Intrinsic observers for perfect incompressible fluids and particle imaging velocimetry

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Outline

Particle Imaging Velocimetry (PIV)

Perfect incompressible fluids and geodesics

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3D Particle Tracking Velocimetry: example of experimental setup

3D Particle Tracking: examples of 3D trajectories

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From: PhD of Jochen Willneff (2003)
From 3D trajectories to velocities

- **Lagrangian point of view.** Denote by $\phi(t, x) \in \mathbb{R}^3$ the Cartesian position at time $t$ of the particle that was at $x \in \mathbb{R}^3$ at time 0 ($\phi(0, x) \equiv x$). 3D Particle Tracking provides $\phi(t, x)$ sampled in time and in space.

- **Eulerian point of view.** Differentiation versus $t$ provides $\mathbf{\tilde{v}}(t, x)$, the velocity field at time $t$ and position $x$ (kinematic relation)

$$\frac{\partial \phi}{\partial t}(t, x) = \mathbf{\tilde{v}}(t, \phi_t(x))$$

If we assume the fluid perfect, homogeneous and incompressible, then $\mathbf{\tilde{v}}$ is tangent to the boundary $\partial \Omega$ and obeys to the **Euler equations** inside the domain $\Omega$:

$$\frac{\partial \mathbf{\tilde{v}}}{\partial t} + \mathbf{\tilde{v}} \cdot \nabla \mathbf{\tilde{v}} = -\nabla \alpha, \quad \nabla \cdot \mathbf{\tilde{v}} = 0.$$

The scalar field $\alpha$ (pressure) depends implicitly on $\mathbf{\tilde{v}}$ via the incompressibility conditions.
Euler equations as geodesics equations\textsuperscript{4}

\[ \vec{v}(t, \cdot) = g \]

\[ \tilde{R}_g \vec{v}(t, \cdot) = \tilde{R}_g \tilde{\xi}(t, \cdot) \]

\[ \nabla_{\vec{v}} \tilde{\xi} = \frac{\partial \tilde{\xi}}{\partial t} + (\vec{v} \cdot \nabla) \tilde{\xi} + \nabla \alpha, \quad \text{with} \quad \vec{v}(t, \cdot) \text{ and } \tilde{\xi}(t, \cdot) \in \mathcal{U} \]

\begin{itemize}
  \item G: "Lie group" of volume preserving diffeomorphisms \( g \) on \( \Omega \)
  \item \( TG_{I_d} = \mathcal{U} \) is the Lie algebra of vector fields in \( \Omega \) of zero divergence and tangent to \( \partial \Omega \).
\end{itemize}

The metric on \( G \) defined by the following scalar product:

\[ < \tilde{\xi}, \tilde{\nu} >_g = \int \int \int_{\Omega} \tilde{\xi}(g(x)) \cdot \tilde{\nu}(g(x)) \ dx = \int \int \int_{\Omega} \tilde{\xi}(x) \cdot \tilde{\nu}(x) \ dx \]

is invariant versus right translation: \( R_g : h \in G \rightarrow h \circ g \in G \).

\textbf{Covariant derivative reads:}

The covariant derivative $\nabla_{\vec{v}}\xi$

The covariant differentiation, with respect to $\vec{v}$, of $\xi(t, \cdot) \in \mathcal{U}$ corresponding to an element of $T_{\phi_t}G_{\vec{v}}$, is given by

$$\nabla_{\vec{v}}\xi = \frac{\partial \xi}{\partial t} + (\vec{v} \cdot \nabla) \xi + \nabla \alpha$$

where $\alpha$ is a real function such that $\frac{\partial \xi}{\partial t} + (\vec{v} \cdot \nabla) \xi + \nabla \alpha$ belongs to $\mathcal{U}$

$(\Delta \alpha + \nabla \cdot ((\vec{v} \cdot \nabla) \xi) = 0$ and $\nabla \alpha + (\vec{v} \cdot \nabla) \xi$ tangent to $\partial \Omega$).

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PIV, geodesics and velocity observers for mechanical systems

Geodesics correspond to mechanical systems those Lagrangian coincides with kinetic energy: if \( q \) is a set of coordinates on the configuration manifold \( M \),
\[ L(q, \dot{q}) = \frac{1}{2} g_{ij}(q) \dot{q}^i \dot{q}^j \]
yields to the second-order ODE:
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i}(q, \dot{q}) \right) = \frac{d}{dt} \left( g_{ij}(q) \dot{q}^j \right) = \frac{1}{2} \frac{\partial g_{kj}}{\partial q_i} \dot{q}_k \dot{q}_j = \frac{\partial L}{\partial q_i}(q, \dot{q})
\]
that reads geometrically \( \dot{q} = v \), \( \nabla_v v = 0 \) where \( \nabla_v v \) is the covariant derivative.

Similarities between velocity observer for mechanical systems and PIV:

- measured positions \( q^i(t) \rightarrow \) the 3D-trajectories \( \phi(t, x) \);
- \( \dot{q} = v \rightarrow \frac{\partial \phi}{\partial t} (t, x) = \vec{v}(t, \phi(t, x)) \);
- ODE \( \nabla_v v = 0 \rightarrow \) PDE \( \nabla_{\vec{v}} \vec{v} = 0 \);
- estimation of \( v = \dot{q} \rightarrow \) estimation of the velocity field \( \vec{v} \).
Velocity observer for mechanical systems

For any constant gains $\alpha > 0$ and $\beta > 0$, the following intrinsic observer is locally convergent:

$$\dot{\hat{q}} = \hat{v} - \alpha \text{grad}_{\hat{q}} F(\hat{q}, q)$$
$$\nabla_{\hat{q}} \dot{\hat{v}} = -\beta \text{grad}_{\hat{q}} F(\hat{q}, q) + R(\hat{v}, \text{grad}_{\hat{q}} F(\hat{q}, q))\hat{v}$$

where: $F(\hat{q}, q)$ is half of the square of the geodesic distance between $q$ and $\hat{q}$; $R$ is the curvature tensor. Here $\nabla$ and $\text{grad}_q$ are the Levi-Civita connexion and the gradient operator associated to the Riemannian structure derived from the $g_{ij}$’s.

- For $\hat{q}$ close to $q$, $\text{grad}_{\hat{q}} F(\hat{q}, q) \approx \hat{q}^j - q^j$
- When $q$ lives on a Lie Group, the above asymptotic observers simplify a little\(^6\).

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Heuristic extension to perfect incompressible fluid

Replace \( \hat{q} - q \) by \( \hat{\phi} - \phi \) and use curvature formulae given in \(^8\):

\[
\frac{\partial \hat{\phi}}{\partial t}(t, x) = \hat{\nabla}(t, \hat{\phi}(t, x)) - \alpha \hat{\varepsilon}(t, \hat{\phi}(t, x))
\]

\[
\frac{\partial \hat{\nabla}}{\partial t} + \left( (\hat{\nabla} - \alpha \hat{\varepsilon}) \cdot \nabla \right) \hat{\nabla} = -\nabla \eta - \beta \hat{\varepsilon} + (\hat{\varepsilon} \cdot \nabla) \nabla \hat{\rho} - (\hat{\nabla} \cdot \nabla) \nabla \hat{\eta}
\]

where:

- \( \hat{\varepsilon} \in \mathcal{U} \) corresponds to the position errors \( \hat{q} - q \), i.e., \( \hat{\varepsilon}(t, \phi(t, x)) \approx \hat{\phi}(t, x) - \phi(t, x) \); Right invariance implies that in the second equation \( \hat{\varepsilon} \approx \hat{\phi}(t, \phi_t^{-1}(x)) - x \).

- the gradient field \( \nabla \eta \) ensures \( \frac{\partial \hat{\nabla}}{\partial t} \in \mathcal{U} \); \( (\hat{\varepsilon} \cdot \nabla) \nabla \hat{\rho} - (\hat{\nabla} \cdot \nabla) \nabla \hat{\eta} \) is the curvature term \( R(\hat{\nabla}, \hat{q} - q)\hat{\nabla} \); \( \nabla \hat{\rho} \) is such that \( \nabla \hat{\rho} + (\hat{\nabla} \cdot \nabla) \hat{\nabla} \in \mathcal{U} \); \( \nabla \hat{\eta} \) is such that \( \nabla \hat{\eta} + (\hat{\nabla} \cdot \nabla) \hat{\varepsilon} \in \mathcal{U} \).

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Concluding remarks

▶ How to increase precision of $\hat{v}$ (turbulence investigations)? interesting question relying on image processing, $SE(3)$ invariance and the PDE underlying fluid mechanics.

▶ Invariance and geometry should play a central role in such data assimilation processes and filtering (for recent investigations on invariant asymptotic observers see $^9$).

▶ For perfect fluids, intrinsic asymptotic observers could be of some interest for velocity estimation: they are based on geometry.

▶ Possible extension to compressible perfect fluids (use $^{10}$).

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