Modeling and Control of the LKB Photon-Box:  
Introduction

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\[^1\] LKB: Laboratoire Kastler Brossel, ENS, Paris. Several slides have been used during the IHP course (fall 2010) given with Mazyar Mirrahimi (INRIA) see: http://cas.ensmp.fr/~rouchon/QuantumSyst/index.html
Outline

1. Control of a classical harmonic oscillator

2. Control of a quantum harmonic oscillator: the LKB photon-box in closed-loop

3. Measurement process in the LKB-photon box

4. Control input in the LKB photon-box

5. Outline of the lectures
For the harmonic oscillator of pulsation $\omega$ with measured position $y$, controlled by the force $u$ and subject to an additional unknown force $w$.

\[ x = (x_1, x_2) \in \mathbb{R}^2, \quad y = x_1 \]

\[ \frac{d}{dt} x_1 = x_2, \quad \frac{d}{dt} x_2 = -\omega^2 x_1 + u + w \]
Feedback for classical systems

Proportional Integral Derivative (PID) for \( \frac{d^2}{dt^2} y = -\omega^2 y + u + w \) with the set point \( v = y^{\text{set point}} \)

\[
    u = -K_p(y - y^{\text{set point}}) - K_d \frac{dy}{dt}(y - y^{\text{set point}}) - K_{\text{int}} \int (y - y^{\text{set point}})
\]

with the positive gains \( (K_p, K_d, K_{\text{int}}) \) tuned as follows \( (0 < \Omega_0 \sim \omega, 0 < \xi \sim 1, 0 < \epsilon \ll 1) \):

\[
    K_p = \Omega_0^2, \quad K_d = 2\xi\Omega_0, \quad K_{\text{int}} = \epsilon\Omega_0^3.
\]
Control of a classical harmonic oscillator

- **Controllability**: the control $u$ can steer the state $x$ to any location ($\frac{d}{dt} x_1 = x_2$, $\frac{d}{dt} x_2 = -\omega^2 x_1 + u$).

- **Observability**: from the knowledge of $u$ and $y$ one can recover without ambiguity the state $x$. ($y = x_1$ and $x_2 = \frac{d}{dt} y$).

- **Feed-forward** $u = u^r(t)$ associated to reference trajectory $t \mapsto (x^r(t), u^r(t), y^r(t))$ (performance).

- **Feed-back** $u = u^r(t) + \Delta u$ where $\Delta u$ depends on the measured output error $\Delta y = y - y^r(t)$ (stability).

- **Stability and robustness**: asymptotic regime for $t$ large of $\Delta x$ and $\Delta y$, sensitivity to perturbations and errors.
Control of quantum harmonic oscillator: LKB photon-box

Control “u” = α

Output “y”
Detection in |g⟩ or |e⟩

Simple schematic of LKB experiment for control of cavity field

A discrete-time system: non-linear Markov chain of state |ψ⟩

\[
|ψ⟩_{k+1} = \begin{cases} 
\frac{Dα M_g |ψ⟩_k}{||M_g |ψ⟩_k||_H} \\
\frac{Dα M_e |ψ⟩_k}{||M_e |ψ⟩_k||_H}
\end{cases}
\]

Detect. in |g⟩ \left( \text{proba.} \ ||M_g |ψ⟩_k||^2_H \right)

Detect. in |e⟩ \left( \text{proba.} \ ||M_e |ψ⟩_k||^2_H \right)
Photon-box: simulations in closed-loop

PhotonBox

\[ P(u) = (P) \]

Coherent pulse

Real Pop

Detector

Lyap

Est Pop

QuantumFilter_Controller

Pop_Est

\[ \text{Real Pop} \]

\[ \text{Detector} \]

\[ \text{Lyap} \]

\[ \text{Est Pop} \]
Simple schematic of LKB experiment for measurement of cavity field
Photon-box (2) : atom-field entanglement

**Initial state** Atom in $|g\rangle$ and cavity in $|\psi\rangle \in \mathcal{H}$ where

$$\mathcal{H} = \left\{ \sum_{k=n}^{\infty} c_n |n\rangle \mid (c_n) \in l^2(\mathbb{C}) \right\}.$$ 

We can write the initial state as

$$|g\rangle \otimes |\psi\rangle \in \mathbb{C}^2 \otimes \mathcal{H}.$$ 

**State before detection** A joint unitary evolution implies an entangled state

$$|g\rangle \otimes \mathcal{M}_g |\psi\rangle + |e\rangle \otimes \mathcal{M}_e |\psi\rangle$$

where $\mathcal{M}_g$ and $\mathcal{M}_e$ are operators acting on $\mathcal{H}$. The unitarity condition implies:

$$\mathcal{M}_g^\dagger \mathcal{M}_g + \mathcal{M}_e^\dagger \mathcal{M}_e = 1$$

**Example of non-resonant interaction**

$$\mathcal{M}_g = \cos(\vartheta \mathbf{N} + \varphi), \quad \mathcal{M}_e = \sin(\vartheta \mathbf{N} + \varphi), \quad \mathbf{N} = \text{diag}(n)$$
Final state is inseparable: we can not write

$$|g\rangle \otimes M_g |\psi\rangle + |e\rangle \otimes M_e |\psi\rangle \neq \left(\tilde{\alpha} |g\rangle + \tilde{\beta} |e\rangle\right) \otimes \left(\sum_n \tilde{c}_n |n\rangle\right).$$

We can not associate to the cavity (nor to the atom) a well-defined wavefunction just before the measurement.

However, we can still compute the probability of having the atom in $|g\rangle$ or in $|e\rangle$:

$$P_g = \left\| M_g |\psi\rangle \right\|_{H}^2, \quad P_e = \left\| M_e |\psi\rangle \right\|_{H}^2.$$
Photon-box (4): measurement and collapse

Measurement in $|g\rangle$

$$|g\rangle \otimes M_g |\psi\rangle + |e\rangle \otimes M_e |\psi\rangle \rightarrow \frac{|g\rangle \otimes M_g |\psi\rangle}{\|M_g |\psi\rangle\|_H},$$

Measurement in $|e\rangle$

$$|g\rangle \otimes M_g |\psi\rangle + |e\rangle \otimes M_e |\psi\rangle \rightarrow \frac{|e\rangle \otimes M_e |\psi\rangle}{\|M_e |\psi\rangle\|_H},$$
**Stochastic evolution:** $\psi_k$ the wave function after the measurement of atom number $k - 1$.

\[
|\psi\rangle_{k+1} = \begin{cases} 
\frac{D_\alpha M_g |\psi\rangle_k}{||M_g |\psi\rangle_k||_H} & \text{Detect. in } |g\rangle \left(\text{proba. } ||M_g |\psi\rangle_k||_H^2\right) \\
\frac{D_\alpha M_e |\psi\rangle_k}{||M_e |\psi\rangle_k||_H} & \text{Detect. in } |e\rangle \left(\text{proba. } ||M_e |\psi\rangle_k||_H^2\right)
\end{cases}
\]

We have a Markov chain
Photon-box (6): imperfect measurement

The atom-detector does not always detect the atoms. Therefore 3 outcomes:

Atom in $|g\rangle$, Atom in $|e\rangle$, No detection

Best estimate for the no-detection case

$$E (|\psi\rangle_+ | |\psi\rangle) = \left\| M_g |\psi\rangle \right\|_H M_g |\psi\rangle + \left\| M_e |\psi\rangle \right\|_H M_e |\psi\rangle$$

This is not a well-defined wavefunction

Barycenter in the sense of geodesics of $S(H)$

not invariant with respect to a change of global phase

We need a barycenter in the sense of the projective space

$$\mathbb{CP}(H) \equiv S(H)/S^1$$
Photon-box (7): density matrix language

Projector over the state $|\psi\rangle$: $P_{|\psi\rangle} = |\psi\rangle \langle \psi |$

Detection in $|g\rangle$: the projector is given by

$$P_{|\psi+\rangle} = \frac{M_g |\psi\rangle \langle \psi | M_g^\dagger}{\|M_g |\psi\rangle\|_H^2} = \frac{M_g |\psi\rangle \langle \psi | M_g^\dagger}{\langle \psi | M_g^\dagger M_g | \psi \rangle^2} = \frac{M_g |\psi\rangle \langle \psi | M_g^\dagger}{\text{Tr} \left( M_g |\psi\rangle \langle \psi | M_g^\dagger \right)}$$

Detection in $|e\rangle$: the projector is given by

$$P_{|\psi+\rangle} = \frac{M_e |\psi\rangle \langle \psi | M_e^\dagger}{\text{Tr} \left( M_e |\psi\rangle \langle \psi | M_e^\dagger \right)}$$

Probabilities:

$$p_g = \text{Tr} \left( M_g |\psi\rangle \langle \psi | M_g^\dagger \right) \quad \text{and} \quad p_e = \text{Tr} \left( M_e |\psi\rangle \langle \psi | M_e^\dagger \right)$$
Photon-box (8): density matrix language

Imperfect detection: barycenter

\[ |\psi\rangle \langle \psi| \rightarrow p_g \frac{M_g |\psi\rangle \langle \psi| M_g^\dagger}{\text{Tr} \left( M_g |\psi\rangle \langle \psi| M_g^\dagger \right)} + p_e \frac{M_e |\psi\rangle \langle \psi| M_e^\dagger}{\text{Tr} \left( M_e |\psi\rangle \langle \psi| M_e^\dagger \right)} \]

\[ = M_g |\psi\rangle \langle \psi| M_g^\dagger + M_e |\psi\rangle \langle \psi| M_e^\dagger. \]

This is not anymore a projector: no well-defined wave function

Adapted state space

\[ \mathcal{X} = \{ \rho \in \mathcal{L}(\mathcal{H}) \mid \rho^\dagger = \rho, \rho \geq 0, \text{Tr} (\rho) = 1 \} \]
A classical control input

The control input \( u = \alpha \) is classical and acts on the state \( |\psi\rangle \) according to the unitary transformation \( D_\alpha \) (displacement of amplitude \( \alpha \)):

\[
|\psi\rangle \mapsto D_\alpha |\psi\rangle = e^{\alpha a^\dagger - \alpha^* a} |\psi\rangle.
\]
Outline of the lectures

**Introduction:** LKB Photon-Box, experimental data and simulations, non-linear state feedback stabilizing Fock states.

**Spin systems:** two-level systems, Dirac notations, Pauli matrices, density matrix as a Bloch vector, RWA, averaging, Rabi oscillation, adiabatic invariance and propagator.

**Spin-Spring systems:** harmonic oscillator, creation/annihilation operators, unitary displacement operator, coherent states, Jaynes-Cummings model, composite systems and tensor products, RWA and dressed states, dispersive and resonant propagators.

**Quantum Non-Demolition (QND) measurement:** LKB photon Box, QND photon counting, Positive Operator Valued Measurement (POVM), discrete-time quantum trajectories and Markov chains, Kraus maps.

**Feedback stabilization with QND measures:** martingales and Lyapunov functions, stochastic convergence, construction of strict control Lyapunov function, feedback stabilization.

**State estimations:** quantum filtering, ideal case, experimental case including detection errors, Bayes law.
Main references

- Mathematical system theory and control:

- Quantum physics and information
  - Serge Haroche Lectures at Collège de France (in French): www.cqed.org
  - D. Steck. Quantum and atom optics (notes for a course 2010)
    http://atomoptics.uoregon.edu/ dsteck/teaching/quantum-optics
Specific references on the Photon-Box


