Modeling and Control of the LKB Photon-Box: Spin-Spring Systems

Pierre Rouchon

pierre.rouchon@mines-paristech.fr

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1LKB: Laboratoire Kastler Brossel, ENS, Paris. Several slides have been used during the IHP course (fall 2010) given with Mazyar Mirrahimi (INRIA) see: http://cas.ensmp.fr/~rouchon/QuantumSyst/index.html
1. Spring systems: the quantum harmonic oscillator

2. Spin-spring systems: the Jaynes-Cummings model
Harmonic oscillator\(^2\) (1): quantization and correspondence principle

Classical Hamiltonian formulation of \(\frac{d^2}{dt^2} x = -\omega^2 x\)

\[
\frac{d}{dt} x = \omega p = \frac{\partial H}{\partial p}, \quad \frac{d}{dt} p = -\omega x = -\frac{\partial H}{\partial x}, \quad H = \frac{\omega}{2} (p^2 + x^2).
\]

Quantization: probability wave function \(|\psi\rangle_t \sim \psi(x, t)\)_{x \in \mathbb{R}}\) with \(|\psi\rangle_t \sim \psi(., t) \in L^2(\mathbb{R}, \mathbb{C})\) obeys to the Schrödinger equation \((\hbar = 1\) in all the lectures) \[
i \frac{d}{dt} |\psi\rangle = H |\psi\rangle, \quad H = \omega(P^2 + X^2) = -\frac{\omega}{2} \frac{\partial^2}{\partial x^2} + \frac{\omega}{2} x^2
\]

where \(H\) results from \(H\) by replacing \(x\) by position operator \(\sqrt{2}X\) and \(p\) by impulsion operator \(\sqrt{2}P = -i \frac{\partial}{\partial x}\).

PDE model: \(i \frac{\partial \psi}{\partial t}(x, t) = -\frac{\omega}{2} \frac{\partial^2 \psi}{\partial x^2}(x, t) + \frac{\omega}{2} x^2 \psi(x, t), \quad x \in \mathbb{R}\).

Averaged position $\langle X \rangle_t = \langle \psi | X | \psi \rangle$ and impulsion $\langle P \rangle_t = \langle \psi | P | \psi \rangle$:

$$\langle X \rangle_t = \frac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} x |\psi|^2 dx, \quad \langle P \rangle_t = -\frac{i}{\sqrt{2}} \int_{-\infty}^{+\infty} \psi^* \frac{\partial \psi}{\partial x} dx.$$ 

Annihilation $a$ and creation operators $a^\dagger$:

$$a = X + iP = \frac{1}{\sqrt{2}} \left( x + \frac{\partial}{\partial x} \right), \quad a^\dagger = X - iP = \frac{1}{\sqrt{2}} \left( x - \frac{\partial}{\partial x} \right)$$

Commutation relationships:

$$[X, P] = \frac{i}{2}, \quad [a, a^\dagger] = 1, \quad H = \omega(P^2 + X^2) = \omega \left( a^\dagger a + \frac{1}{2} \right).$$

Set $X_\lambda = \frac{1}{2} \left( e^{-i\lambda} a + e^{i\lambda} a^\dagger \right)$ for any angle $\lambda$:

$$\left[ X_\lambda, X_{\lambda + \frac{\pi}{2}} \right] = \frac{i}{2}.$$ 

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3 We assume everywhere that for each $t$, $x \mapsto \psi(x, t)$ is of the Schwartz class (fast decay at infinity + smooth).
Harmonic oscillator (3): spectral decomposition and Fock states

\[ [a, a^\dagger] = 1 \] and \( \text{Ker}(a) \) of dimension one imply that the spectrum of \( N = a^\dagger a \) is non-degenerate and is \( \mathbb{N} \). More we have the useful commutations for any entire function \( f \):

\[
af(N) = f(N + 1) a, \quad f(N) a^\dagger = a^\dagger f(N + 1).
\]

Fock state with \( n \) photon(s): the eigen-state of \( N \) associated to the eigen-value \( n \):

\[
N |n\rangle = n |n\rangle, \quad a |n\rangle = \sqrt{n} |n - 1\rangle, \quad a^\dagger |n\rangle = \sqrt{n + 1} |n + 1\rangle.
\]

The ground state \( |0\rangle \) (0 photon state or vacuum state) satisfies \( a |0\rangle = 0 \) and corresponds to the Gaussian function:

\[
|0\rangle \sim \psi_0(x) = \frac{1}{\pi^{1/4}} \exp(-x^2/2).
\]

The operator \( a \) (resp. \( a^\dagger \)) is the annihilation (resp. creation) operator since it transfers \( |n\rangle \) to \( |n - 1\rangle \) (resp. \( |n + 1\rangle \)) and thus decreases (resp. increases) the quantum number \( n \) by one unit.
Harmonic oscillator (4): displacement operator

Quantization of $\frac{d^2}{dt^2}x = -\omega^2 x - \omega\sqrt{2}u$

$$H = \omega \left( a^\dagger a + \frac{1}{2} \right) + u(a + a^\dagger).$$

The associated controlled PDE

$$i \frac{\partial \psi}{\partial t}(x, t) = -\frac{\omega}{2} \frac{\partial^2 \psi}{\partial x^2}(x, t) + \left( \frac{\omega}{2} x^2 + \sqrt{2}ux \right) \psi(x, t).$$

Glauber displacement operator $D_\alpha$ (unitary) with $\alpha \in \mathbb{C}$:

$$D_\alpha = e^{\alpha a^\dagger - \alpha^* a} = e^{2i\Im \alpha X - 2i\Re \alpha P}$$

From Baker-Campbell Hausdorff formula valid for any operators $A$ and $B$,

$$e^A Be^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \frac{1}{3!} [A, [A, [A, B]]] + \ldots$$

we get the Glauber formula when $[A, [A, B]] = [B, [A, B]] = 0$:

$$e^{A+B} = e^A e^B e^{-\frac{1}{2} [A, B]}.$$

(show that $C_t = e^{t(A+B)} - e^{tA} e^{tB} e^{-\frac{t^2}{2} [A, B]}$ satisfies $\frac{d}{dt} C = (A + B)C$)
Harmonic oscillator (5): identities resulting from Glauber formula

With $A = \alpha a^\dagger$ and $B = -\alpha^* a$, Glauber formula gives:

$$D_\alpha = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} e^{-\alpha^* a} = e^{\frac{|\alpha|^2}{2}} e^{-\alpha^* a} e^{\alpha a^\dagger}$$

$$D_{-\alpha} a D_\alpha = a + \alpha \quad \text{and} \quad D_{-\alpha} a^\dagger D_\alpha = a^\dagger + \alpha^*.$$

With $A = 2i\Im \alpha X \sim i\sqrt{2}\Im \alpha x$ and $B = -2i\Re \alpha P \sim -\sqrt{2}\Re \alpha \frac{\partial}{\partial x}$, Glauber formula gives:

$$D_\alpha = e^{-i\Re \alpha \Im x} e^{i\sqrt{2}\Im \alpha x} e^{-\sqrt{2}\Re \alpha \frac{\partial}{\partial x}}$$

$$(D_\alpha | \psi \rangle)_{x,t} = e^{-i\Re \alpha \Im x} e^{i\sqrt{2}\Im \alpha x} \psi(x - \sqrt{2}\Re \alpha, t)$$

**Exercice**

*For any $\alpha, \beta, \epsilon \in \mathbb{C}$, prove that*

$$D_{\alpha+\beta} = e^{\frac{\alpha^* \beta - \alpha \beta^*}{2}} D_\alpha D_\beta$$

$$D_{\alpha+\epsilon} D_{-\alpha} = \left( 1 + \frac{\alpha \epsilon^* - \alpha^* \epsilon}{2} \right) 1 + \epsilon a^\dagger - \epsilon^* a + O(|\epsilon|^2)$$

$$\left( \frac{d}{dt} D_\alpha \right) D_{-\alpha} = \left( \frac{\alpha^* \alpha - \alpha \alpha^*}{2} \frac{d}{dt} \right) 1 + \left( \frac{d}{dt} \alpha \right) a^\dagger - \left( \frac{d}{dt} \alpha^* \right) a.$$
Harmonic oscillator (6): lack of controllability

Take $|\psi\rangle$ solution of the controlled Schrödinger equation

$$i \frac{d}{dt} |\psi\rangle = \left( \omega \left( a^\dagger a + \frac{1}{2} \right) + u(a + a^\dagger) \right) |\psi\rangle.$$ 

Set $\langle a \rangle = \langle \psi | a | \psi \rangle$. Then

$$\frac{d}{dt} \langle a \rangle = -i \omega \langle a \rangle - iu.$$

From $a = X + iP$, we have $\langle a \rangle = \langle X \rangle + i \langle P \rangle$ where

$$\langle X \rangle = \langle \psi | X | \psi \rangle \in \mathbb{R} \quad \text{and} \quad \langle P \rangle = \langle \psi | P | \psi \rangle \in \mathbb{R}.$$

Consequently:

$$\frac{d}{dt} \langle X \rangle = \omega \langle P \rangle, \quad \frac{d}{dt} \langle P \rangle = -\omega \langle X \rangle - u.$$

Consider the change of frame $|\psi\rangle = e^{-i\theta_t} D \langle a \rangle_t |\chi\rangle$ with

$$\theta_t = \int_0^t \left( |\langle a \rangle|^2 + u \Re(\langle a \rangle) \right), \quad D \langle a \rangle_t = e^{\langle a \rangle_t a^\dagger - \langle a \rangle_t^* a},$$

Then $|\chi\rangle$ obeys the autonomous Schrödinger equation

$$i \frac{d}{dt} |\chi\rangle = \omega a^\dagger a |\chi\rangle.$$

The dynamics of $|\psi\rangle$ can be decomposed into two parts:

- a controllable part of dimension two for $\langle a \rangle$
- an uncontrollable part of infinite dimension for $|\chi\rangle$. 


Coherent states

\[ |\alpha\rangle = D_\alpha |0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{+\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad \alpha \in \mathbb{C} \]

are the states reachable from vacuum set. They are also the eigen-state of \(a\): \(a |\alpha\rangle = \alpha |\alpha\rangle\).

A widely known result in quantum optics\(^5\): classical currents and sources (generalizing the role played by \(u\)) only generate classical light (quasi-classical states of the quantized field generalizing the coherent state introduced here)

We just propose here a control theoretic interpretation in terms of reachable set from vacuum\(^6\)

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The composite system lives on the Hilbert space $\mathbb{C}^2 \otimes L^2(\mathbb{R}; \mathbb{C}) \sim \mathbb{C}^2 \otimes l^2(\mathbb{C})$ with the Jaynes-Cummings Hamiltonian

$$H_{JC} = \frac{\omega_{eg}}{2} \sigma_z + \omega_c (a^{\dagger} a + \frac{1}{2}) + i \frac{\Omega(t)}{2} \sigma_x (a^{\dagger} - a),$$

with $\omega_c/2\pi \approx \omega_{eg}/2\pi$ around 50 GHz. Gaussian radial profile of the cavity quantized mode (for $t = 0$ atom at cavity center):

$$\Omega(t) = \Omega_0 \exp \left( - \left( \frac{vt}{w} \right)^2 \right)$$

where $v$ is the atom velocity (250 m/s), $w$ is the width (6 mm), $\Omega_0/2\pi$ around 50 kHz. Thus we have also $\Omega(t) \ll \omega_c, \omega_{eg}$. 

We consider the change of frame:

$$|\psi\rangle = e^{-i\omega_0 t(a^\dagger a + \frac{1}{2})} e^{-i\omega_0 t \sigma_z} |\phi\rangle.$$ 

The system becomes

$$i \frac{d}{dt} |\phi\rangle = H_{\text{int}} |\phi\rangle$$

with

$$H_{\text{int}} = \frac{\Delta}{2} \sigma_z + i \frac{\Omega(t)}{2} (e^{-i\omega_0 t} |g\rangle \langle e| + e^{i\omega_0 t} |e\rangle \langle g|)(e^{i\omega_0 t} a^\dagger - e^{-i\omega_0 t} a),$$

where \(\Delta = \omega_{eg} - \omega_c\) much smaller than \(\omega_c\) (\(\Delta/2\pi\) around 250 kHz, same order as \(\Omega(t)\)).

The secular terms of \(H_{\text{int}}\) are given by (RWA, first order approximation):

$$H_{\text{rwa}} = \frac{\Delta}{2} (|e\rangle \langle e| - |g\rangle \langle g|) + i \frac{\Omega(t)}{2} (|g\rangle \langle e| a^\dagger - |e\rangle \langle g| a).$$

Since \(\Omega(t) = \Omega_0 \exp \left(-\left(\frac{vt}{w}\right)^2\right)\) we have, with \(\Delta > 0\) of the same order of \(\Omega_0\), for all \(t\), \(|\frac{d}{dt} \Omega(t)| \ll \Delta \Omega(t)\): adiabatic coupling between atom/cavity, called also dispersive interaction.
The quantum state $|\phi\rangle$ satisfying

$$\frac{d}{dt} |\phi\rangle = -i \left( \frac{\Delta}{2} (|e\rangle \langle e| - |g\rangle \langle g|) + i \frac{\Omega(t)}{2} (|g\rangle \langle e| a^\dagger - |e\rangle \langle g| a) \right) |\phi\rangle$$

is described by two elements of $L^2(\mathbb{R}, \mathbb{C})$, $\phi_g$ and $\phi_e$,

$$|\phi\rangle = (\phi_g(x, t), \phi_e(x, t)) \quad \text{with} \quad \|\phi_g\|^2_{L^2} + \|\phi_e\|^2_{L^2} = 1$$

and the time evolution is given by the coupled Partial Differential Equations (PDE’s)

$$\frac{\partial \phi_g}{\partial t} = i \frac{\Delta}{2} \phi_g + \frac{\Omega(t)}{2\sqrt{2}} \left( x - \frac{\partial}{\partial x} \right) \phi_e$$

$$\frac{\partial \phi_e}{\partial t} = -i \frac{\Delta}{2} \phi_e - \frac{\Omega(t)}{2\sqrt{2}} \left( x + \frac{\partial}{\partial x} \right) \phi_g$$

since $a = \frac{1}{\sqrt{2}} (x + \frac{\partial}{\partial x})$ and $a^\dagger = \frac{1}{\sqrt{2}} (x - \frac{\partial}{\partial x})$. 
The unitary operator \( U \) solution of

\[
\frac{d}{dt} U_t = -i \left( \frac{\Delta}{2} (|e\rangle \langle e| - |g\rangle \langle g|) + i \frac{\Omega(t)}{2} (|g\rangle \langle e| a^\dagger - |e\rangle \langle g| a) \right) U_t
\]

starting from \( U_{-T} = I \) with \( vT/w \gg 1 \) (\( \Omega(-T) \approx 0 \)), satisfies in a good approximation

\[
U_T = |g\rangle \langle g| e^{i\phi(N)} + |e\rangle \langle e| e^{-i\phi(N+1)}
\]

with \( \phi(n) \) being the analytic function (light-induced phase):

\[
\phi(n) = \frac{1}{2} \int_{-T}^{+T} \sqrt{\Delta^2 + n \Omega^2(t)} dt
\]

Proof: for each \( n \), use invariance of the space \( (|g, n+1\rangle, |e, n\rangle) \) and the adiabatic propagator for the spin system with an adiabatic Hamiltonian of the form \( \frac{\Delta}{2} \sigma_z + \frac{\nu(\epsilon t)}{2} \sigma_y \).
Exercise on Jaynes-Cumming propagator (1)

Let us consider the Jaynes-Cumming propagator $U_C$

$$-i\tau \left( \frac{\Delta (|e\rangle \langle e| - |g\rangle \langle g|)}{2} + i \Omega (|g\rangle \langle e| a^\dagger - |e\rangle \langle g| a) \right)$$

$$U_C = e^\tau$$

where $\tau$ is an interaction time, $\Delta$ and $\Omega$ are constant.

■ Show by recurrence on integer $k$ that

$$\left( \Delta (|e\rangle \langle e| - |g\rangle \langle g|) + i\Omega (|g\rangle \langle e| a^\dagger - |e\rangle \langle g| a) \right)^{2k} =$$

$$|e\rangle \langle e| \left( \Delta^2 + (N + 1)\Omega^2 \right)^k + |g\rangle \langle g| \left( \Delta^2 + N\Omega^2 \right)^k$$

and that

$$\left( \Delta (|e\rangle \langle e| - |g\rangle \langle g|) + i\Omega (|g\rangle \langle e| a^\dagger - |e\rangle \langle g| a) \right)^{2k+1} =$$

$$|e\rangle \langle e| \Delta \left( \Delta^2 + (N + 1)\Omega^2 \right)^k - |g\rangle \langle g| \Delta \left( \Delta^2 + N\Omega^2 \right)^k$$

$$+ i\Omega \left( |g\rangle \langle e| \left( \Delta^2 + N\Omega^2 \right)^k a^\dagger - |e\rangle \langle g| a \left( \Delta^2 + N\Omega^2 \right)^k \right).$$
Deduce that

\[ U_C = \langle g | g \rangle \left( \cos \left( \frac{\tau \sqrt{\Delta^2 + N\Omega^2}}{2} \right) + i \frac{\Delta \sin \left( \frac{\tau \sqrt{\Delta^2 + N\Omega^2}}{2} \right)}{\sqrt{\Delta^2 + N\Omega^2}} \right) + \langle e | e \rangle \left( \cos \left( \frac{\tau \sqrt{\Delta^2 + (N+1)\Omega^2}}{2} \right) - i \frac{\Delta \sin \left( \frac{\tau \sqrt{\Delta^2 + (N+1)\Omega^2}}{2} \right)}{\sqrt{\Delta^2 + (N+1)\Omega^2}} \right) + \langle g | e \rangle \left( \frac{\Omega \sin \left( \frac{\tau \sqrt{\Delta^2 + N\Omega^2}}{2} \right)}{\sqrt{\Delta^2 + N\Omega^2}} \right) a^\dagger - \langle e | g \rangle a \left( \frac{\Omega \sin \left( \frac{\tau \sqrt{\Delta^2 + N\Omega^2}}{2} \right)}{\sqrt{\Delta^2 + N\Omega^2}} \right) \]

where \( N = a^\dagger a \) the photon-number operator (\( a \) is the photon annihilator operator).
In the resonant case, $\Delta = 0$, prove that:

$$U_C = |g\rangle \langle g| \cos \left(\frac{\theta}{2} \sqrt{N}\right) + |e\rangle \langle e| \cos \left(\frac{\theta}{2} \sqrt{N + 1}\right)$$

$$+ |g\rangle \langle e| \left(\frac{\sin \left(\frac{\theta}{2} \sqrt{N}\right)}{\sqrt{N}}\right) a^\dagger - |e\rangle \langle g| a \left(\frac{\sin \left(\frac{\theta}{2} \sqrt{N}\right)}{\sqrt{N}}\right)$$

and check that $U_C^\dagger U_C = I$. 