(z) \) and its derivatives,

\[ x_1 = z, \quad x_2 = w(z), \quad x_3 = \frac{dw}{dz}, \quad \ldots \quad , \quad x_n = \frac{d^{n-2}w}{dz^{n-2}}. \]

In this case, the general solution of \( \frac{d}{dt} x = f_1(x)u_1 + f_2u_2 \) reads in terms of \( t \mapsto z(t) \) any \( C^1 \) time function and any \( C^{n-2} \) function of \( z \), \( z \mapsto w(z) \). The quantities \( x_1 = z(t) \) and \( x_2 = w(z(t)) \) play here a special role. We call them the flat output.
An elementary definition based on inversion

- **Explicit control systems:** $\frac{d}{dt} x = f(x, u)$ ($x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$) is flat, iff, exist $\alpha \in \mathbb{N}$ and $h(x, u, \ldots, u^{(\alpha)}) \in \mathbb{R}^m$ such that the generic solution of

$$\frac{d}{dt} x = f(x, u), \quad y = h(x, u, \ldots, u^{(\alpha)})$$

reads ($\beta \in \mathbb{N}$)

$$x = A(y, \ldots, y^{(\beta)}), \quad u = B(y, \ldots, y^{(\beta+1)})$$

- **Under-determined systems:** $F(x, \ldots, x^{(r)}) = 0$ ($x \in \mathbb{R}^n$, $F \in \mathbb{R}^{n-m}$) is flat, iff, exist $\alpha \in \mathbb{N}$ and $h(x, \ldots, x^{(\alpha)}) \in \mathbb{R}^m$ such that the generic solution of

$$F(x, \ldots, x^{(r)}) = 0, \quad y = h(x, \ldots, x^{(\alpha)}) \quad \text{reads} \quad x = A(y, \ldots, y^{(\beta)})$$

$y$ is called a **flat output**: Fliess and co-workers 1991, ....

**Integrable** under-determined differential systems: Monge (1784), Darboux, Goursat, Hilbert (1912), Cartan (1914).
Flat systems (Fliess-et-al, 1992,…,1999)

A basic definition extending remark of Isidori-Moog-DeLuca (CDC86) on dynamic feedback linearization (Charlet-Lévine-Marino (1989)):

\[
\frac{d}{dt} x = f(x, u)
\]

is flat, iff, exist \( m = \dim(u) \) output functions \( y = h(x, u, \ldots, u^{(p)}) \), \( \dim(h) = \dim(u) \), such that the inverse of \( u \mapsto y \) has no dynamics, i.e.,

\[
x = \Lambda \left( y, \dot{y}, \ldots, y^{(q)} \right), \quad u = \Upsilon \left( y, \dot{y}, \ldots, y^{(q+1)} \right).
\]

Take $\frac{d}{dt} x = f(x, u)$, $(x, u) \in X \times U \subset \mathbb{R}^n \times \mathbb{R}^m$. It generates a system $(F, M)$, (D-variety) where

$$M := X \times U \times \mathbb{R}_m^\infty$$

with the vector field $F(x, u, u^1, \ldots) := (f(x, u), u^1, u^2, \ldots)$. $(F, M)$ is equivalent to $(G, N)$ ($\dot{z} = g(z, v)$: $N := Z \times V \times \mathbb{R}_m^\infty$ with the vector field $G(z, v, v^1, \ldots) := (g(z, v), v^1, v^2, \ldots)$) iff there exists an invertible transformation $\Phi : M \mapsto N$ such that

$$\forall \xi := (x, u, u^1, \ldots) \in M, \quad G(\Phi(\xi)) = D\Phi(\xi) \cdot F(\xi).$$
Equivalence and flatness (extrinsic point of view)

Elimination of $u$ from the $n$ state equations $\frac{d}{dt}x = f(x, u)$ provides an under-determinate system of $n - m$ equations with $n$ unknowns

$$ F \left( x, \frac{d}{dt}x \right) = 0. $$

An endogenous transformation $x \mapsto z$ is defined by

$$ z = \Phi(x, \dot{x}, \ldots, x^{(p)}), \quad x = \Psi(z, \dot{z}, \ldots, z^{(q)}) $$

(nonlinear analogue of uni-modular matrices, the "integral free" transformations of Hilbert).

Two systems are equivalents, iff, exists an endogenous transformation exchanging the equations.

A system equivalent to the trivial equation $z_1 = 0$ with $z = (z_1, z_2)$ is flat with $z_2$ the flat output.
The time dependent definition

We present here the simplest version of this definition (Murray and co-workers (SIAM JCO 1998)):

\[ \frac{d}{dt} x = f(t, x, u) \]

is flat, iff, exist \( m = \dim(u) \) output functions \( y = h(t, x, u, \ldots, u^{(p)}) \), \( \dim(h) = \dim(u) \), such that the inverse of \( u \mapsto y \) has no dynamics, i.e.,

\[ x = \Lambda \left( t, y, \dot{y}, \ldots, y^{(q)} \right), \quad u = \Upsilon \left( t, y, \dot{y}, \ldots, y^{(q+1)} \right). \]
The general \( n \)-trailer system for \( n \geq 2 \) is not flat.

Proof: by pure chance, the characterization of codimension 2 contact systems is also a characterization of drifless flat systems (Cartan 1914, Martin-R. 1994) (adding integrator, endogenous or exogenous or singular dynamic feedbacks are useless here).
When the number $n$ of trailers becomes large...
The nonholonomic snake: a trivial delay system.

Implicit partial differential nonlinear system:

\[ \left\| \frac{\partial P}{\partial r} \right\| = 1, \quad \frac{\partial P}{\partial r} \wedge \frac{\partial P}{\partial t} = 0. \]

General solution via \( s \mapsto Q(s) \) arbitrary smooth:

\[ P(r, t) = Q(s(t) + L - r) \equiv \sum_{k \geq 0} \frac{(L - r)^k}{k!} \frac{dQ^k}{ds^k}(s(t)). \]
Two linearized pendulum in series

Flat output $y = u + l_1 \theta_1 + l_2 \theta_2$:

$$\theta_2 = -\frac{\ddot{y}}{g}, \quad \theta_1 = -\frac{m_1(y - l_2 \theta_2)}{(m_1 + m_2)g} + \frac{m_2}{m_1 + m_2} \theta_2$$

and $u = y - l_1 \theta_1 - l_2 \theta_2$ is a linear combination of $(y, y^{(2)}, y^{(4)})$. 
Flat output $y = u + l_1 \theta_1 + \ldots + l_n \theta_n$:

$$u = y + a_1 y^{(2)} + a_2 y^{(4)} + \ldots + a_n y^{(2n)}.$$ 

When $n$ tends to $\infty$ the system tends to a partial differential equation.
The heavy chain

\[ \frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial z} \left( g \frac{\partial X}{\partial z} \right) \]

\[ X(L, t) = U(t) \]

Flat output \( y(t) = X(0, t) \) with

\[ U(t) = \frac{1}{2\pi} \int_0^{2\pi} y \left( t - 2\sqrt{L/g} \sin \zeta \right) d\zeta \]

---

With the same flat output, for a discrete approximation \((n \text{ pendulums in series, } n \text{ large})\) we have

\[u(t) = y(t) + a_1 \ddot{y}(t) + a_2 y^{(4)}(t) + \ldots + a_n y^{(2n)}(t),\]

for a continuous approximation (the heavy chain) we have

\[U(t) = \frac{1}{2\pi} \int_0^{2\pi} y \left( t + 2\sqrt{\frac{L}{g}} \sin \zeta \right) d\zeta.\]

Why? Because formally

\[y(t + 2\sqrt{\frac{L}{g}} \sin \zeta) = y(t) + \ldots + \frac{\left(2\sqrt{\frac{L}{g}} \sin \zeta\right)^n}{n!} y^{(n)}(t) + \ldots\]

But integral formula is preferable (divergence of the series\.\.\.).
The general solution of the PDE
\[ \frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial z} \left( gz \frac{\partial X}{\partial z} \right) \]
is
\[ X(z, t) = \frac{1}{2\pi} \int_0^{2\pi} y \left( t - 2\sqrt{z/g} \sin \zeta \right) d\zeta \]
where \( t \mapsto y(t) \) is any time function.

Proof: replace \( \frac{d}{dt} \) by \( s \), the Laplace variable, to obtain a singular second order ODE in \( z \) with bounded solutions. Symbolic computations and operational calculus on
\[ s^2 X = \frac{\partial}{\partial z} \left( gz \frac{\partial X}{\partial z} \right). \]
Symbolic computations in the Laplace domain

Thanks to $x = 2\sqrt{\frac{z}{g}}$, we get

$$x \frac{\partial^2 X}{\partial x^2}(x, t) + \frac{\partial X}{\partial x}(x, t) - x \frac{\partial^2 X}{\partial t^2}(x, t) = 0.$$  

Use Laplace transform of $X$ with respect to the variable $t$

$$x \frac{\partial^2 \hat{X}}{\partial x^2}(x, s) + \frac{\partial \hat{X}}{\partial x}(x, s) - xs^2 \hat{X}(x, s) = 0.$$  

This is the Bessel equation defining $J_0$ and $Y_0$:

$$\hat{X}(z, s) = A(s) \ J_0(2\sqrt{s} \sqrt{z/g}) + B(s) \ Y_0(2\sqrt{s} \sqrt{z/g}).$$  

Since we are looking for a bounded solution at $z = 0$ we have $B(s) = 0$ and (remember that $J_0(0) = 1$):

$$\hat{X}(z, s) = J_0(2\sqrt{s} \sqrt{z/g}) \hat{X}(0, s).$$
\[ \hat{X}(z, s) = J_0(2i s \sqrt{z/g}) \hat{X}(0, s). \]

Using Poisson’s integral representation of \( J_0 \)

\[ J_0(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} \exp(i\zeta \sin \theta) \, d\theta, \quad \zeta \in \mathbb{C} \]

we have

\[ J_0(2i s \sqrt{z/g}) = \frac{1}{2\pi} \int_0^{2\pi} \exp(2s \sqrt{z/g} \sin \theta) \, d\theta. \]

In terms of Laplace transforms, this last expression is a combination of delay operators:

\[ X(z, t) = \frac{1}{2\pi} \int_0^{2\pi} y(t + 2 \sqrt{z/g} \sin \theta) \, d\theta \]

with \( y(t) = X(0, t) \).
Explicit parameterization of the heavy chain

The general solution of

\[
\frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial z} \left( gz \frac{\partial X}{\partial z} \right), \quad U(t) = X(L, t)
\]

reads

\[
X(z, t) = \frac{1}{2\pi} \int_0^{2\pi} y(t + 2\sqrt{z/g \sin \theta}) \, d\theta
\]

There is a one to one correspondence between the (smooth) solutions of the PDE and the (smooth) functions \( t \mapsto y(t) \).
Heavy chain with a variable section

\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{\tau'(z)}{g} \frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial z} \left( \tau(z) \frac{\partial X}{\partial z} \right) \\
X(L, t) = u(t)
\end{array} \right.
\end{aligned}
\]
The general solution of

\[
\begin{align*}
\left\{ \frac{\tau'(z)}{g} \frac{\partial^2 X}{\partial t^2} &= \frac{\partial}{\partial z} \left( \tau(z) \frac{\partial X}{\partial z} \right) \\
X(L, t) &= u(t)
\right. \right.
\end{align*}
\]

where \( \tau(z) \geq 0 \) is the tension in the rope, can be parameterized by an arbitrary time function \( y(t) \), the position of the free end of the system \( y = X(0, t) \), via delay and advance operators with compact support.
Sketch of the proof.

Main difficulty: $\tau(0) = 0$. The bounded solution $B(z, s)$ of

$$
\frac{\partial}{\partial z} \left( \tau(z) \frac{\partial X}{\partial z} \right) = \frac{s^2 \tau'(z)}{g} X
$$

is an entire function of $s$, is of exponential type and

$$
\mathbb{R} \ni \omega \mapsto B(z, \omega)
$$

is $L^2$ modulo some $J_0$. By the Paley-Wiener theorem $B(z, s)$ can be described via

$$
\int_a^b K(z, \zeta) \exp(s\zeta) \, d\zeta.
$$
is equivalent to

\[
\frac{d^2 x}{dt^2} = \frac{g}{l}(u - x)
\]

The following maps exchange the trajectories:

\[
\begin{align*}
X(z, t) &= \frac{1}{2\pi} \int_0^{2\pi} x \left( t - 2\sqrt{z/g \sin \zeta} \right) \, d\zeta \\
U(t) &= \frac{1}{2\pi} \int_0^{2\pi} x \left( t - 2\sqrt{L/g \sin \zeta} \right) \, d\zeta
\end{align*}
\]
The Indian rope.

\[
\frac{\partial}{\partial z} \left( gz \frac{\partial X}{\partial z} \right) + \frac{\partial^2 X}{\partial t^2} = 0
\]

\[
X(L, t) = U(t)
\]

The equation becomes elliptic and the Cauchy problem is not well posed in the sense of Hadamard. Nevertheless formulas are still valid with a complex time and \( y \) holomorphic

\[
X(z, t) = \frac{1}{2\pi} \int_0^{2\pi} y \left( t - \left( 2\sqrt{z/g} \sin \zeta \right) \sqrt{-1} \right) d\zeta.
\]
A computation due to Holmgren\textsuperscript{6}

Take the 1D-heat equation, $\frac{\partial \theta}{\partial t}(x, t) = \frac{\partial^2 \theta}{\partial x^2}(x, t)$ for $x \in [0, 1]$ and set, formally, $\theta = \sum_{i=0}^{\infty} a_i(t) \frac{x_i}{i!}$. Since,

$$\frac{\partial \theta}{\partial t} = \sum_{i=0}^{\infty} \frac{da_i}{dt} \left( \frac{x_i}{i!} \right), \quad \frac{\partial^2 \theta}{\partial x^2} = \sum_{i=0}^{\infty} a_{i+2} \left( \frac{x_i}{i!} \right)$$

the heat equation $\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}$ reads $\frac{d}{dt} a_i = a_{i+2}$ and thus

$$a_{2i+1} = a_1^{(i)}, \quad a_{2i} = a_0^{(i)}$$

With two arbitrary smooth time-functions $f(t)$ and $g(t)$, playing the role of $a_0$ and $a_1$, the general solution reads:

$$\theta(x, t) = \sum_{i=0}^{\infty} f^{(i)}(t) \left( \frac{x^{2i}}{(2i)!} \right) + g^{(i)}(t) \left( \frac{x^{2i+1}}{(2i+1)!} \right).$$

Convergence issues?

Gevrey functions

- A $C^\infty$-function $[0, T] \ni t \mapsto f(t)$ is of Gevrey-order $\alpha$ when,
  \[ \exists \, M, A > 0, \quad \forall \, t \in [0, T], \forall \, i \geq 0, \quad |f^{(i)}(t)| \leq MA^i \Gamma(1 + \alpha i) \]
  where $\Gamma$ is the gamma function with $n! = \Gamma(n + 1)$, $\forall \, n \in \mathbb{N}$.
- Analytic functions correspond to Gevrey-order $\leq 1$.
- When $\alpha > 1$, the set of $C^\infty$-functions with Gevrey-order $\alpha$ contains non-zero functions with compact supports. Prototype of such functions:
  \[ t \mapsto f(t) = \begin{cases} 
  \exp \left( - \left( \frac{1}{t(1-t)} \right)^{-\frac{1}{\alpha-1}} \right) & \text{if } t \in ]0, 1[ \\
  0 & \text{otherwise.} 
  \end{cases} \]

---

Gevrey functions and exponential decay

- Take, in the complex plane, the open bounded sector $S$ whose vertex is the origin. Assume that $f$ is analytic on $S$ and admits an exponential decay of order $\sigma > 0$ and type $A$ in $S$:

$$\exists C, \rho > 0, \quad \forall z \in S, \quad |f(z)| \leq C|z|^{\rho} \exp \left( \frac{-1}{A|z|^{\sigma}} \right)$$

Then in any closed sub-sector $\tilde{S}$ of $S$ with origin as vertex, exists $M > 0$ such that

$$\forall z \in \tilde{S}/\{0\}, \quad |f^{(i)}(z)| \leq MA^{i} \Gamma \left( 1 + i \left( \frac{1}{\sigma} + 1 \right) \right)$$

- **Rule of thumb**: if a piece-wise analytic $f$ admits an exponential decay of order $\sigma$ then it is of Gevrey-order

$$\alpha = \frac{1}{\sigma} + 1.$$  

---

Gevrey space and ultra-distributions\(^9\)

Denote by \( \mathcal{D}_\alpha \) the set of functions \( \mathbb{R} \mapsto \mathbb{R} \) of order \( \alpha > 1 \) and with compact supports. As for the class of \( C^\infty \) functions, most of the usual manipulations remain in \( \mathcal{D}_\alpha \):

- \( \mathcal{D}_\alpha \) is stable by addition, multiplication, derivation, integration, ....
- if \( f \in \mathcal{D}_\alpha \) and \( F \) is an analytic function on the image of \( f \), then \( F(f) \) remains in \( \mathcal{D}_\alpha \).
- if \( f \in \mathcal{D}_\alpha \) and \( F \in L^1_{loc}(\mathbb{R}) \) then the convolution \( f \ast F \) is of Gevrey-order \( \alpha \) on any compact interval.

As for the construction of \( \mathcal{D}' \), the space of distributions (the dual of \( \mathcal{D} \) the space of \( C^\infty \) functions of compact supports), one can construct \( \mathcal{D}'_\alpha \supset \mathcal{D}' \), a space of **ultra-distributions**, the dual of \( \mathcal{D}_\alpha \subset \mathcal{D} \).

Symbolic computations: $s := d/dt, s \in \mathbb{C}$

The general solution of $\theta'' = s\theta$ reads ($':= d/dx$)

$$\theta = \cosh(x\sqrt{s}) \ f(s) + \frac{\sinh(x\sqrt{s})}{\sqrt{s}} \ g(s)$$

where $f(s)$ and $g(s)$ are the two constants of integration. Since cosh and sinh gather the even and odd terms of the series defining exp, we have

$$\cosh(x\sqrt{s}) = \sum_{i \geq 0} s^i \frac{x^{2i}}{(2i)!}, \quad \frac{\sinh(x\sqrt{s})}{\sqrt{s}} = \sum_{i \geq 0} s^i \frac{x^{2i+1}}{(2i + 1)!}$$

and we recognize $\theta = \sum_{i=0}^{\infty} f^{(i)}(t) \left(\frac{x^{2i}}{(2i)!}\right) + g^{(i)}(t) \left(\frac{x^{2i+1}}{(2i+1)!}\right)$. For each $x$, the operators $\cosh(x\sqrt{s})$ and $\sinh(x\sqrt{s})/\sqrt{s}$ are ultra-distributions of $\mathcal{D}'_2$:

$$\sum_{i \geq 0} \frac{(-1)^i x^{2i}}{(2i)!} \delta^{(i)}(t), \quad \sum_{i \geq 0} \frac{(-1)^i x^{2i+1}}{(2i + 1)!} \delta^{(i)}(t)$$

with $\delta$, the Dirac distribution.
Entire functions of $s = d/dt$ as ultra-distributions

- $\mathbb{C} \ni s \mapsto P(s) = \sum_{i \geq 0} a_i s^i$ is an entire function when the radius of convergence is infinite.

- If its order at infinity is $\sigma > 0$ and its type is finite, i.e., $\exists M, K > 0$ such that $\forall s \in \mathbb{C}$, $|P(s)| \leq M \exp(K|s|^\sigma)$, then

  $$\exists A, B > 0 \mid \forall i \geq 0, \quad |a_i| \leq A \frac{B^i}{\Gamma(i/\sigma + 1)}.$$  

cosh($\sqrt{s}$) and sinh($\sqrt{s}$)/$\sqrt{s}$ are entire functions of order $\sigma = 1/2$ and of type 1.

- Take $P(s)$ of order $\sigma < 1$ with $s = d/dt$. Then $P \in \mathcal{D}'_{1/\sigma}$: $P(s)f(s)$ corresponds, in the time domain, to absolutely convergent series

  $$P(s)y(s) \equiv \sum_{i=0}^{\infty} a_i f^{(i)}(t)$$

  when $t \mapsto f(t)$ is a $C^\infty$-function of Gevrey-order $\alpha < 1/\sigma$. 
Motion planning for the 1D heat equation

\[ \frac{\partial x}{\partial x} \theta(0, t) = 0 \]
\[ \theta(x, t) \]
\[ \theta(1, t) = u \]

The data are:

1. the model relating the control input \( u(t) \) to the state, \( (\theta(x, t))_{x \in [0,1]} \):
\[
\frac{\partial \theta}{\partial t}(x, t) = \frac{\partial^2 \theta}{\partial x^2}(x, t), \quad x \in [0, 1] \\
\frac{\partial \theta}{\partial x}(0, t) = 0 \quad \theta(1, t) = u(t).
\]

2. A transition time \( T > 0 \), the initial (resp. final) state:
\[ [0, 1] \ni x \mapsto p(x) \text{ (resp. } q(x)) \]

The goal is to find the open-loop control \( [0, T] \ni t \mapsto u(t) \) steering \( \theta(x, t) \) from the initial profile \( \theta(x, 0) = p(x) \) to the final profile \( \theta(x, T) = q(x) \).
Series solutions

Set, formally

\[\theta = \sum_{i=0}^{\infty} a_i(t) \frac{x^i}{i!}, \quad \frac{\partial \theta}{\partial t} = \sum_{i=0}^{\infty} \frac{da_i}{dt} \left( \frac{x^i}{i!} \right), \quad \frac{\partial^2 \theta}{\partial x^2} = \sum_{i=0}^{\infty} a_{i+2} \left( \frac{x^i}{i!} \right)\]

and \(\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}\) reads \(\frac{d}{dt} a_i = a_{i+2}\). Since \(a_1 = \frac{\partial \theta}{\partial x} (0, t) = 0\) and \(a_0 = \theta(0, t)\) we have

\[a_{2i+1} = 0, \quad a_{2i} = a_0^{(i)}\]

Set \(y := a_0 = \theta(0, t)\) we have, in the time domain,

\[\theta(x, t) = \sum_{i=0}^{\infty} \left( \frac{x^{2i}}{(2i)!} \right) y^{(i)}(t), \quad u(t) = \sum_{i=0}^{\infty} \left( \frac{1}{(2i)!} \right) y^{(i)}(t)\]

that also reads in the Laplace domain \((s = d/dt)\):

\[\theta(x, s) = \cosh(x \sqrt{s}) \ y(s), \quad u(s) = \cosh(\sqrt{s})y(s)\].
An explicit parameterization of trajectories

For any $C^\infty$-function $y(t)$ of Gevrey-order $\alpha < 2$, the time function

$$u(t) = \sum_{i=1}^{+\infty} \frac{y^{(i)}(t)}{(2i)!}$$

is well defined and smooth. The $(x, t)$-function

$$\theta(x, t) = \sum_{i=1}^{+\infty} \frac{y^{(i)}(t)}{(2i)!} x^{2i}$$

is also well defined (entire versus $x$ and smooth versus $t$). Moreover for all $t$ and $x \in [0, 1]$, we have, whatever $t \mapsto y(t)$ is,

$$\frac{\partial \theta}{\partial t}(x, t) = \frac{\partial^2 \theta}{\partial x^2}(x, t), \quad \frac{\partial \theta}{\partial x}(0, t) = 0, \quad \theta(1, t) = u(t)$$

An infinite dimensional analogue of differential flatness.\(^{10}\)

Motion planning of the heat equation

Take \( \sum_{i \geq 0} a_i \frac{\xi^i}{i!} \) and \( \sum_{i \geq 0} b_i \frac{\xi^i}{i!} \) entire functions of \( \xi \). With \( \sigma > 1 \)

\[
y(t) = \left( \sum_{i \geq 0} a_i \frac{t^i}{i!} \right) \left( e^{\frac{-T^\sigma}{(T-t)^\sigma}} \right) + \left( \sum_{i \geq 0} b_i \frac{t^i}{i!} \right) \left( e^{\frac{-T^\sigma}{t^\sigma}} + e^{\frac{-T^\sigma}{(T-t)^\sigma}} \right)
\]

the series

\[
\theta(x, t) = \sum_{i=1}^{+\infty} \frac{y^{(i)}(t)}{(2i)!} x^{2i} , \quad u(t) = \sum_{i=1}^{+\infty} \frac{y^{(i)}(t)}{(2i)!} .
\]

are convergent and provide a trajectory from

\[
\theta(x, 0) = \sum_{i \geq 0} a_i \frac{x^{2i}}{(2i)!} \quad \text{to} \quad \theta(x, T) = \sum_{i \geq 0} b_i \frac{x^{2i}}{(2i)!}
\]

Real-time motion planning for the heat equation

Take $\sigma > 1$ and $\epsilon > 0$. Consider the positive function

$$\phi_\epsilon(t) = \frac{\exp\left(-\frac{\epsilon^2}{(t(t+\epsilon))^\sigma}\right)}{A_\epsilon}$$

for $t \in [-\epsilon, 0]$ prolonged by 0 outside $[-\epsilon, 0]$ and where the normalization constant $A_\epsilon > 0$ is such that $\int \phi_\epsilon = 1$.

For any $L^1_{loc}$ signal $t \mapsto Y(t)$, set $y_r = \phi_\epsilon \ast Y$: its order $1 + 1/\sigma$ is less than 2. Then $\theta_r = \cosh(\sqrt{s})y_r$ reads

$$\theta_r(x, t) = \Phi_{x, \epsilon} \ast Y(t), \quad u_r(t) = \Phi_{1, \epsilon} \ast Y(t),$$

where for each $x$, $\Phi_{x, \epsilon} = \cosh(\sqrt{s})\phi_\epsilon$ is a smooth time function with support contained in $[-\epsilon, 0]$. Since $u_r(t)$ and the profile $\theta_r(\cdot, t)$ depend only on the values of $Y$ on $[t - \epsilon, t]$, such computations are well adapted to real-time generation of reference trajectories $t \mapsto (\theta_r, u_r)$ (see matlab code \texttt{heat.m}).
Quantum particle inside a moving box

Schrödinger equation in a Galilean frame:

\[ i \frac{\partial \phi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \phi}{\partial z^2}, \quad z \in [v - \frac{1}{2}, v + \frac{1}{2}] \]

\[ \phi(v - \frac{1}{2}, t) = \phi(v + \frac{1}{2}, t) = 0 \]

Particle in a moving box of position $v$

- In a Galilean frame

$$\frac{\partial \phi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \phi}{\partial z^2}, \quad z \in [v - \frac{1}{2}, v + \frac{1}{2}],$$

$$\phi(v - \frac{1}{2}, t) = \phi(v + \frac{1}{2}, t) = 0$$

where $v$ is the position of the box and $z$ is an absolute position.

- In the box frame $x = z - v$:

$$\frac{\partial \psi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + \ddot{v}x \psi, \quad x \in [-\frac{1}{2}, \frac{1}{2}],$$

$$\psi(-\frac{1}{2}, t) = \psi(\frac{1}{2}, t) = 0$$
Tangent linearization around state $\bar{\psi}$ of energy $\bar{\omega}$

With\textsuperscript{13} $-\frac{1}{2} \frac{\partial^2 \bar{\psi}}{\partial x^2} = \bar{\omega} \bar{\psi}$, $\bar{\psi}(-\frac{1}{2}) = \bar{\psi}(\frac{1}{2}) = 0$ and with

$$\psi(x, t) = \exp(-i\bar{\omega}t)(\bar{\psi}(x) + \Psi(x, t))$$

$\Psi$ satisfies

$$i \frac{\partial \psi}{\partial t} + \bar{\omega} \psi = -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + \ddot{v}x(\bar{\psi} + \Psi)$$

$$0 = \psi(-\frac{1}{2}, t) = \psi(\frac{1}{2}, t).$$

Assume $\psi$ and $\ddot{v}$ small and neglecte the second order term $\ddot{v}x\psi$:

$$i \frac{\partial \psi}{\partial t} + \bar{\omega} \psi = -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + \ddot{v}x\bar{\psi}, \quad \psi(-\frac{1}{2}, t) = \psi(\frac{1}{2}, t) = 0.$$

\textsuperscript{13}Remember that $\int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{\psi}^2(x)dx = 1$. 

Operational computations \( s = d/dt \)

The general solution of (' stands for \( d/dx \))

\[
(\imath s + \bar{\omega})\psi = -\frac{1}{2} \psi'' + s^2 v x \bar{\psi}
\]

is

\[
\psi = A(s, x)a(s) + B(s, x)b(s) + C(s, x)\nu(s)
\]

where

\[
A(s, x) = \cos \left( x\sqrt{2\imath s + 2\bar{\omega}} \right)
\]

\[
B(s, x) = \frac{\sin \left( x\sqrt{2\imath s + 2\bar{\omega}} \right)}{\sqrt{2\imath s + 2\bar{\omega}}}
\]

\[
C(s, x) = (\imath s x \bar{\psi}(x) + \bar{\psi}'(x)).
\]
Case $x \mapsto \bar{\phi}(x)$ even

The boundary conditions imply

$$A(s, 1/2)a(s) = 0, \quad B(s, 1/2)b(s) = -\psi'(1/2)v(s).$$

$a(s)$ is a torsion element: the system is not controllable. Nevertheless, for steady-state controllability, we have

$$b(s) = -\bar{\psi}'(1/2) \frac{\sin \left(\frac{1}{2} \sqrt{-2\iota s + 2\bar{\omega}}\right)}{\sqrt{-2\iota s + 2\bar{\omega}}} y(s)$$

$$v(s) = \frac{\sin \left(\frac{1}{2} \sqrt{2\iota s + 2\bar{\omega}}\right)}{\sqrt{2\iota s + 2\bar{\omega}}} \frac{\sin \left(\frac{1}{2} \sqrt{-2\iota s + 2\bar{\omega}}\right)}{\sqrt{-2\iota s + 2\bar{\omega}}} y(s)$$

$$\Psi(s, x) = B(s, x)b(s) + C(s, x)v(s)$$
Series and convergence

\[ \nu(s) = \frac{\sin \left( \frac{1}{2} \sqrt{2} s + 2 \bar{\omega} \right)}{\sqrt{2} s + 2 \bar{\omega}} \frac{\sin \left( \frac{1}{2} \sqrt{-2} s + 2 \bar{\omega} \right)}{\sqrt{-2} s + 2 \bar{\omega}} y(s) = F(s)y(s) \]

where the entire function \( s \mapsto F(s) \) is of order \( \frac{1}{2} \),

\[ \exists K, M > 0, \forall s \in \mathbb{C}, \quad |F(s)| \leq K \exp(M|s|^{1/2}). \]

Set \( F(s) = \sum_{n \geq 0} a_n s^n \) where \( |a_n| \leq K^n / \Gamma(1 + 2n) \) with \( K > 0 \) independent of \( n \). Then \( F(s)y(s) \) corresponds, in the time domain, to

\[ \sum_{n \geq 0} a_n y^{(n)}(t) \]

that is convergent when \( t \mapsto y(t) \) is \( C^\infty \) of Gevrey-order \( \alpha < 2 \).
Steady state controllability

Steering from $\Psi = 0$, $v = 0$ at time $t = 0$, to $\Psi = 0$, $v = D$ at $t = T$ is possible with the following $C^\infty$-function of Gevrey-order $\sigma + 1$:

$$[0, T] \ni t \mapsto y(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ \bar{D} \frac{\exp\left(-\left(\frac{T}{t}\right)^{\frac{1}{\sigma}}\right)}{\exp\left(-\left(\frac{T}{t}\right)^{\frac{1}{\sigma}}\right) + \exp\left(-\left(\frac{T}{T-t}\right)^{\frac{1}{\sigma}}\right)} & \text{for } 0 < t < T \\ \bar{D} & \text{for } t \geq T \end{cases}$$

with $\bar{D} = \frac{2\bar{\omega}D}{\sin^2(\sqrt{\bar{\omega}}/2)}$. The fact that this $C^\infty$-function is of Gevrey-order $\sigma + 1$ results from its exponential decay of order $1/\sigma$ around 0 and $T$. 
Practical computations via Cauchy formula

Using the "magic" Cauchy formula

\[ y^{(n)}(t) = \frac{\Gamma(n+1)}{2i\pi} \oint_{\gamma} \frac{y(t + \xi)}{\xi^{n+1}} \, d\xi \]

where \( \gamma \) is a closed path around zero, \( \sum_{n \geq 0} a_n y^{(n)}(t) \) becomes

\[ \sum_{n \geq 0} a_n \frac{\Gamma(n+1)}{2i\pi} \oint_{\gamma} \frac{y(t + \xi)}{\xi^{n+1}} \, d\xi = \frac{1}{2i\pi} \oint_{\gamma} \left( \sum_{n \geq 0} a_n \frac{\Gamma(n+1)}{\xi^{n+1}} \right) y(t + \xi) \, d\xi. \]

But

\[ \sum_{n \geq 0} a_n \frac{\Gamma(n+1)}{\xi^{n+1}} = \int_{D_\delta} F(s) \exp(-s\xi) \, ds = B_1(F)(\xi) \]

is the Borel/Laplace transform of \( F \) in direction \( \delta \in [0, 2\pi] \).
Practical computations via Cauchy formula (end)
(matlab code Qbox.m)

In the time domain $F(s)y(s)$ corresponds to

$$\frac{1}{2\iota \pi} \oint_{\gamma} B_1(F)(\xi)y(t + \xi) \, d\xi$$

where $\gamma$ is a closed path around zero. Such integral representation is very useful when $y$ is defined by convolution with a real signal $Y$,

$$y(\zeta) = \frac{1}{\varepsilon \sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(\frac{-\zeta^2}{2\varepsilon^2}\right) Y(t) \, dt$$

where $\mathbb{R} \ni t \mapsto Y(t) \in \mathbb{R}$ is any measurable and bounded function. Approximate motion planning with:

$$v(t) = \int_{-\infty}^{+\infty} \left[ \frac{1}{\iota \varepsilon (2\pi)^{3/2}} \oint_{\gamma} B_1(F)(\xi) \exp\left(-\frac{(\xi - \tau)^2}{2\varepsilon^2}\right) \, d\xi \right] Y(t - \tau) \, d\tau.$$
A free-boundary Stefan problem

\[ \frac{\partial \theta}{\partial t}(x, t) = \frac{\partial^2 \theta}{\partial x^2}(x, t) - \nu \frac{\partial \theta}{\partial x}(x, t) - \rho \theta^2(x, t), \quad x \in [0, y(t)] \]

\[ \theta(0, t) = u(t), \quad \theta(y(t), t) = 0 \]

\[ \frac{\partial \theta}{\partial x}(y(t), t) = -\frac{d}{dt}y(t) \]

with \( \nu, \rho \geq 0 \) parameters.

---

Series solutions

- Set \( \theta(x, t) = \sum_{i=0}^{\infty} a_i(t) \frac{(x-y(t))^i}{i!} \) in

\[
\frac{\partial \theta}{\partial t}(x, t) = \frac{\partial^2 \theta}{\partial x^2}(x, t) - \nu \frac{\partial \theta}{\partial x}(x, t) - \rho \theta^2(x, t), \quad x \in [0, y(t)]
\]

\[
\theta(0, t) = u(t), \quad \theta(y(t), t) = 0, \quad \frac{\partial \theta}{\partial x}(y(t), t) = -\frac{d}{dt} y(t)
\]

Then \( \frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} \) yields

\[
a_{i+2} = \frac{d}{dt} a_i - a_{i-1} \frac{d}{dt} y + \nu a_{i+1} + \rho \sum_{k=0}^{i} \binom{i}{k} a_{i-k} a_k
\]

and the boundary conditions: \( a_0 = 0 \) and \( a_1 = -\frac{d}{dt} y \).

- The series defining \( \theta \) admits a strictly positive radius of convergence as soon as \( y \) is of Gevrey-order \( \alpha \) strictly less than 2.
Growth of the liquide zone with $\theta \geq 0$

$\nu = 0.5$, $\rho = 1.5$, $y$ goes from 1 to 2.
Conclusion for PDE

- For other 1D PDE of engineering interest with motion planning see the book of J. Rudolph: Flatness Based Control of Distributed Parameter Systems (Shaker-Germany, 2003)
- For tracking and feedback stabilization on linear 1D diffusion and wave equations, see the book of M. Krstić and A. Smyshlyaev: Boundary Control of PDEs: a Course on Backstepping Designs (SIAM, 2008).
- Open questions:
  - Combine divergent series and smallest-term summation (see the PhD of Th. Meurer: Feedforward and Feedback Tracking Control of Diffusion-Convection-Reaction Systems using Summability Methods (Stuttgart, 2005)).
  - 2D heat equation with a scalar control $u(t)$: with modal decomposition and symbolic computations, we get $u(s) = P(s)y(s)$ with $P(s)$ an entire function (coding the spectrum) of order 1 but infinite type $|P(s)| \leq M \exp(K|s| \log(|s|))$. It yields divergence series for any $C^\infty$ function $y \neq 0$ with compact support.
**(u(s) = P(s)y(s) for 1D and 2D heat equations**

- **1D heat equation: eigenvalue asymptotics** \( \lambda_n \sim -n^2 \):

  Prototype: 
  \[
  P(s) = \prod_{n=1}^{+\infty} \left(1 - \frac{s}{n^2}\right) = \frac{\sinh(\pi \sqrt{s})}{\pi \sqrt{s}}
  \]
  entire function of order 1/2.

- **2D heat equation in a domain** \( \Omega \) with a single scalar control \( u(t) \) on the boundary \( \partial \Omega_1 \) \((\partial \Omega = \partial \Omega_1 \cup \partial \Omega_2)\):

  \[
  \frac{\partial \theta}{\partial t} = \Delta \theta \text{ on } \Omega, \quad \theta = u(t) \text{ on } \partial \Omega_1, \quad \frac{\partial \theta}{\partial n} = 0 \text{ on } \partial \Omega_2
  \]
  Eigenvalue asymptotics \( \lambda_n \sim -n \)

  Prototype: 
  \[
  P(s) = \prod_{n=1}^{+\infty} \left(1 + \frac{s}{n}\right) \exp(-s/n) = \frac{\exp(-\gamma s)}{s \Gamma(s)}
  \]
  entire function of order 1 but of infinite type\(^{15}\)

\(^{15}\)For the links between the distributions of the zeros and the order at infinity of entire functions see the book of B.Ja Levin: Distribution of Zeros of Entire Functions; AMS, 1972.
Symbolic computations with Laplace variable $s = \frac{d}{dt}$

- **Wave 1D**: $u = \cosh(s)y$. General case is similar: $u = P(s)y$ where the zeros of $P$ are the eigen-values $\pm i\omega_n$ with asymptotic $\omega_n \sim n$; $P(s)$ entire function of order 1 and finite type (in time domain: advance/delay operator with compact support).

- **Diffusion 1D**: $u = \cosh(\sqrt{s})u$. General case is similar: $u = P(s)y$ where the zeros of $P$ are the eigen-values $-\lambda_n$ with asymptotic $\lambda_n \sim n^2$; $P(s)$ entire function of order $1/2$ (in time domain: ultra-distribution made of an infinite sum of Dirac derivatives applied on Gevrey functions with compact support of order $< 2$).

- **Wave 2D**: since $\omega_n \sim \sqrt{n}$, $P$ entire with order 2 but infinite type; prototype $P(s) = \prod_{n=1}^{+\infty} \left(1 - \frac{s^2}{n}\right) \exp(s^2/n) = -\frac{\exp(\gamma s^2)}{s^2 \Gamma(-s^2)}$.

- **Diffusion 2D**: since $\lambda_n \sim -n$, $P$ entire with order 1 but infinite type; prototype $P(s) = \prod_{n=1}^{+\infty} \left(1 + \frac{s}{n}\right) \exp(-s/n) = \frac{\exp(-\gamma s)}{s \Gamma(s)}$.

**Open Question**: interpretation of $P(s)$ in time domain as operator on a set of time functions $y(t)$...
Wave 1D with internal damping

\[
\frac{\partial^2 H}{\partial t^2} = \frac{\partial^2 H}{\partial x^2} + \epsilon \frac{\partial^3 H}{\partial x^2 \partial t}
\]

\[H(0, t) = 0, \quad H(1, t) = u(t)\]

where the eigenvalues are the zeros of

\[P(s) = \cosh \left( \frac{s}{\sqrt{\epsilon s + 1}} \right).\]

Approximate controllability depends on the functional space chosen to have a well-posed Cauchy problem\(^\text{16}\)

\(^\text{16}\)Rosier-R, CAO’06. 13th IFAC Workshop on Control Applications of Optimisation. 2006.
Dispersive wave 1D (Maxwell-Lorentz)

Propagation of electro-magnetic wave in a partially transparent medium:

\[
\frac{\partial^2}{\partial t^2}(E + D) = c^2 \frac{\partial^2}{\partial x^2} E, \quad \frac{\partial^2 D}{\partial t^2} = \omega_0^2 (\epsilon E - D)
\]

where \( \omega_0 \) is associated to an adsorption ray and \( \epsilon \) is the coupling constant between medium of polarization \( P \) and travelling field \( E \)

- The eigenvalues rely on the analytic function (\( s = d/dt \) Laplace variable, \( L \) length)

\[
Q^{\pm}(s, L) = \exp \left( \pm \frac{Ls}{c} \sqrt{1 + \frac{\epsilon s^2}{\omega_0^2 + s^2}} \right)
\]

The essential singularity in \( s = \pm \omega_0 \) yields an accumulation of eigenvalues around \( \pm \omega_0 \).

- Few works on this kind of PDE with spectrum that accumulates at finite distance.
The flatness characterization problem

\[ \frac{d}{dt} x = f(x, u) \] is said \( r \)-flat if exists a flat output \( y \) only function of \((x, u, \dot{u}, \ldots, u^{(r-1)})\); \( 0 \)-flat means \( y = h(x) \).

Example:

\[ x_1^{(\alpha_1)} = u_1, \quad x_2^{(\alpha_2)} = u_2, \quad \frac{d}{dt} x_3 = u_1 u_2 \]

is \([r := \min(\alpha_1, \alpha_2) - 1] \)-flat with

\[ y_1 = x_3 + \sum_{i=1}^{\alpha_1} (-1)^i x_1^{(\alpha_1-i)} u_2^{(i-1)} \]
\[ y_2 = x_2, \]

Conjecture: there is no flat output depending on derivatives of \( u \) of order less than \( r - 1 \).

The main difficulty: for \( \frac{d}{dt} x = f(x, u) \) with \( y = h(x, u, \ldots, u^{(p)}) \) as flat output, we do not know an upper-bound on \( p \) with respect to \( n = \text{dim}(x), m = \text{dim}(u), \ldots \).
Systems linearizable by static feedback

- A system which is linearizable by static feedback and coordinate change is flat: geometric necessary and sufficient conditions by Jakubczyk and Respondek (1980) (see also Hunt et al. (1983)).

- When there is only one control input, flatness reduces to static feedback linearizability (Charlet et al. (1989))
Affine control systems of small co-dimension

- Affine systems of codimension 1.

\[ \frac{d}{dt} x = f_0(x) + \sum_{j=1}^{n-1} u_j g_j(x), \quad x \in \mathbb{R}^n, \]

is 0-flat as soon as it is controllable, Charlet et al. (1989)

- Affine systems with 2 inputs and 4 states. Necessary and sufficient conditions for 1-flatness (Pomet (1997)) give a good idea of the complexity of checking \( r \)-flatness even for \( r \) small.
Driftless systems with two controls.

\[
\frac{d}{dt} x = f_1(x)u_1 + f_2(x)u_2
\]

is flat if and only if the generic rank of \( E_k \) is equal to \( k + 2 \) for \( k = 0, \ldots, n - 2 \) where

\[
\begin{align*}
E_0 &:= \text{span}\{f_1, f_2\} \\
E_{k+1} &:= \text{span}\{E_k, [E_k, E_k]\}, \quad k \geq 0.
\end{align*}
\]


A flat two-input driftless system satisfying some additional regularity conditions (Murray (1994)) can be put into the \textit{chained system}

\[
\begin{align*}
\frac{d}{dt} x_1 &= u_1, \\
\frac{d}{dt} x_2 &= u_2, \\
\frac{d}{dt} x_3 &= x_2u_1, \quad \ldots, \\
\frac{d}{dt} x_n &= x_{n-1}u_1.
\end{align*}
\]
Codimension 2 driftless systems

\[ \frac{d}{dt} x = \sum_{i=1}^{n-2} u_i f_i(x), \quad x \in \mathbb{R}^n \]

is flat as soon as it is controllable (Martin and R. (1995))

- Tools: exterior differential systems.
- Many nonholonomic control systems are flat.
The ruled-manifold criterion (R. (1995))

- Assume $\dot{x} = f(x, u)$ is flat. The projection on the $p$-space of the submanifold $p = f(x, u)$, where $x$ is considered as a parameter, is a ruled submanifold for all $x$.

- Otherwise stated: eliminating $u$ from $\dot{x} = f(x, u)$ yields a set of equations $F(x, \dot{x}) = 0$: for all $(x, p)$ such that $F(x, p) = 0$, there exists $a \in \mathbb{R}^n, a \neq 0$ such that

  $$\forall \lambda \in \mathbb{R}, \quad F(x, p + \lambda a) = 0.$$

- Proof elementary and derived from Hilbert (1912).

- Restricted version proposed by Sluis (1993).

Why static linearization coincides with flatness for single input systems? Because a ruled-manifold of dimension 1 is just a straight line.
Proving that a multi-input system is not flat

\[
\begin{align*}
\frac{d}{dt} x_1 &= u_1, & \frac{d}{dt} x_2 &= u_2, & \frac{d}{dt} x_3 &= (u_1)^2 + (u_2)^3
\end{align*}
\]

is not flat The submanifold \( p_3 = p_1^2 + p_2^3 \) is not ruled: there is no \( a \in \mathbb{R}^3, a \neq 0 \), such that

\[
\forall \lambda \in \mathbb{R}, p_3 + \lambda a_3 = (p_1 + \lambda a_1)^2 + (p_2 + \lambda a_2)^3.
\]

Indeed, the cubic term in \( \lambda \) implies \( a_2 = 0 \), the quadratic term \( a_1 = 0 \) hence \( a_3 = 0 \).

The system \( \frac{d}{dt} x_3 = \left( \frac{d}{dt} x_1 \right)^2 + \left( \frac{d}{dt} x_2 \right)^2 \) does not define a ruled submanifold of \( \mathbb{R}^3 \): it is not flat in \( \mathbb{R} \). But it defines a ruled submanifold in \( \mathbb{C}^3 \): in fact it is flat in \( \mathbb{C} \), with the flat output

\[
\begin{align*}
y_1 &= x_3 - (\dot{x}_1 - \dot{x}_2 \sqrt{-1})(x_1 + x_2 \sqrt{-1}) \\
y_2 &= x_1 + x_2 \sqrt{-1}.
\end{align*}
\]
Take two explicit analytic systems $\frac{d}{dt}x = f(x, u)$ and $\frac{d}{dt}z = g(z, v)$ with $\text{dim } u = \text{dim } v$ but not necessarily $\text{dim } x$ equals to $\text{dim } z$. Assume that they are equivalent via a possible dynamic state feedback. Then we have

- if $\text{dim } x < \text{dim } z$ then $\frac{d}{dt}x = f(x, u)$ is ruled.
- if $\text{dim } z < \text{dim } x$ then $\frac{d}{dt}z = g(z, v)$ is ruled.
- if $\text{dim } x = \text{dim } z$ either they are equivalent by static feedback or they are both ruled.

The system $\frac{d}{dt}x = f(x, u)$ (resp. $\frac{d}{dt}z = g(z, v)$ is said ruled when after elimination of $u$ (resp. $v$), the implicit system $F(x, \frac{d}{dt}x) = 0$ (resp. $G(x, \frac{d}{dt}x) = 0$) is ruled in the sense of the ruled manifold criterion explained here above.
Invariance versus actions of the group $SE(2)$.

Flat outputs are not unique: $(\xi = x_n, \zeta = y_n + \frac{d}{dt} x_n)$ is another flat output since $x_n = \xi$ and $y_n = \zeta - \frac{d}{dt} \xi$.

The flat output $(x_n, y_n)$ formed by the cartesian coordinates of $P_n$ seems more adapted than $(\xi, \zeta)$: the output map $h$ is equivariant.
Why the flat output $z := (x, y)$ is better than the flat output $	ilde{z} := (x, y + \dot{x})$?

Each symmetry of the system induces a transformation on the flat output $z$

\[
\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} z_1 \cos \alpha - z_2 \sin \alpha + a \\ z_1 \sin \alpha + z_2 \cos \alpha + b \end{pmatrix}
\]

which does not involve derivatives of $z$
This point transformation, generates an endogenous transformation $(z, \dot{z}, \ldots) \mapsto (Z, \dot{Z}, \ldots)$ that is holonomic.
Why the flat output $z := (x, y)$ is better than the flat output $\tilde{z} := (x, y + \dot{x})$?

On the contrary

\[
\begin{pmatrix} x \\ y + \dot{x} \end{pmatrix} = \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} \tilde{Z}_1 \\ \tilde{Z}_2 \end{pmatrix} = \begin{pmatrix} X \\ Y + \dot{X} \end{pmatrix} = \begin{pmatrix} \tilde{z}_1 \cos \alpha + (\dot{\tilde{z}}_1 - \tilde{z}_2) \sin \alpha + a \\ \tilde{z}_1 \sin \alpha + \tilde{z}_2 \cos \alpha + (\ddot{\tilde{z}}_1 - \dot{\tilde{z}}_2) \sin \alpha + b \end{pmatrix}
\]

is not a point transformation and does not give to a holonomic transformation. It is endogenous since its inverse is

\[
\begin{pmatrix} \tilde{Z}_1 \\ \tilde{Z}_2 \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} = \begin{pmatrix} (\tilde{Z}_1 - a) \cos \alpha - (\dot{\tilde{Z}}_1 - \dot{\tilde{Z}}_2) \sin \alpha \\ (\tilde{Z}_1 - a) \sin \alpha + (\tilde{Z}_2 - b) \cos \alpha - (\ddot{\tilde{Z}}_1 - \ddot{\tilde{Z}}_2) \sin \alpha \end{pmatrix}
\]
Symmetry preserving flat output

- Take the implicit system \( F(x, \ldots, x^{(r)}) = 0 \) with flat output \( y = h(x, \ldots, x^{(\alpha)}) \in \mathbb{R}^m \) (i.e. \( x = A(y, \ldots, y^{(\beta)}) \))
- Assume that the group \( G \) acting on the \( x \)-space via the family of diffeomorphism \( X = \phi_g(x) \) (\( x = \phi_g^{-1}(X) \)) leaves the ideal associated to the set of equation \( F = 0 \) invariante:

\[
F(x, \ldots, x^{(r)}) = 0 \iff F \left( \left( \phi_g(x), \ldots, \phi_g^{(r)}(x, \ldots, x^{(r)}) \right) \right) = 0
\]

- Question: we wonder if exists always an equivariante flat output \( \bar{y} = \bar{h}(x, \ldots, x^{(\bar{\alpha})}) \), i.e. such that exists an action of \( G \) on the \( y \)-space via the family of diffeomorphisms \( \bar{Y} = \rho_g(\bar{y}) \) satisfying

\[
\rho_g(y) \equiv h \left( \phi_g(x), \ldots \phi_g^{(\bar{\alpha})}(x, \ldots, x^{(\bar{r})}) \right).
\]

Two different flat outputs correspond via a "non-linear uni-modular transformation ":

\[
\bar{y} = \psi(y, \ldots, y^{(\mu)}) \quad \text{with inverse} \quad y = \bar{\psi}(\bar{y}, \ldots, \bar{y}^{(\bar{\mu})})
\]
Flat outputs as potentials and gauge degree of freedom

Maxwell’s equations in vacuum imply that the magnetic field $H$ is divergent free:

$$\frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial x_2} + \frac{\partial H_3}{\partial x_3} = 0$$

When $H = \nabla \times A$ the constraint $\nabla \cdot H = 0$ is automatically satisfied.

The potential $A$ is a priori not uniquely defined, but up to an arbitrary gradient field, the gauge degree of freedom. The symmetries indicate how to use this degree of freedom to fix a “natural” potential.

For flat systems: a flat output is a “potential” for the underdetermined differential equation $\dot{x} - f(x, u) = 0$; endogenous transformations on the flat output correspond to gauge degrees of freedom.
Open problems

- $\frac{d}{dt} x = f(x, u)$ with $y = h(x, u, ..., u^{(r)})$, $r$-flatness: bounds on $r$ with respect to $\dim(x)$ and $\dim(u)$.

- Symmetries and flat-output preserving symmetries: are time-invariant systems flat with a time invariant flat output map (a first step to prove that linearization via exogenous dynamics feedback, implies flatness).

- Are the intrinsic and extrinsic definitions of flat systems equivalent?

- Flatness of JBP example
The system

\[ \frac{d}{dt} x_3 - x_2 - \left( \frac{d}{dt} x_1 \right) \left( \frac{d}{dt} x_2 - x_3 \frac{d}{dt} x_1 \right)^2 = 0 \]

is ruled with a single linear direction

\[ a(x, \dot{x}) = (1, x_3, (\dot{x}_2 - x_3 \dot{x}_1)^2)^T. \]

There is no flat output \( y \) depending only on \( x \) and \( \dot{x} \) (this system is not 1-flat)

Conjecture: this system is not flat.