Modeling and Control of the LKB Photon-Box: [1] Feedback stabilization with Quantum Non-Demolition (QND) Measurements

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[1] LKB: Laboratoire Kastler Brossel, ENS, Paris.
Several slides have been used during the IHP course (fall 2010) given with Mazyar Mirrahimi (INRIA) see:
http://cas.ensmp.fr/~rouchon/QuantumSyst/index.html

The controlled non-linear Markov chain

Attached to $\mathcal{M}_g = \cos(\varphi_0 + \vartheta N)$ and $\mathcal{M}_e = \sin(\varphi_0 + \vartheta N)$ we have the controlled Markov chain:

$$\rho_{k+1} = \mathbb{D}_{\alpha_k}(\rho_{k+\frac{1}{2}}), \qquad \rho_{k+\frac{1}{2}} = \mathbb{M}_{s_k}(\rho_k) = \frac{\mathcal{M}_{s_k} \rho_k \mathcal{M}_{s_k}^\dagger}{\operatorname{Tr}\left(\mathcal{M}_{s_k} \rho_k \mathcal{M}_{s_k}^\dagger\right)}$$

where

input: $\alpha_k \in \mathbb{R}$ drives a unitary operation on the cavity-field: $\mathbb{D}_\alpha(\rho) := D_\alpha \rho D_\alpha^\dagger$, $D_\alpha = \exp(\alpha(a^\dagger - a))$.

state: ρ_k the density matrix of the cavity-field; it resumes all the past.

output: $s_k \in \{g, e\}$ is a stochastic variable, associated to probabilities $p_{g,k}$ and $p_{e,k}$ depending on ρ_k,

$$p_{g,k} = \operatorname{Tr}\left(\mathcal{M}_g \rho_k \mathcal{M}_g^\dagger\right) \quad \text{and} \quad p_{e,k} = \operatorname{Tr}\left(\mathcal{M}_e \rho_k \mathcal{M}_e^\dagger\right),$$

and given by the detector outcome at time k.

Outline

1. Open-loop convergence properties
2. Feedback stabilization
   - A first feedback scheme
   - Construction of strict control Lyapunov functions
   - Quantum filter and separation principle
3. Stability, Lyapunov functions and martingales
   - Deterministic systems: continuous-time ODE
   - Stochastic systems: discrete-time Markov chain

Restriction to finite dimensional subspace spanned by the $n^{\max}+1$ first modes $\{|0\rangle, |1\rangle, \ldots, |n^{\max}\rangle\}$.

$$\mathbf{N} = \mathrm{diag}(0, 1, \ldots, n^{\max}), \qquad a|0\rangle = 0, \quad a|n\rangle = \sqrt{n}\,|n-1\rangle.$$

The truncated creation operator $a^\dagger$ is the Hermitian conjugate of a. We still have $\mathbf{N} = a^\dagger a$, but truncation does not preserve the usual commutation $[a, a^\dagger] = 1$ (this is only valid when $n^{\max} = \infty$).

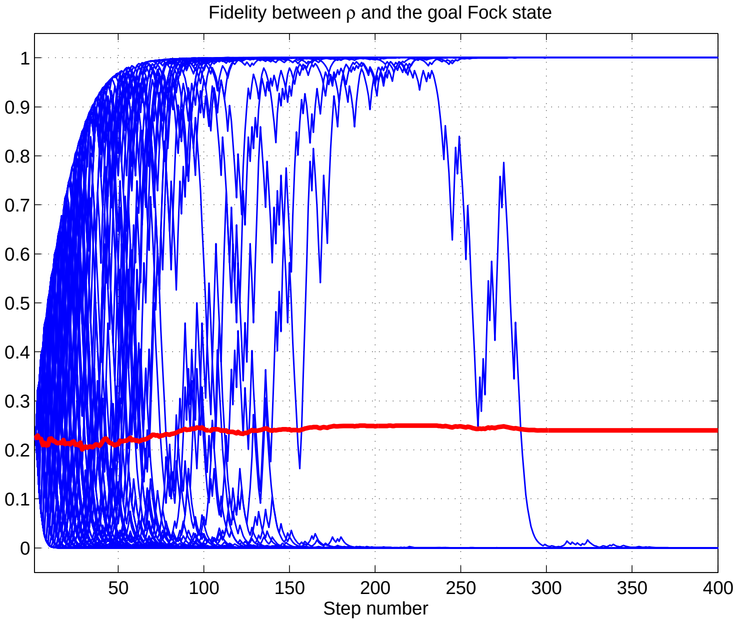
The Markov chain of state ρ ($\rho^\dagger = \rho$, $\rho \geq 0$ and $\mathrm{Tr}(\rho) = 1$):

$$\rho_{k+1} = \begin{cases} \mathbb{M}_g(\rho_k) = \frac{\mathcal{M}_g \rho_k \mathcal{M}_g^\dagger}{\mathrm{Tr}\left(\mathcal{M}_g \rho_k \mathcal{M}_g^\dagger\right)}, & \text{prob. } p_{g,k} = \mathrm{Tr}\left(\mathcal{M}_g \rho_k \mathcal{M}_g^\dagger\right); \\ \mathbb{M}_e(\rho_k) = \frac{\mathcal{M}_e \rho_k \mathcal{M}_e^\dagger}{\mathrm{Tr}\left(\mathcal{M}_e \rho_k \mathcal{M}_e^\dagger\right)}, & \text{prob. } p_{e,k} = \mathrm{Tr}\left(\mathcal{M}_e \rho_k \mathcal{M}_e^\dagger\right). \end{cases}$$

with $\mathcal{M}_g$ and $\mathcal{M}_e$ diagonal operators (dispersive atom/cavity interaction)

$$\mathcal{M}_g = \cos(\varphi_0 + N\vartheta), \quad \mathcal{M}_e = \sin(\varphi_0 + N\vartheta)$$

100 Monte-Carlo simulations with $\alpha_k \equiv 0$ ($\langle 3|\rho_k|3\rangle$ versus k)



Open-loop convergence of the truncated model [3]

Theorem

Consider the Markov process defined above with an initial density matrix ρ_0. Assume that the parameters φ_0, ϑ are chosen in order to have $\mathcal{M}_g = \cos(\varphi_0 + N\vartheta)$, $\mathcal{M}_e = \sin(\varphi_0 + N\vartheta)$ invertible and such that the spectrum of $\mathcal{M}_g^\dagger \mathcal{M}_g = \mathcal{M}_g^2$ and $\mathcal{M}_e^\dagger \mathcal{M}_e = \mathcal{M}_e^2$ are not degenerate. Then

1. *for any $n \in \{0, \ldots, n^{\max}\}$, $Tr(\rho_k |n\rangle\langle n|) = \langle n| \rho_k |n\rangle$ is a martingale*
2. *ρ_k converges with probability 1 to one of the $n^{\max} + 1$ Fock state $|n\rangle\langle n|$ with $n \in \{0, \ldots, n^{\max}\}$.*
3. *the probability to converge towards the Fock state $|n\rangle\langle n|$ is given by $Tr(\rho_0 |n\rangle\langle n|) = \langle n| \rho_0 |n\rangle$.*

Proof[2] : for stat.2 use $V^{\text{open-loop}}(\rho) = \sum_{n=0}^{n^{\max}} (\mathrm{Tr}(|n\rangle\langle n| \rho))^2$ and $\forall x_\mu, \theta_\mu \in [0,1]$

$$\sum_\mu \theta_\mu = 1 \implies \sum_\mu \theta_\mu (x_\mu)^2 = \left(\sum_\mu \theta_\mu x_\mu \right)^2 + \sum_{\mu,\nu} \theta_\mu \theta_\nu \frac{(x_\mu - x_\nu)^2}{2}$$

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[2] See H.Amini, M. Mirrahimi, PR: http://arxiv.org/abs/1103.1365.

[3] For the **infinite dimensional** Markov chain see R. Somaraju, M. Mirrahimi, PR: http://arxiv.org/abs/1103.1724

Lyapunov control for stabilizing $\bar{\rho} = |\bar{n}\rangle\langle\bar{n}|$

Choosing α_k such that $\mathbb{E}\left(\mathrm{Tr}\left(\rho_k\bar{\rho}\right)\right)$ is increasing.

We have

$$\rho_{k+\frac{1}{2}} = \begin{cases} \dfrac{M_g\rho_k M_g^\dagger}{\mathrm{Tr}\left(M_g\rho_k M_g^\dagger\right)}, & \text{with probability} \quad \mathrm{Tr}\left(M_g\rho_k M_g^\dagger\right), \\ \dfrac{M_e\rho_k M_e^\dagger}{\mathrm{Tr}\left(M_e\rho_k M_e^\dagger\right)}, & \text{with probability} \quad \mathrm{Tr}\left(M_e\rho_k M_e^\dagger\right), \end{cases}$$

So

$$\begin{aligned} \mathbb{E}\left(\mathrm{Tr}\left(\rho_{k+\frac{1}{2}}\bar{\rho}\right) \mid \rho_k\right) &= \mathrm{Tr}\left(|\bar{n}\rangle\langle\bar{n}|\, M_g\rho_k M_g^\dagger\right) + \mathrm{Tr}\left(|\bar{n}\rangle\langle\bar{n}|\, M_e\rho_k M_e^\dagger\right) \\ &= \mathrm{Tr}\left(|\bar{n}\rangle\langle\bar{n}|\,\rho_k\right), \end{aligned}$$

as

$$M_g^\dagger |\bar{n}\rangle\langle\bar{n}|\, M_g + M_e^\dagger |\bar{n}\rangle\langle\bar{n}|\, M_e = \left(\cos^2 + \sin^2\right)|\bar{n}\rangle\langle\bar{n}| = |\bar{n}\rangle\langle\bar{n}|\,.$$

Lyapunov control: continued

Furthermore

$$\rho_{k+1} = D(\alpha_k)\rho_{k+\frac{1}{2}} D(-\alpha_k),$$

and BCH formula

$$D_\alpha \rho D_\alpha^\dagger = e^{\alpha a^\dagger - \alpha^* a} \rho e^{-(\alpha a^\dagger - \alpha^* a)} = \rho + [\alpha a^\dagger - \alpha^* a, \rho] + O(|\alpha|^2).$$

So

$$\mathrm{Tr}\left(\rho_{k+1}\bar{\rho}\right) = \mathrm{Tr}\left(\rho_{k+\frac{1}{2}}\bar{\rho}\right) + \alpha_k \mathrm{Tr}\left([|\bar{n}\rangle\langle\bar{n}|, a^\dagger]\rho_{k+\frac{1}{2}}\right) - \alpha_k^* \mathrm{Tr}\left([|\bar{n}\rangle\langle\bar{n}|, a]\rho_{k+\frac{1}{2}}\right) + O(|\alpha_k|^2$$

Therefore, taking

$$\alpha_k = \epsilon \mathrm{Tr}\left(|\bar{n}\rangle\langle\bar{n}| [\rho_{k+\frac{1}{2}}, a]\right) = \epsilon \left(\mathrm{Tr}\left([|\bar{n}\rangle\langle\bar{n}|, a^\dagger]\rho_{k+\frac{1}{2}}\right)\right)^*,$$

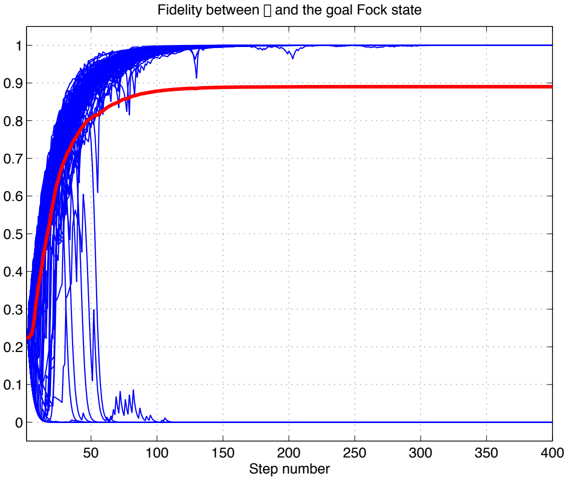
for sufficiently small $\epsilon > 0$, we have

$$\mathrm{Tr}\left(\rho_{k+1}\bar{\rho}\right) \geq \mathrm{Tr}\left(\rho_k\bar{\rho}\right) \quad \Longrightarrow \quad \mathbb{E}\left(\mathrm{Tr}\left(\rho_{k+1}\bar{\rho}\right) \mid \rho_k\right) \geq \mathrm{Tr}\left(\rho_k\bar{\rho}\right)$$

$\mathrm{Tr}\left(\rho_k\bar{\rho}\right)$ is a sub-martingale

Bad attractors

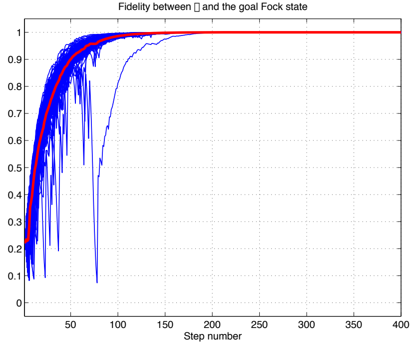
We do not have semi-global stabilization ...



$\mathrm{Tr}\left(\rho_k \bar{\rho}\right)$ converges almost surely towards a random variable with values 0 or 1

Modified feedback law [4]

$$\alpha_k = \begin{cases} \epsilon \mathrm{Tr}\left(\bar{\rho}[\rho_{k+\frac{1}{2}}, a]\right) & \text{if } \mathrm{Tr}\left(\bar{\rho}\rho_{k+\frac{1}{2}}\right) \geq \eta \\ \underset{|\alpha|\leq\bar{\alpha}}{\mathrm{argmax}}\ \mathrm{Tr}\left(\bar{\rho}\mathbb{D}_\alpha(\rho_{k+\frac{1}{2}})\right) & \text{if } \mathrm{Tr}\left(\bar{\rho}\rho_{k+\frac{1}{2}}\right) < \eta \end{cases}$$



[4] See I. Dotsenko et al., Phys. Rev. A, 2009. See also, M. Mirrahimi, R. Van Handel, SIAM JCO 2007, for a similar feedback in continuous time.

Closed-loop convergence

Closed-loop Markov chain:

$$, \qquad \rho_{k+1} = \mathbb{D}_{\alpha_k}(\rho_{k+\frac{1}{2}}), \qquad \rho_{k+\frac{1}{2}} = \mathbb{M}_{s_k}(\rho_k)$$

with

$$\alpha_k = \begin{cases} \epsilon \operatorname{Tr}\left(\bar{\rho}[\rho_{k+\frac{1}{2}}, a]\right) & \text{if } \operatorname{Tr}\left(\bar{\rho}\rho_{k+\frac{1}{2}}\right) \geq \eta \\ \underset{|\alpha|\leq\bar{\alpha}}{\operatorname{argmax}} \operatorname{Tr}\left(\bar{\rho}\mathbb{D}_\alpha(\rho_{k+\frac{1}{2}})\right) & \text{if } \operatorname{Tr}\left(\bar{\rho}\rho_{k+\frac{1}{2}}\right) < \eta \end{cases}$$

Theorem

Consider the above closed-loop quantum system. For small enough parameters $\epsilon, \eta > 0$ in the feedback scheme, the trajectories converge almost surely toward the target Fock state $\bar{\rho}$.

Proof's scheme

Four steps:

1. First, we show that for small enough η, the trajectories starting within the set $\mathcal{S}_{<\eta} = \{\rho \mid \mathrm{Tr}\,(\bar{\rho}\rho) < \eta\}$ always reach in one step the set $\mathcal{S}_{\geq 2\eta} = \{\rho \mid \mathrm{Tr}\,(\bar{\rho}\rho) \geq 2\eta\}$;
2. next, we show that the trajectories starting within the set $\mathcal{S}_{\geq 2\eta}$, will never hit the set $\mathcal{S}_{<\eta}$ with a uniformly non-zero probability $p_\eta > 0$ (Doob's inequality);
3. we prove an inequality showing that, for small enough ϵ, $\mathcal{V}(\rho_k) = f(\mathrm{Tr}\,(\bar{\rho}\rho_k))$ with $f(x) = \frac{x^2+x}{2}$ is a sub-martingale within $\mathcal{S}_{\geq\eta} = \{\rho \mid \mathrm{Tr}\,(\bar{\rho}\rho) \geq \eta\}$;
4. finally, we combine the previous step and the Kushner's invariance principle, to prove that almost all trajectories remaining inside $\mathcal{S}_{\geq\eta}$ converge towards $\bar{\rho}$.

Step 2: Doob's inequality

Doob's Inequality

Let $\{X_n\}$ be a Markov chain on state space $\mathcal{X}$. Suppose that there is a non-negative function $V(x)$ satisfying $\mathbb{E}\left(V(X_1) \mid X_0 = x\right) - V(x) = -k(x)$, where $k(x) \geq 0$ on the set $\{x : V(x) < \lambda\} \equiv Q_\lambda$. Then

$$\mathbb{P}\left(\sup_{\infty > n \geq 0} V(X_n) \geq \lambda \,\middle|\, X_0 = x\right) \leq \frac{V(x)}{\lambda}.$$

Here we take $V(\rho_k) = 1 - \operatorname{Tr}(\bar{\rho}\rho_k)$ which is a super-martingale. We have:

$$\mathbb{P}\left(\sup_{k' \geq k}(1 - \operatorname{Tr}(\bar{\rho}\rho_{k'})) \geq 1 - \eta \,\middle|\, \rho_k \in \mathcal{S}_{\geq 2\eta}\right) \leq \frac{1 - \operatorname{Tr}(\bar{\rho}\rho_k)}{1 - \eta} \leq \frac{1 - 2\eta}{1 - \eta},$$

and thus

$$\mathbb{P}\left(\inf_{k' \geq k} \operatorname{Tr}(\bar{\rho}\rho_{k'}) > \eta \,\middle|\, \operatorname{Tr}(\bar{\rho}\rho_k) \geq 2\eta\right)$$
$$= 1 - \mathbb{P}\left(\sup_{k' \geq k}(1 - \operatorname{Tr}(\bar{\rho}\rho_{k'})) \geq 1 - \eta \,\middle|\, \operatorname{Tr}(\bar{\rho}\rho_k) \geq 2\eta\right)$$
$$\geq 1 - \frac{1 - 2\eta}{1 - \eta} = \frac{\eta}{1 - \eta} = p_\eta.$$

Strict control-Lyapunov function[5] (1)

For any function λ, consider the open-loop martingale

$$V_\lambda(\rho) = \text{Tr}\left(\lambda(N)\rho\right) = \sum_{n=1}^{d} \lambda_n \text{Tr}\left(|n\rangle\langle n|\,\rho\right) = \sum_{n=1}^{d} \lambda_n \langle n|\,\rho\,|n\rangle\,.$$

($\lambda(N)$ is a fixed point of the adjoint Kraus map).
For each Fock state $\rho\,|n\rangle\langle n|$, $\alpha = 0$ is a critical point of $\alpha \mapsto V_\lambda(\mathbb{D}_\alpha(\rho)$, $\left.\frac{dV_\lambda\left(\mathbb{D}_\alpha(|n\rangle\langle n|)\right)}{d\alpha}\right|_{\alpha=0} = 0$, and

$$\left.\frac{d^2 V_\lambda\left(\mathbb{D}_\alpha(|n\rangle\langle n|)\right)}{d\alpha^2}\right|_{\alpha=0} = \text{Tr}\left([[a^\dagger - a, [a^\dagger - a, \lambda(N)]]\,|n\rangle\langle n|\right) = \text{Tr}\left(R\lambda(N)\,|n\rangle\langle n|\right)$$

where $R =$ is a tridiagonal Laplacian matrix with $\dim(\ker R) = 1$ with entries

$$R_{n-1,n} = 2n, \quad , R_{n,n} = -4n-2, \quad R_{n+1,n} = 2n+2.$$

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[5] See H.Amini, M. Mirrahimi, PR: CDC/ECC 2011 http://arxiv.org/abs/1103.1365.

Strict control-Lyapunov function (2)

Take a goal Fock state $|\bar{n}\rangle$ and, for each $n \neq \bar{n}$, $\sigma_n > 0$. By inverting The Laplacian R, we define λ_n such that, for any $n \neq 0$,

$$\left.\frac{d^2 V_\lambda\left(\mathbb{D}_\alpha(|n\rangle\langle n|)\right)}{d\alpha^2}\right|_{\alpha=0} = \sigma_n > 0.$$

Then $\left.\frac{d^2 V_\lambda\left(\mathbb{D}_\alpha(|\bar{n}\rangle\langle \bar{n}|)\right)}{d\alpha^2}\right|_{\alpha=0} = -\sum_{n\neq\bar{n}} \sigma_n < 0$. Moreover $n \mapsto \lambda(n)$ is strictly increasing from 0 to $\bar{n}$ and strictly decreasing from $\bar{n}$ to n^{max}.

Then, for $\epsilon > 0$ small enough

$$W_\epsilon(\rho) = \epsilon V^{\text{open-loop}}(\rho) + V_\lambda(\rho)$$

becomes a strict Lyapunov function with the feedback

$$\alpha_k = K(\rho_{k+\frac{1}{2}}) = \underset{\alpha\in[-\bar{\alpha},\bar{\alpha}]}{\text{argmax}}\Big(W_\epsilon\left(\mathbb{D}_\alpha\left(\rho_{k+\frac{1}{2}}\right)\right)\Big),$$

for any $\bar{\alpha} > 0$.

Strict control-Lyapunov function (3)

In closed-loop W_ϵ is a strict sub-martingale since, for $\rho_k \neq |\bar{n}\rangle\langle\bar{n}|$,

$$\mathbb{E}\left(W_\epsilon(\rho_{k+1})|\rho_k\right) > W_\epsilon(\rho_k)$$

because we have

$$\begin{aligned}
&\mathbb{E}\left(W_\epsilon(\rho_{k+1})|\rho_k\right) - W_\epsilon(\rho_k) = \\
&\quad \sum_{\mu\in\{g,e\}} p_{\mu,\rho_k}\left(\max_{\alpha\in[-\bar{\alpha},\bar{u}]}\Big(W_\epsilon\big(\mathbb{D}_\alpha(\mathbb{M}_\mu(\rho_k))\big)\Big) - W_\epsilon(\rho_k)\right) = \\
&\qquad \sum_{\mu\in\{g,e\}} p_{\mu,\rho_k}\Big(W_\epsilon\big(\mathbb{M}_\mu(\rho_k)\big) - W_\epsilon(\rho_k)\Big) + \\
&\qquad \sum_{\mu\in\{g,e\}} p_{\mu,\rho_k}\left(\max_{\alpha\in[-\bar{\alpha},\bar{\alpha}]}\Big(W_\epsilon\big(\mathbb{D}_\alpha(\mathbb{M}_\mu(\rho_k))\big)\Big) - W_\epsilon\big(\mathbb{M}_\mu(\rho_k)\big)\right)
\end{aligned}$$

The blue sum is strictly positive, excepted when ρ_k is a Fock state (see open-loop convergence). The red sum is always non-negative. When ρ_k is a Fock state, the red sum vanishes only for $\rho_k = |\bar{n}\rangle\langle\bar{n}|$.

Quantum filter for feedback control

$$\rho_{k+1} = \mathbb{M}_{s_k}(\rho_{k+\frac{1}{2}}), \qquad \rho_{k+\frac{1}{2}} = \mathbb{D}_{\alpha_k}(\rho_k).$$

We wish to find the control α_k as a function of the k first measured jumps. In this aim we need to estimate the state of the system.

We consider here the ideal case (no measurement uncertainties nor decoherence): Best estimate is given by the system dynamics itself.

Quantum filter

$$\rho^{\text{est}}_{k+1} = \mathbb{M}_{s_k}(\rho^{\text{est}}_{k+\frac{1}{2}}), \qquad \rho^{\text{est}}_{k+\frac{1}{2}} = \mathbb{D}_{\alpha_k}(\rho^{\text{est}}_k),$$

where the values for $s_k \in \{g, e\}$ are given by the measurement results and α_k is a function of ρ^{est}_k: $\alpha_k = \alpha(\rho^{\text{est}}_k)$.

A quantum separation principle[6]

System+Filter dynamics:

$$\rho_{k+\frac{1}{2}} = \mathbb{M}_{s_k}(\rho_k), \qquad \rho_{k+1} = \mathbb{D}_{\alpha_k}(\rho_{k+\frac{1}{2}}),$$
$$\rho^{\text{est}}_{k+\frac{1}{2}} = \mathbb{M}_{s_k}(\rho^{\text{est}}_k), \qquad \rho^{\text{est}}_{k+1} = \mathbb{D}_{\alpha_k}(\rho^{\text{est}}_{k+\frac{1}{2}})$$

where s_k takes the values g or e with probabilities $p_{g,k}$ and $p_{e,k}$ given by

$$p_{g,k} = \text{Tr}\left(\mathcal{M}_g \rho_k \mathcal{M}_g^\dagger\right), \qquad p_{e,k} = \text{Tr}\left(\mathcal{M}_e \rho_k \mathcal{M}_e^\dagger\right)$$

and where $\alpha_k = \alpha(\rho^{\text{est}}_{k+\frac{1}{2}})$.

Theorem: a quantum separation principle

Consider a closed-loop system of the above form. Assume moreover that, whenever $\rho^{\text{est}}_0 = \rho_0$ (so that the quantum filter coincides with the closed-loop dynamics, $\rho^{\text{est}} \equiv \rho$), the closed-loop system converges almost surely towards a fixed pure state $\bar{\rho}$. Then, for any choice of the initial state ρ^{est}_0, such that $\ker \rho^{\text{est}}_0 \subset \ker \rho_0$, the trajectories of the system-filter converge almost surely towards the same pure state: $\rho_k, \rho^{\text{est}}_k \to \bar{\rho}$.

[6] See R. Van Handel: Filtering, Stability, and Robustness. PhD thesis,

Proof (1)

$\mathbb{E}\left(\mathrm{Tr}\left(\rho_k \bar{\rho}\right) \mid \rho_0, \rho_0^{\mathrm{est}}\right)$ depends linearly on ρ_0 even though we are applying a feedback control.

Indeed, we can write

$$\alpha_k = \alpha(\rho_0^{\mathrm{est}}, s_0, \ldots, s_{k-1}),$$

and simple computations imply

$$\mathbb{E}\left(\mathrm{Tr}\left(\bar{\rho}\rho_k\right) \mid \rho_0, \rho_0^{\mathrm{est}}\right) = \sum_{s_0,\ldots,s_{k-1}} \mathrm{Tr}\left(\bar{\rho}\, \widetilde{\mathbb{M}}_{s_{k-1}} \circ \mathbb{D}_{\alpha_{k-1}} \circ \ldots \circ \widetilde{\mathbb{M}}_{s_0} \circ \mathbb{D}_{\alpha_0}(\rho_0)\right)$$

where

$$\widetilde{\mathbb{M}}_s \rho = \mathcal{M}_s \rho \mathcal{M}_s^{\dagger}.$$

So, we easily have the linearity of $\mathbb{E}\left(\mathrm{Tr}\left(\rho_k \bar{\rho}\right) \mid \rho_0, \rho_0^{\mathrm{est}}\right)$ with respect to ρ_0.

The rest of the proof follows from the assumption $\ker \rho_0^{\mathrm{est}} \subset \ker \rho_0$ which implies the existence of a constant $\gamma > 0$ and a density matrix ρ_0^c, such that

$$\rho_0^{\mathrm{est}} = \gamma \rho_0 + (1-\gamma)\rho_0^c.$$

Proof (2)

We know that if both the system and filter start at ρ_0^{est}, we have the almost sure convergence. This, together with dominated convergence theorem implies

$$\lim_{k\to\infty} \mathbb{E}\left(\text{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0^{\text{est}}, \rho_0^{\text{est}}\right) = 1.$$

By the linearity of $\mathbb{E}\left(\text{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0, \rho_0^{\text{est}}\right)$ with respect to ρ_0, we have

$$\mathbb{E}\left(\text{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0^{\text{est}}, \rho_0^{\text{est}}\right) = \gamma\mathbb{E}\left(\text{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0, \rho_0^{\text{est}}\right) + (1-\gamma)\mathbb{E}\left(\text{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0^c, \rho_0^{\text{est}}\right),$$

and as both $\mathbb{E}\left(\text{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0, \rho_0^{\text{est}}\right)$ and $\mathbb{E}\left(\text{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0^c, \rho_0^{\text{est}}\right)$ are less than or equal to one, we necessarily have that both of them converge to 1:

$$\lim_{k\to\infty} \mathbb{E}\left(\text{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0, \rho_0^{\text{est}}\right) = 1.$$

This implies the almost sure convergence of the physical system towards the pure state $\bar{\rho}$.

Lyapunov stability for ODE

$\bar{x} \in \mathbb{R}^n$ is an equilibrium of $\frac{d}{dt}x = v(x)$, when $v(\bar{x}) = 0$.

Stability

Equilibrium $\bar{x} \in \mathbb{R}^n$ is stable iff $\forall \epsilon > 0$, $\exists \eta > 0$ such that $\forall x^0$, $\|x^0 - \bar{x}\| \leq \eta$, the solution of the Cauchy problem $\frac{d}{dt}x = v(x, t)$ starting from x^0 at $t = 0$ satisfies

$$\|x(t) - \bar{x}\| \leq \epsilon, \quad \forall t \geq 0$$

Asymptotic stability

The equilibrium $\bar{x} \in \mathbb{R}^n$ is said locally asymptotically stable iff it is stable and moreover, $\exists \eta > 0$ such that

$$\|x^0 - \bar{x}\| \leq \eta, \text{ implies } x(t) \longrightarrow \bar{x}$$

when $t \longrightarrow +\infty$

First Lyapunov method [7]

Spectrum and local stability

The equilibrium $\bar{x}$ of $\frac{d}{dt}x = v(x)$ is *locally asymptotically stable* if the eigenvalues of the Jacobian matrix at $\bar{x}$,

$$\left(\frac{\partial v_i}{\partial x_j}\right)_{\bar{x}},$$

are all with strictly negative real parts.
The equilibrium $\bar{x}$ is unstable if at least one of the eigenvalues of the Jacobian matrix admits a strictly positive real part

[7] See H.K. Khalil, Nonlinear Systems (Prentice Hall, 2001).

Second Lyapunov method[8]

Lyapunov functions and Lasalle's invariance principle

$\mathbb{R}^n \ni x \mapsto v(x) \in \mathbb{R}^n$ C^1 versus x. Take $\mathbb{R}^n \ni x \mapsto V(x) \in \mathbb{R}^+$ a C^1 function of x. Assume that

1. $\lim_{\|x\| \mapsto +\infty} V(x) = +\infty$
2. V decreases along all solutions of $\frac{d}{dt}x = v(x)$:

$$\frac{d}{dt}V(x) = \nabla V(x) \cdot v(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(x)\; v_i(x) \leq 0, \quad \text{for all } x.$$

Then, for all initial condition x^0, the solution $\frac{d}{dt}x = v(x)$ is defined for any $t > 0$ (no finite-time explosion) and converges towards the largest invariant set contained in $\left\{x \in \mathbb{R}^n \mid \frac{d}{dt}V(x) = 0\right\}$.

[8] See H.K. Khalil, Nonlinear Systems (Prentice Hall, 2001).

Stability and convergence of stochastic processes (1)

Convergence of a random process

Consider $(X_k)_{k\in\mathbb{N}}$, a discrete-time sequence of random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in a Banach space $\mathcal{X}$. The random process X_k is said to,

1. converge in probability towards the constant $\bar{x} \in \mathcal{X}$ if for all $\epsilon > 0$,

$$\lim_{k\to\infty} \mathbb{P}\left(\|X_k - \bar{x}\| > \epsilon\right) = \lim_{k\to\infty} \mathbb{P}\left(\omega \in \Omega \mid \|X_k(\omega) - \bar{x}\| > \epsilon\right) = 0;$$

2. converge almost surely towards the constant $\bar{x}$ if

$$\mathbb{P}\left(\lim_{k\to\infty} X_k = \bar{x}\right) = \mathbb{P}\left(\omega \in \Omega \mid \lim_{k\to\infty} X_k(\omega) = \bar{x}\right) = 1;$$

3. converge in mean towards the constant $\bar{x}$ if

$$\lim_{k\to\infty} \mathbb{E}\left(\|X_k - \bar{x}\|\right) = 0.$$

Mean convergence implies convergence in probability.
Almost sure convergence implies convergence in probability.

Stability and convergence of stochastic processes (2)

Markov process

The sequence $(X_k)_{k=1}^{\infty}$ is called a Markov process, if for $k' > k$ and any measurable real function $f(x)$ with $\sup_x |f(x)| < \infty$,

$$\mathbb{E}\left(f(X_{k'}) \mid X_1, \ldots, X_k\right) = \mathbb{E}\left(f(X_{k'}) \mid X_k\right).$$

Martingales

Consider a measurable real function $V(x)$ and $(X_k)_{k \in \mathbb{N}}$ a Markov chain on $\mathcal{X}$. $V(X_k)_{k=1}^{\infty}$ is a *super-martingale*, a *sub-martingale* or a martingale, if $\mathbb{E}\left(\|V(X_k)\|\right) < \infty$ for $k > 0$, and if, respectively,

$$\mathbb{E}\left(V(X_{k+1}) \mid X_k\right) \leq V(X_k) \qquad (\mathbb{P} \text{ almost surely}), \qquad \forall k > 0,$$

or

$$\mathbb{E}\left(V(X_{k+1}) \mid X_k\right) \geq V(X_k) \qquad (\mathbb{P} \text{ almost surely}), \qquad \forall k > 0,$$

or finally,

$$\mathbb{E}\left(V(X_{k+1}) \mid X_k\right) = V(X_k) \qquad (\mathbb{P} \text{ almost surely}), \qquad \forall k > 0,$$

Stability and convergence of stochastic processes (3)

Doob's Inequality

Let $\{X_k\}$ be a Markov chain on state space $\mathcal{X}$. Suppose that there is a non-negative function $V(x)$ satisfying $\mathbb{E}\left(V(X_1) \mid X_0 = x\right) - V(x) = -k(x)$, where $k(x) \geq 0$ on the set $\{x : V(x) < \lambda\} \equiv Q_\lambda$. Then, for all $x \in Q_\lambda$,

$$\mathbb{P}\left(\sup_{\infty > k \geq 0} V(X_k) \geq \lambda \;\middle|\; X_0 = x\right) \leq \frac{V(x)}{\lambda}.$$

Corollary: stability in probability

Consider the same assumptions as in the above theorem. Assume moreover that there exists $\bar{x} \in \mathcal{X}$ such that $V(\bar{x}) = 0$ and that $V(X) \neq 0$ for all x different from $\bar{x}$. Then the Doob's inequality implies that the Markov process X_k is **stable in probability** around $\bar{x}$, i.e.

$$\lim_{x \to \bar{x}} \mathbb{P}\left(\sup_k \|X_k - \bar{x}\| \geq \epsilon \mid X_0 = x\right) = 0, \qquad \forall \epsilon > 0.$$

Stability and convergence of stochastic processes (4)

Kushner's invariance Theorem

Consider the same assumptions as that of the Doob's inequality. Let $\mu_0 = \sigma$ be concentrated on a state $x_0 \in Q_\lambda$, i.e. $\sigma(x_0) = 1$. Assume that $0 \leq f(X_k) \to 0$ in Q_λ implies that $X_k \to \{x \mid f(x) = 0\} \cap Q_\lambda \equiv F_\lambda$. For the trajectories never leaving Q_λ, X_k converges to F_λ almost surely. Also, the associated conditioned probability measures $\tilde{\mu}_k$ tend to the largest invariant set of measures $M_\infty \subset M$ whose support set is in F_λ. Finally, for the trajectories never leaving Q_λ, X_k converges, in probability, to the support set of M_∞.

Corollary: global stability

Consider the same assumptions as in the above theorem and assume moreover that $\bar{x} \in \mathcal{X}$ is the only point in Q_λ such that $V(\bar{x}) = 0$ and furthermore that the set F_λ is reduced to $\{\bar{x}\}$ (strict Lyapunov function). Then the equilibrium $\bar{x}$ is **globally stable in probability** in the set Q_λ, i.e. $\bar{x}$ is stable in probability and moreover

$$\mathbb{P}\left(\lim_{k\to\infty} X_k = \bar{x} \mid X_k \text{ never leaves } Q_\lambda\right) = 1.$$