Intrinsic observers for perfect incompressible fluids and particle imaging velocimetry

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Outline

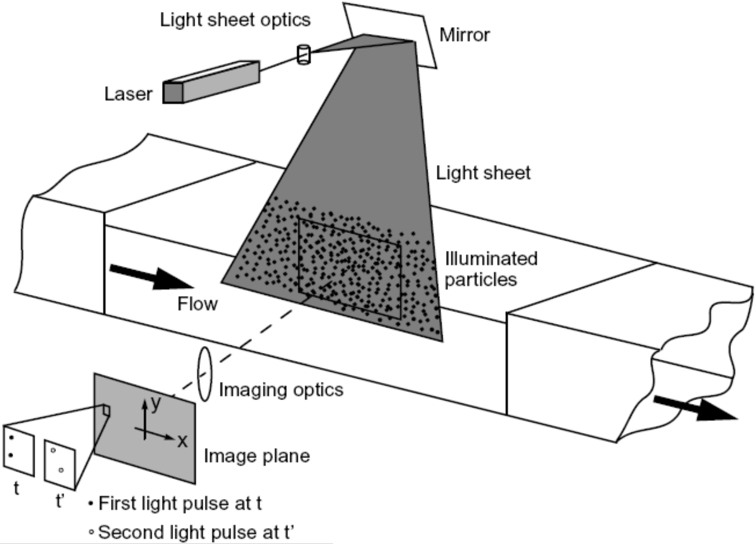
Particle Imaging Velocimetry (PIV)

Perfect incompressible fluids and geodesics

Velocity observer for mechanical systems

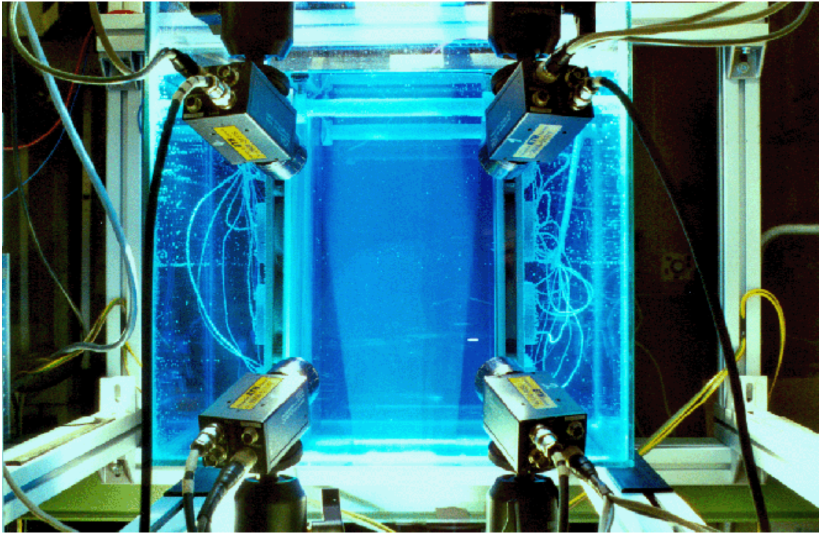
Heuristic extension to PIV

Particle imaging velocimetry[1]



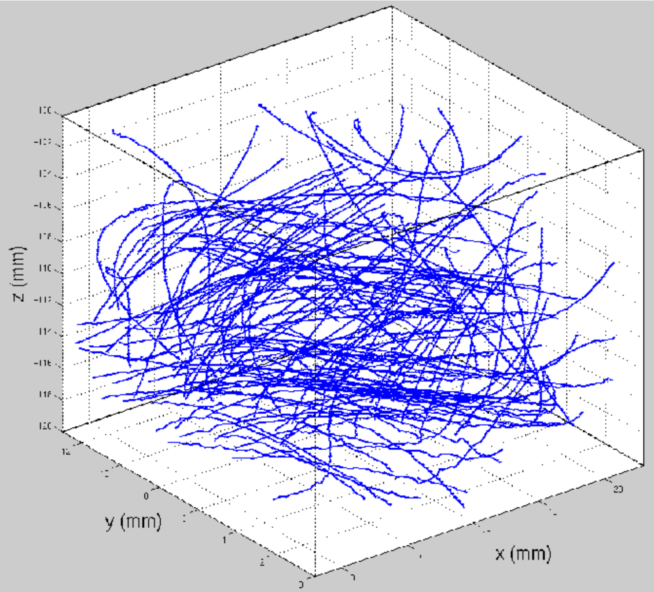
[1] From: "Particle tracking velocimetry" of Wikipedia and PhD "A Spatio-Temporal Matching Algorithm for 3D Particle Tracking Velocimetry" by Jochen Willneff (2003) (Diss. ETH No. 15276).

3D Particle Tracking Velocimetry: example of experimental setup[2]



[2] From: PhD of Jochen Willneff (2003)

3D Particle Tracking: examples of 3D trajectories [3]



[3] From: PhD of Jochen Willneff (2003)

From 3D trajectories to velocities

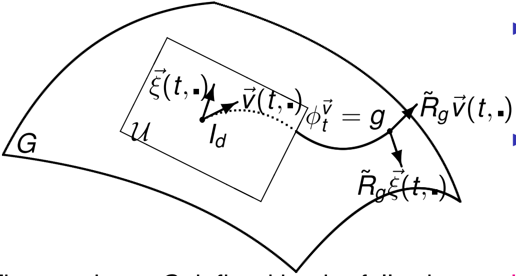
- Lagrangian point of view. Denote by $\phi(t,x) \in \mathbb{R}^3$ the Cartesian position at time t of the particle that was at $x \in \mathbb{R}^3$ at time 0 ($\phi(0,x) \equiv x$). 3D Particle Tracking provides $\phi(t,x)$ sampled in time and in space.
- Eulerian point of view. Differentiation versus t provides $\vec{v}(t,x)$, the velocity field at time t and position x (kinematic relation)
$$\frac{\partial \phi}{\partial t}(t,x) = \vec{v}(t, \phi_t(x))$$

If we assume the fluid perfect, homogeneous and incompressible, then $\vec{v}$ is tangent to the boundary $\partial\Omega$ and obeys to the Euler equations inside the domain Ω:

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\nabla \alpha, \quad \nabla \cdot \vec{v} = 0.$$

The scalar field α (pressure) depends implicitly on $\vec{v}$ via the incompressibility conditions.

Euler equations as geodesics equations[4]



- G: "Lie group" of volume preserving diffeomorphisms g on Ω
- $TG_{I_d} = \mathcal{U}$ is the Lie algebra of vector fields in Ω of zero divergence and tangent to $\partial\Omega$.

The metric on G defined by the following scalar product:

$$< \vec{\xi}, \vec{v} >_g = \int\int\int_\Omega \vec{\xi}(g(x)) \cdot \vec{v}(g(x))\, dx = \int\int\int_\Omega \vec{\xi}(x) \cdot \vec{v}(x)\, dx$$

is invariant versus right translation: $R_g:\ h \in G \rightarrow h \circ g \in G$.
Covariant derivative reads:

$$\nabla_{\vec{v}}\vec{\xi} = \frac{\partial\vec{\xi}}{\partial t} + (\vec{v}\cdot\nabla)\vec{\xi} + \nabla\alpha, \quad \text{with} \quad \vec{v}(t,.) \text{ and } \vec{\xi}(t,.) \in \mathcal{U}$$

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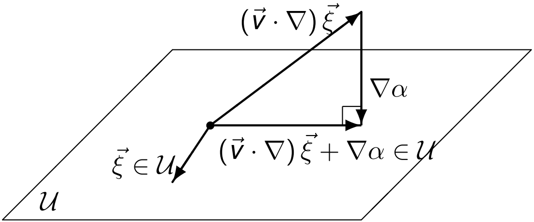
[4] V.I. Arnol'd. Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier*, 16:319–361, 1966.

The covariant derivative[5] $\nabla_{\vec{v}}\vec{\xi}$



The covariant differentiation, with respect to $\vec{v}$, of $\vec{\xi}(t,\cdot)\in\mathcal{U}$ corresponding to an element of $TG_{\phi_t^{\vec{v}}}$, is given by

$$\nabla_{\vec{v}}\vec{\xi} = \frac{\partial\vec{\xi}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{\xi} + \nabla\alpha$$

where α is a real function such that $\dfrac{\partial\vec{\xi}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{\xi} + \nabla\alpha$ belongs to $\mathcal{U}$ ($\Delta\alpha + \nabla\cdot((\vec{v}\cdot\nabla)\,\vec{\xi}) = 0$ and $\nabla\alpha + (\vec{v}\cdot\nabla)\,\vec{\xi}$ tangent to $\partial\Omega$).

[5] J.J. Moreau, J.J.: Une méthode de cinématique fonctionnelle en hydrodynamique. C.R. Acad. Sci. Paris, pp:2156–2158, Nov 1959.

PIV, geodesics and velocity observers for mechanical systems

Geodesics correspond to mechanical systems those Lagrangian coincides with kinetic energy: if q is a set of coordinates on the configuration manifold M, $L(q,\dot{q}) = \frac{1}{2} g_{ij}(q)\dot{q}^i\dot{q}^j$ yields to the second-order ODE:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}(q,\dot{q})\right) = \frac{d}{dt}\left(g_{ij}(q)\dot{q}^j\right) = \frac{1}{2}\frac{\partial g_{kj}}{\partial q_i}\dot{q}_k\dot{q}_j = \frac{\partial L}{\partial q_i}(q,\dot{q})$$

that reads geometrically $\dot{q} = v$, $\nabla_v v = 0$ where $\nabla_v v$ is the covariant derivative.

Similarities between velocity observer for mechanical systems and PIV:

- measured positions $q^i(t) \longrightarrow$ the 3D-trajectories $\phi(t,x)$;
- $\dot{q} = v \longrightarrow \frac{\partial \phi}{\partial t}(t,x) = \vec{v}(t,\phi(t,x))$;
- ODE $\nabla_v v = 0 \longrightarrow$ PDE $\nabla_{\vec{v}}\vec{v} = 0$;
- estimation of $v = \dot{q} \longrightarrow$ estimation of the velocity field $\vec{v}$.

Velocity observer for mechanical systems [7]

For any constant gains $\alpha > 0$ and $\beta > 0$, the following intrinsic observer is locally convergent:

$$\begin{aligned} \dot{\hat{q}} &= \hat{v} - \alpha \, \text{grad}_{\hat{q}} F(\hat{q}, q) \\ \nabla_{\dot{\hat{q}}} \hat{v} &= -\beta \, \text{grad}_{\hat{q}} F(\hat{q}, q) + R(\hat{v}, \text{grad}_{\hat{q}} F(\hat{q}, q)) \hat{v} \end{aligned}$$

where: $F(\hat{q}, q)$ is half of the square of the geodesic distance between q and $\hat{q}$; R is the curvature tensor. Here ∇ and grad_q are the Levi-Civita connexion and the gradient operator associated to the Riemnanian structure derived from the g_{ij}'s.

- For $\hat{q}$ close to q, $\text{grad}_{\hat{q}} F(\hat{q}, q) \approx \hat{q}^i - q^i$
- When q lives on a Lie Group, the above asymptotic observers simplify a little[6].

[6] D. H. S. Maithripala, W. P. Dayawansa, and J. M. Berg. Intrinsic observer-based stabilization for simple mechanical systems on Lie groups. *SIAM J. Control and Optim.*, 44:1691–1711, 2005.

[7] N. Aghannan and PR. An intrinsic observer for a class of Lagrangian systems. *IEEE AC*, 48(6):936–945, 2003.

Heuristic extension to perfect incompressible fluid

Replace $\hat{q} - q$ by $\hat{\phi} - \phi$ and use curvature formulae given in [8]:

$$\frac{\partial \hat{\phi}}{\partial t}(t, x) = \hat{\vec{v}}(t, \hat{\phi}(t, x)) - \alpha \vec{e}(t, \hat{\phi}(t, x))$$

$$\frac{\partial \hat{\vec{v}}}{\partial t} + \left((\hat{\vec{v}} - \alpha \vec{e}) \cdot \nabla\right) \hat{\vec{v}} = -\nabla \eta - \beta \vec{e} + (\vec{e} \cdot \nabla)\nabla \hat{p} - (\hat{\vec{v}} \cdot \nabla)\nabla \hat{\eta}$$

where:

- $\vec{e} \in \mathcal{U}$ corresponds to the position errors $\hat{q} - q$, i.e., $\vec{e}(t, \phi(t, x)) \approx \hat{\phi}(t, x) - \phi(t, x)$; Right invariance implies that in the second equation $\vec{e} \approx \hat{\phi}(t, \phi_t^{-1}(x)) - x$.
- the gradient field $\nabla \eta$ ensures $\frac{\partial \hat{\vec{v}}}{\partial t} \in \mathcal{U}$; $(\vec{e} \cdot \nabla)\nabla \hat{p} - (\hat{\vec{v}} \cdot \nabla)\nabla \hat{\eta}$ is the curvature term $R(\hat{v}, \hat{q} - q)\hat{v}$; $\nabla \hat{p}$ is such that $\nabla \hat{p} + (\hat{\vec{v}} \cdot \nabla)\hat{\vec{v}} \in \mathcal{U}$; $\nabla \hat{\eta}$ is such that $\nabla \hat{\eta} + (\hat{\vec{v}} \cdot \nabla)\vec{e} \in \mathcal{U}$.

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[8] PR. Jacobi equation, Riemannian curvature and the motion of a perfect incompressible fluid. *European Journal of Mechanics /B Fluids*, 11:317–336, 1992.

Concluding remarks

- How to increase precision of $\hat{\vec{v}}$ (turbulence investigations)? interesting question relying on image processing, $SE(3)$ invariance and the PDE underlying fluid mechanics.
- Invariance and geometry should play a central role in such data assimilation processes and filtering (for recent investigations on invariant asymptotic observers see [9]).
- For perfect fluids, intrinsic asymptotic observers could be of some interest for velocity estimation: they are based on geometry.
- Possible extension to compressible perfect fluids (use [10]).

[9] S. Bonnabel, Ph. Martin, PR: Symmetry-preserving observers. IEEE Trans. Automatic Control. Vol 53, pp:2514-2526, 2008.
S. Bonnabel, D. Auroux: Symmetry-preserving nudging: theory and application to a shallow water model. CDPS 2009.

[10] D.G. Ebin: The Motion of Sightly Compressible Fluids Viewed as a Motion With Strong Constraining Force. Annals of Math. Vol.105, pp:141–200,1977.
PR: Dynamique des fluides parfaits, principe de moindre action, stabilité lagrangienne. Technical Report 13/3446 EN, ONERA, 1991.