Modeling and Control of the LKB Photon-Box: [1] Spin-Spring Systems

Pierre Rouchon

pierre.rouchon@mines-paristech.fr

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[1] LKB: Laboratoire Kastler Brossel, ENS, Paris.
Several slides have been used during the IHP course (fall 2010) given with Mazyar Mirrahimi (INRIA) see:
http://cas.ensmp.fr/~rouchon/QuantumSyst/index.html

Outline

Harmonic oscillator[2] (1): quantization and correspondence principle

Classical Hamiltonian formulation of $\frac{d^2}{dt^2}x = -\omega^2 x$

$$\frac{d}{dt}x = \omega p = \frac{\partial \mathbb{H}}{\partial p}, \quad \frac{d}{dt}p = -\omega x = -\frac{\partial \mathbb{H}}{\partial x}, \quad \mathbb{H} = \frac{\omega}{2}(p^2 + x^2).$$

Quantization: probability wave function $|\psi\rangle_t \sim (\psi(x,t))_{x\in\mathbb{R}}$ with $|\psi\rangle_t \sim \psi(.,t) \in L^2(\mathbb{R},\mathbb{C})$ obeys to the Schrödinger equation ($\hbar = 1$ in all the lectures)

$$i\frac{d}{dt}|\psi\rangle = H|\psi\rangle, \quad H = \omega(P^2 + X^2) = -\frac{\omega}{2}\frac{\partial^2}{\partial x^2} + \frac{\omega}{2}x^2$$

where H results from $\mathbb{H}$ by replacing x by position operator $\sqrt{2}X$ and p by impulsion operator $\sqrt{2}P = -i\frac{\partial}{\partial x}$.

PDE model: $i\frac{\partial\psi}{\partial t}(x,t) = -\frac{\omega}{2}\frac{\partial^2\psi}{\partial x^2}(x,t) + \frac{\omega}{2}x^2\psi(x,t), \quad x \in \mathbb{R}$.

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[2] Two references: C. Cohen-Tannoudji, B. Diu, and F. Laloë. *Mécanique Quantique*, volume I& II. Hermann, Paris, 1977.
M. Barnett and P. M. Radmore. *Methods in Theoretical Quantum Optics*. Oxford University Press, 2003.

Averaged position $\langle X\rangle_t = \langle\psi|X|\psi\rangle$ and impulsion $\langle P\rangle_t = \langle\psi|P|\psi\rangle$ [3]:

$$\langle X\rangle_t = \frac{1}{\sqrt{2}}\int_{-\infty}^{+\infty} x|\psi|^2 dx,, \quad \langle P\rangle_t = -\frac{i}{\sqrt{2}}\int_{-\infty}^{+\infty} \psi^* \frac{\partial\psi}{\partial x} dx.$$

Annihilation a and creation operators $a^\dagger$:

$$a = X + iP = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right), \quad a^\dagger = X - iP = \frac{1}{\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right)$$

Commutation relationships:

$$[X, P] = \frac{i}{2}, \quad [a, a^\dagger] = 1, \quad H = \omega(P^2 + X^2) = \omega\left(a^\dagger a + \frac{1}{2}\right).$$

Set $X_\lambda = \frac{1}{2}\left(e^{-i\lambda} a + e^{i\lambda} a^\dagger\right)$ for any angle λ:

$$\left[X_\lambda, X_{\lambda+\frac{\pi}{2}}\right] = \frac{i}{2}.$$

[3] We assume everywhere that for each t, $x \mapsto \psi(x,t)$ is of the Schwartz class (fast decay at infinity + smooth).

Harmonic oscillator (3): spectral decomposition and Fock states

$[a, a^\dagger] = 1$ and Ker(a) of dimension one imply that the spectrum of $N = a^\dagger a$ is non-degenerate and is $\mathbb{N}$. More we have the useful commutations for any entire function f:

$$a\, f(N) = f(N + I)\, a, \quad f(N)\, a^\dagger = a^\dagger\, f(N + I).$$

Fock state with n photon(s): the eigen-state of N associated to the eigen-value n:

$$N\,|n\rangle = n\,|n\rangle\,, \quad a|n\rangle = \sqrt{n}\,|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle.$$

The ground state $|0\rangle$ (0 photon state or vacuum state) satisfies $a|0\rangle = 0$ and corresponds to the Gaussian function:

$$|0\rangle \sim \psi_0(x) = \frac{1}{\pi^{1/4}} \exp(-x^2/2).$$

The operator a (resp. $a^\dagger$) is the annihilation (resp. creation) operator since it transfers $|n\rangle$ to $|n-1\rangle$ (resp. $|n+1\rangle$) and thus decreases (resp. increases) the quantum number n by one unit.

Quantization of $\frac{d^2}{dt^2}x = -\omega^2 x - \omega\sqrt{2}u$

$$H = \omega\left(a^\dagger a + \frac{1}{2}\right) + u(a + a^\dagger).$$

The associated controlled PDE

$$i\frac{\partial \psi}{\partial t}(x,t) = -\frac{\omega}{2}\frac{\partial^2 \psi}{\partial x^2}(x,t) + \left(\frac{\omega}{2}x^2 + \sqrt{2}ux\right)\psi(x,t).$$

Glauber displacement operator D_α (unitary) with $\alpha \in \mathbb{C}$:

$$D_\alpha = e^{\alpha a^\dagger - \alpha^* a} = e^{2i\Im\alpha X - 2\imath\Re\alpha P}$$

From Baker-Campbell Hausdorf formula valid for any operators A and B,

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots$$

we get the Glauber formula when $[A,[A,B]] = [B,[A,B]] = 0$:

$$e^{A+B} = e^A\, e^B\, e^{-\frac{1}{2}[A,B]}.$$

(show that $C_t = e^{t(A+B)} - e^{tA}\, e^{tB}\, e^{-\frac{t^2}{2}[A,B]}$ satisfies $\frac{d}{dt}C = (A+B)C$)

With $A = \alpha a^\dagger$ and $B = -\alpha^* a$, Glauber formula gives:

$$D_\alpha = e^{-\frac{|\alpha|^2}{2}} \; e^{\alpha a^\dagger} e^{-\alpha^* a} = e^{+\frac{|\alpha|^2}{2}} \; e^{-\alpha^* a} e^{\alpha a^\dagger}$$

$$D_{-\alpha} a D_\alpha = a + \alpha \quad \text{and} \quad D_{-\alpha} a^\dagger D_\alpha = a^\dagger + \alpha^*.$$

With $A = 2i\Im\alpha X \sim i\sqrt{2}\Im\alpha x$ and $B = -2\imath\Re\alpha P \sim -\sqrt{2}\Re\alpha\frac{\partial}{\partial x}$, Glauber formula gives[4]:

$$D_\alpha = e^{-i\Re\alpha\Im\alpha} \; e^{i\sqrt{2}\Im\alpha x} e^{-\sqrt{2}\Re\alpha\frac{\partial}{\partial x}}$$

$$\left(D_\alpha \left|\psi\right\rangle\right)_{x,t} = e^{-i\Re\alpha\Im\alpha} \; e^{i\sqrt{2}\Im\alpha x} \psi(x - \sqrt{2}\Re\alpha, t)$$

Exercice

For any $\alpha, \beta, \epsilon \in \mathbb{C}$, prove that

$$D_{\alpha+\beta} = e^{\frac{\alpha^*\beta - \alpha\beta^*}{2}} D_\alpha D_\beta$$

$$D_{\alpha+\epsilon} D_{-\alpha} = \left(1 + \tfrac{\alpha\epsilon^* - \alpha^*\epsilon}{2}\right) \mathbf{1} + \epsilon a^\dagger - \epsilon^* a + O(|\epsilon|^2)$$

$$\left(\tfrac{d}{dt} D_\alpha\right) D_{-\alpha} = \left(\frac{\alpha \frac{d}{dt}\alpha^* - \alpha^* \frac{d}{dt}\alpha}{2}\right) \mathbf{1} + \left(\tfrac{d}{dt}\alpha\right) a^\dagger - \left(\tfrac{d}{dt}\alpha^*\right) a.$$

Harmonic oscillator (6): lack of controllability

Take $|\psi\rangle$ solution of the controlled Schrödinger equation $i\frac{d}{dt}|\psi\rangle = \left(\omega\left(a^\dagger a + \frac{1}{2}\right) + u(a + a^\dagger)\right)|\psi\rangle$. Set $\langle a\rangle = \langle\psi|a\psi\rangle$. Then

$$\frac{d}{dt}\langle a\rangle = -i\omega\langle a\rangle - iu.$$

From $a = X + iP$, we have $\langle a\rangle = \langle X\rangle + i\langle P\rangle$ where $\langle X\rangle = \langle\psi|X|\psi\rangle \in \mathbb{R}$ and $\langle P\rangle = \langle\psi|P|\psi\rangle \in \mathbb{R}$. Consequently:

$$\frac{d}{dt}\langle X\rangle = \omega\langle P\rangle, \quad \frac{d}{dt}\langle P\rangle = -\omega\langle X\rangle - u.$$

Consider the change of frame $|\psi\rangle = e^{-i\theta_t} D_{\langle a\rangle_t} |\chi\rangle$ with

$$\theta_t = \int_0^t \left(|\langle a\rangle|^2 + u\Re(\langle a\rangle)\right), \quad D_{\langle a\rangle_t} = e^{\langle a\rangle_t a^\dagger - \langle a\rangle_t^* a},$$

Then $|\chi\rangle$ obeys to autonomous Schrödinger equation

$$i\frac{d}{dt}|\chi\rangle = \omega a^\dagger a|\chi\rangle.$$

The dynamics of $|\psi\rangle$ can be decomposed into two parts:

- a controllable part of dimension two for $\langle a\rangle$
- an uncontrollable part of infinite dimension for $|\chi\rangle$.

Coherent states

$$|\alpha\rangle = D_\alpha |0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{+\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle , \quad \alpha \in \mathbb{C}$$

are the states reachable from vacuum set. They are also the eigen-state of a: $a|\alpha\rangle = \alpha |\alpha\rangle$.
A widely known result in quantum optics[5]: classical currents and sources (generalizing the role played by u) only generate classical light (quasi-classical states of the quantized field generalizing the coherent state introduced here)
We just propose here a control theoretic interpretation in terms of reachable set from vacuum[6]

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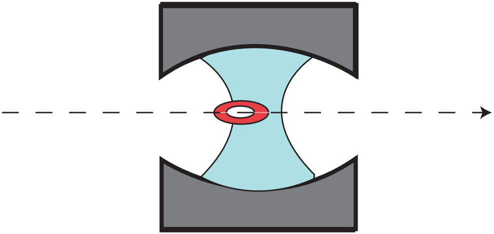
[5] See complement B_{III}, page 217 of C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg. *Photons and Atoms: Introduction to Quantum Electrodynamics*.Wiley, 1989.

[6] see also: MM-PR, IEEE Trans. Automatic Control, 2004 and MM-PR, CDC-ECC, 2005.

Atom-cavity coupling



The composite system lives on the Hilbert space $\mathbb{C}^2 \otimes L^2(\mathbb{R};\mathbb{C}) \sim \mathbb{C}^2 \otimes l^2(\mathbb{C})$ with the Jaynes-Cummings Hamiltonian

$$H_{JC} = \frac{\omega_{eg}}{2}\sigma_z + \omega_c(a^\dagger a + \frac{1}{2}) + i\frac{\Omega(t)}{2}\sigma_x(a^\dagger - a),$$

with $\omega_c/2\pi \approx \omega_{eg}/2\pi$ around 50 GHz.
Gaussian radial profile of the cavity quantized mode (for $t = 0$ atom at cavity center):

$$\Omega(t) = \Omega_0 \exp\left(-\left(\frac{vt}{w}\right)^2\right)$$

where v is the atom velocity (250 m/s), w is the width (6 mm), $\Omega_0/2\pi$ around 50 kHz. Thus we have also $\Omega(t) \ll \omega_c, \omega_{eg}$.

Jaynes-Cumming model: RWA

We consider the change of frame:
$|\psi\rangle = e^{-i\omega_c t(a^\dagger a + \frac{1}{2})} e^{-i\omega_c t \sigma_z} |\phi\rangle$.
The system becomes $i\frac{d}{dt}|\phi\rangle = H_{\rm int}|\phi\rangle$ with

$$H_{\rm int} = \frac{\Delta}{2}\sigma_z + i\frac{\Omega(t)}{2}(e^{-i\omega_c t}|g\rangle\langle e| + e^{i\omega_c t}|e\rangle\langle g|)(e^{i\omega_c t}a^\dagger - e^{-i\omega_c t}a),$$

where $\Delta = \omega_{eg} - \omega_c$ much smaller than ω_c ($\Delta/2\pi$ around 250 kHz, same order as $\Omega(t)$)
The secular terms of $H_{\rm int}$ are given by (RWA, first order approximation):

$$H_{\rm rwa} = \frac{\Delta}{2}(|e\rangle\langle e| - |g\rangle\langle g|) + i\frac{\Omega(t)}{2}(|g\rangle\langle e| a^\dagger - |e\rangle\langle g| a).$$

Since $\Omega(t) = \Omega_0 \exp\left(-\left(\frac{vt}{w}\right)^2\right)$ we have, with $\Delta > 0$ of the same order of Ω_0, for all t, $\left|\frac{d}{dt}\Omega(t)\right| \ll \Delta\Omega(t)$: adiabatic coupling between atom/cavity, called also dispersive interaction.

PDE model behind H_{rwa}

The quantum state $|\phi\rangle$ satisfying

$$\frac{d}{dt}|\phi\rangle = -i\left(\frac{\Delta}{2}(|e\rangle\langle e| - |g\rangle\langle g|) + i\frac{\Omega(t)}{2}(|g\rangle\langle e|\, a^\dagger - |e\rangle\langle g|\, a)\right)|\phi\rangle$$

is described by two elements of $L^2(\mathbb{R},\mathbb{C})$, ϕ_g and ϕ_e,

$$|\phi\rangle = (\phi_g(x,t), \phi_e(x,t)) \quad \text{with} \quad \|\phi_g\|^2_{L^2} + \|\phi_e\|^2_{L^2} = 1$$

and the time evolution is given by the coupled Partial Differential Equations (PDE's)

$$\frac{\partial \phi_g}{\partial t} = i\frac{\Delta}{2}\phi_g + \frac{\Omega(t)}{2\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right)\phi_e$$

$$\frac{\partial \phi_e}{\partial t} = -i\frac{\Delta}{2}\phi_e - \frac{\Omega(t)}{2\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)\phi_g$$

since $a = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)$ and $a^\dagger = \frac{1}{\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right)$.

Jaynes-Cumming model: adiabatic propagator

The unitary operator U solution of

$$\frac{d}{dt}U_t = -i\left(\frac{\Delta}{2}(|e\rangle\langle e| - |g\rangle\langle g|) + i\frac{\Omega(t)}{2}(|g\rangle\langle e| a^\dagger - |e\rangle\langle g| a)\right) U_t$$

starting from $U_{-T} = I$ with $vT/w \gg 1$ ($\Omega(-T) \approx 0$), satisfies in a good approximation

$$U_T = |g\rangle\langle g| e^{i\,\phi(N)} + |e\rangle\langle e| e^{-i\,\phi(N+1)}$$

with $\phi(n)$ being the analytic function (light-induced phase):

$$\phi(n) = \frac{1}{2}\int_{-T}^{+T}\sqrt{\Delta^2 + n\,\Omega^2(t)}dt$$

Proof: for each n, use invariance of the space $(|g, n+1\rangle, |e, n\rangle)$ and the adiabatic propagator for the spin system with an adiabatic Hamiltonian of the form $\frac{\Delta_r}{2}\sigma_z + \frac{v(\epsilon t)}{2}\sigma_y$.

Exercise on Jaynes-Cumming propagator (1)

Let us consider the Jaynes-Cumming propagator U_C

$$U_C = e^{-i\tau\left(\frac{\Delta\left(|e\rangle\langle e|-|g\rangle\langle g|\right)}{2}+i\frac{\Omega\left(|g\rangle\langle e|a^\dagger-|e\rangle\langle g|a\right)}{2}\right)}$$

where τ is an interaction time, Δ and Ω are constant.

- Show by recurrence on integer k that

$$\left(\Delta\left(|e\rangle\langle e| - |g\rangle\langle g|\right) + i\Omega\left(|g\rangle\langle e| a^\dagger - |e\rangle\langle g| a\right)\right)^{2k} = |e\rangle\langle e|\left(\Delta^2+(N+1)\Omega^2\right)^k + |g\rangle\langle g|\left(\Delta^2+N\Omega^2\right)^k$$

and that

$$\left(\Delta\left(|e\rangle\langle e| - |g\rangle\langle g|\right) + i\Omega\left(|g\rangle\langle e| a^\dagger - |e\rangle\langle g| a\right)\right)^{2k+1} = |e\rangle\langle e|\Delta\left(\Delta^2+(N+1)\Omega^2\right)^k - |g\rangle\langle g|\Delta\left(\Delta^2+N\Omega^2\right)^k + i\Omega\left(|g\rangle\langle e|\left(\Delta^2+N\Omega^2\right)^k a^\dagger - |e\rangle\langle g| a\left(\Delta^2+N\Omega^2\right)^k\right).$$

- Deduce that

$$U_C = |g\rangle\langle g| \left(\cos\left(\frac{\tau\sqrt{\Delta^2+N\Omega^2}}{2}\right) + i\frac{\Delta \sin\left(\frac{\tau\sqrt{\Delta^2+N\Omega^2}}{2}\right)}{\sqrt{\Delta^2+N\Omega^2}} \right)$$

$$+ |e\rangle\langle e| \left(\cos\left(\frac{\tau\sqrt{\Delta^2+(N+1)\Omega^2}}{2}\right) - i\frac{\Delta \sin\left(\frac{\tau\sqrt{\Delta^2+(N+1)\Omega^2}}{2}\right)}{\sqrt{\Delta^2+(N+1)\Omega^2}} \right)$$

$$+ |g\rangle\langle e| \left(\frac{\Omega \sin\left(\frac{\tau\sqrt{\Delta^2+N\Omega^2}}{2}\right)}{\sqrt{\Delta^2+N\Omega^2}} \right) a^\dagger - |e\rangle\langle g|\, a \left(\frac{\Omega \sin\left(\frac{\tau\sqrt{\Delta^2+N\Omega^2}}{2}\right)}{\sqrt{\Delta^2+N\Omega^2}} \right)$$

where $N = a^\dagger a$ the photon-number operator (a is the photon annihilator operator).

Exercise on Jaynes-Cumming propagator (3)

- In the resonant case, $\Delta = 0$, prove that:

$$U_C = |g\rangle\langle g| \cos\left(\frac{\Theta}{2}\sqrt{N}\right) + |e\rangle\langle e| \cos\left(\frac{\Theta}{2}\sqrt{N+1}\right) + |g\rangle\langle e| \left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{N}\right)}{\sqrt{N}}\right) a^\dagger - |e\rangle\langle g| a \left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{N}\right)}{\sqrt{N}}\right)$$

and check that $U_C^\dagger U_C = I$.