Modeling and Control of Quantum Systems

Mazyar Mirrahimi  Pierre Rouchon

mazyar.mirrahimi@inria.fr  pierre.rouchon@ensmp.fr

http://cas.ensmp.fr/~rouchon/QuantumSyst/index.html

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Outline

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   - Lie-algebra rank condition
   - A graph sufficient controllability condition

2. Lyapunov control
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3. Optimal control
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Controllability of bilinear Schrödinger equations

Schrödinger equation

\[ i \frac{d}{dt} |\psi\rangle = \left( H_0 + \sum_{k=1}^{m} u_k H_k \right) |\psi\rangle \]

State controllability

For any |\psi_a\rangle and |\psi_b\rangle on the unit sphere of \( \mathcal{H} \), there exist a time \( T > 0 \), a global phase \( \theta \in [0, 2\pi] \), and a piecewise continuous control \( [0, T] \ni t \mapsto u(t) \) such that the solution with initial condition \( |\psi\rangle_0 = |\psi_a\rangle \) satisfies \( |\psi\rangle_T = e^{i\theta} |\psi_b\rangle \).

Controllability of bilinear Schrödinger equations

Propagator equation:

\[
i \frac{d}{dt} U = \left( H_0 + \sum_{k=1}^{m} u_k H_k \right) U, \quad U(0) = 1
\]

We have \( |\psi_t\rangle = U(t) |\psi_0\rangle \).

Operator controllability

For any unitary operator \( V \) on \( \mathcal{H} \), there exist a time \( T > 0 \), a global phase \( \theta \) and a piecewise continuous control \([0, T] \ni t \mapsto u(t)\) such that the solution of propagator equation satisfies \( U_T = e^{i\theta} V \).

Operator controllability implies state controllability
Lie-algebra rank condition

\[ \frac{d}{dt} U = \left( A_0 + \sum_{k=1}^{m} u_k A_k \right) U \]

with \( A_k = H_k / i \) are skew-Hermitian. We define

\[ \mathcal{L}_0 = \text{span}\{A_0, A_1, \ldots, A_m\} \]
\[ \mathcal{L}_1 = \text{span}(\mathcal{L}_0, [\mathcal{L}_0, \mathcal{L}_0]) \]
\[ \mathcal{L}_2 = \text{span}(\mathcal{L}_1, [\mathcal{L}_1, \mathcal{L}_1]) \]

\[ \vdots \]
\[ \mathcal{L} = \mathcal{L}_\nu = \text{span}(\mathcal{L}_{\nu-1}, [\mathcal{L}_{\nu-1}, \mathcal{L}_{\nu-1}]) \]

Lie Algebra Rank Condition

**Operator controllable** if, and only if, the Lie algebra generated by the \( m + 1 \) skew-Hermitian matrices \( \{-iH_0, -iH_1, \ldots, -iH_m\} \) is either \( su(n) \) or \( u(n) \).

**Exercice**

*Show that* \( i \frac{d}{dt} |\psi\rangle = \left( \frac{\omega_{eg}}{2} \sigma_z + \frac{u}{2} \sigma_x \right) |\psi\rangle \), \( |\psi\rangle \in \mathbb{C}^2 \) *is controllable.*
A simple sufficient condition

We consider $H = H_0 + uH_1$, $(|j\rangle)_{j=1,...,n}$ the eigenbasis of $H_0$. We assume $H_0 |j\rangle = \omega_j |j\rangle$ where $\omega_j \in \mathbb{R}$, we consider a graph $G$:

$$V = \{ |1\rangle, \ldots, |n\rangle \}, \quad E = \{ (|j_1\rangle, |j_2\rangle) \mid 1 \leq j_1 < j_2 \leq n, \langle j_1 | H_1 | j_2 \rangle \neq 0 \}.$$ 

$G$ admits a degenerate transition if there exist $(|j_1\rangle, |j_2\rangle) \in E$ and $(|l_1\rangle, |l_2\rangle) \in E$, admitting the same transition frequencies,

$$|\omega_{j_1} - \omega_{j_2}| = |\omega_{l_1} - \omega_{l_2}|.$$ 

A sufficient controllability condition

Remove from $E$, all the edges with identical transition frequencies. Denote by $\bar{E} \subset E$ the reduced set of edges without degenerate transitions and by $\bar{G} = (V, \bar{E})$. If $\bar{G}$ is connected, then the system is operator controllable.
Controllability of a 2-qubit in Ising interaction

The dynamics of the 2-qubit system (state $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$) obey

$$i \frac{d}{dt} |\psi\rangle = (H_0 + uH_1) |\psi\rangle = (Z_1Z_2 + u(X_1 + X_2)) |\psi\rangle \quad (1)$$

with $u \in \mathbb{R}$ as control.

1. Prove that $X_1X_2$ commutes with $H_0$ and with $H_1$.
2. Is the system controllable?
3. Use the spectral basis of $X_1X_2$ and the decomposition
   span${\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}} =$
   span${\{|++\rangle, |--\rangle\}} \oplus$ span${\{|+-\rangle, |-+\rangle\}}$ with $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$, $|--\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$, to deduce a splitting of this system into two separated systems on span${\{|++\rangle, |--\rangle\}}$ and on span${\{|+-\rangle, |-+\rangle\}}$.
4. Prove that one of these sub-systems is controllable and that the other one is not controllable.
Lyapunov control

Bilinear Schrödinger equation:

\[ i \frac{d}{dt} |\psi\rangle = (H_0 + u(t)H_1) |\psi\rangle \]

Control task: to prepare \( |\bar{\psi}\rangle \) such that

\[ H_0 |\bar{\psi}\rangle = \bar{\omega} |\bar{\psi}\rangle. \]

The states \( |\psi\rangle \) and \( e^{i\phi} |\psi\rangle \) represent the same physical states

We add a fictitious control:

\[ i \frac{d}{dt} |\psi\rangle = (H_0 + u(t)H_1) |\psi\rangle + \omega(t) |\psi\rangle \]

\( |\bar{\psi}\rangle \) is a stationary solution for \( u(t) \equiv 0 \) and \( \omega(t) \equiv -\bar{\omega}. \)
Lyapunov control

We look for feedback laws $u(t) = f(|\psi\rangle)$ and $\omega(t) = g(|\psi\rangle)$ such that the solution of

$$i \frac{d}{dt} |\psi\rangle = (H_0 + f(|\psi\rangle)H_1 + g(|\psi\rangle)) |\psi\rangle$$

converges asymptotically towards $|\bar{\psi}\rangle$.

Remark

These feedback laws are calculated off-line and by simulating the closed-loop system and are then applied in open-loop on the real system.
A Lyapunov function

We consider

\[ V(|\psi\rangle) = \frac{1}{2} \| |\psi\rangle - |\bar{\psi}\rangle \|^2 = 1 - \Re(\langle \bar{\psi} | \psi \rangle). \]

We have

\[ \frac{d}{dt} V = -u(t) \Im(\langle \bar{\psi} | H_1 | \psi \rangle) - (\omega(t) + \bar{\omega}) \Im(\langle \bar{\psi} | \psi \rangle) \]

Choice of feedback laws

\[ u(t) \equiv a \Im(\langle \bar{\psi} | H_1 | \psi \rangle) \quad \text{and} \quad \omega(t) \equiv -\bar{\omega} + b \Im(\langle \bar{\psi} | \psi \rangle), \]

where \( a, b > 0 \).
LaSalle’s invariance principle

**Theorem (Lyapunov function and Lasalle invariance principle)**

Take $\Omega \subset \mathbb{R}^n$ an open and non-empty subset of $\mathbb{R}^n$ and $\Omega \ni x \mapsto v(x) \in \mathbb{R}^n$ continuously differentiable function of $x$. Consider $\Omega \ni x \mapsto V(x) \in \mathbb{R}$ a continuously differentiable function of $x$ and assume that

1. there exits $c \in \mathbb{R}$ such that the subset $V_c = \{ x \in \Omega \mid V(x) \leq c \}$ of $\mathbb{R}^n$ is compact (bounded and closed) and non-empty.

2. $V$ is a decreasing time function for solutions of $\frac{d}{dt} x = v(x)$ inside $V_c$:

$$\forall x \in V_c, \quad \frac{d}{dt} V(x) = \nabla V(x) \cdot v(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(x) v_i(x) \leq 0$$

Then for any initial condition $x^0 \in V_c$, the solution of $\frac{d}{dt} x = v(x)$ remains in $V_c$, is defined for all $t > 0$ (no explosion in finite time) and converges towards the largest invariant set included in

$$\{ x \in V_c \mid \frac{d}{dt} V(x) = 0 \}.$$
Application to Schrödinger equation

\[ dV/dt = 0 \text{ and invariance} \]

\[ \Im(\langle \bar{\psi} | \psi \rangle) = 0, \]
\[ \Im(\langle \bar{\psi} | H_1 | \psi \rangle) = 0, \]
\[ \Re(\langle \bar{\psi} | [H_0, H_1] | \psi \rangle) = 0, \]
\[ \vdots \]
\[ \Im(\langle \bar{\psi} | \text{ad}^{2k}_{H_1} H_0 | \psi \rangle) = 0, \]
\[ \Re(\langle \bar{\psi} | \text{ad}^{2k+1}_{H_1} H_0 | \psi \rangle) = 0. \]

Assume that the spectrum of \( H_0 \) is not \( \bar{\omega} \)-degenerate: i.e. \( H_0 \) is not degenerate and for any two eigenvalues \( \omega_\alpha \neq \omega_\beta \),

\[ |\omega_\alpha - \bar{\omega}| \neq |\omega_\beta - \bar{\omega}|; \]

\( \Omega \)-limit set

Intersection of \( S^{2n-1} \) with \( \mathbb{R} |\bar{\psi}\rangle \cup \alpha \mathbb{C} |\psi_\alpha\rangle \), where \( |\psi_\alpha\rangle \) is any eigenvector of \( H_0 \) non co-linear with \( |\bar{\psi}\rangle \) and satisfying \( \langle \bar{\psi} | H_1 | \psi_\alpha \rangle = 0. \)
Convergence Analysis

Theorem

Under the assumption of $H_0$ not $\bar{\omega}$-degenerate and mono-photonic transitions to $|\bar{\psi}\rangle$ ($\langle \bar{\psi} | H_1 | \psi_\alpha \rangle \neq 0$ for all eigenvector $|\psi_\alpha\rangle$ of $H_0$), the $\Omega$-limit set reduces to $\{ |\bar{\psi}\rangle, -|\bar{\psi}\rangle \}$. The equilibrium $-|\bar{\psi}\rangle$ is unstable and the attraction region for the equilibrium $|\psi\rangle$ is exactly $S^{2n-1}/\{-|\bar{\psi}\rangle\}$.

Remark

Assumptions of $H_0$ not $\bar{\omega}$-degenerate and mono-photonic transitions to $|\bar{\psi}\rangle$

$\leftrightarrow$

Controllability of linearized system around $(|\psi\rangle, u, \omega) = (|\bar{\psi}\rangle, 0, -\bar{\omega})$
Main idea: stabilizing around another reference trajectory, around which the linearized system is controllable.

Reference trajectory:

\[ i \frac{d}{dt} |\psi_r\rangle = (H_0 + u_r(t)H_1 + \omega_r(t)) |\psi_r\rangle \]

Same Lyapunov function: \( V(t, |\psi\rangle) = 1 - \Re(\langle \psi_r(t) | \psi \rangle) \).

Feedback laws:

\[
\begin{align*}
    u(t, |\psi\rangle) &= u_r(t) + a \Im(\langle \psi_r(t) | H_1 | \psi \rangle), \\
    \omega(t, |\psi\rangle) &= \omega_r(t) + b \Im(\langle \psi_r(t) | \psi \rangle)
\end{align*}
\]
We consider a drift-less propagator dynamics:

\[
i \frac{d}{dt} U = \left( \omega 1 + \sum_{k=1}^{m} u_k H_k \right) U, \quad U \big|_{t=0} = 1.
\]

**Periodic reference trajectory:** \( u'_k \) and \( \omega_r \) periodic and odd.

**Main idea**

By a Coron’s result, as soon as \( \text{Lie}(H_1, \ldots, H_m) = su(n) \), one can find reference controls \( \omega^r \) and \( u'_k \) around which the linearized system is controllable.

**Lyapunov function:** \( \mathcal{V}(U, U^r) = n - \Re(\text{Tr} (U^\dagger U^r)) \).

**Feedback laws:**

\[
\begin{align*}
    u_k &= u'_k - a_k \Im(\text{Tr} (U^\dagger H_k U^r)), \\
    \omega &= \omega^r - b \Im(\text{Tr} (U^\dagger U^r)).
\end{align*}
\]
Remark

The LaSalle’s invariance principle also works for time-periodic systems; only one needs to be careful about the notion of invariance:

A set $S$ is said to be invariant for the time-periodic system $\frac{d}{dt} x = \nu(x, t)$ if, for all $x_0 \in S$ there exists a time $t_0 > 0$ such that the solution starting from $x_0$ at time $t_0$ remains in the set $S$ for all $t \geq t_0$. 

Two optimal control problems

For given $T$, $|\psi_a\rangle$ and $|\psi_b\rangle$, find the open-loop control $[0, T] \ni t \mapsto u(t)$ such that

$$
\min_{u_k \in L^2([0, T], \mathbb{R})} \frac{1}{2} \int_0^T \left( \sum_{k=1}^m u_k^2 \right) \, dt
$$

$$
i \frac{d}{dt} |\psi\rangle = (H_0 + \sum_{k=1}^m u_k H_k) |\psi\rangle
$$

$$
|\psi\rangle_{t=0} = |\psi_a\rangle, \quad |\langle \psi_b | \psi \rangle|_t^2 = 1
$$

Since the initial and final constraints are difficult to satisfy simultaneously from a numerical point of view, consider the second problem where the final constraint is penalized with $\alpha > 0$:

$$
\min_{u_k \in L^2([0, T], \mathbb{R})} \frac{1}{2} \int_0^T \left( \sum_{k=1}^m u_k^2 \right) \, dt + \frac{\alpha}{2} \left( 1 - |\langle \psi_b | \psi \rangle|_T^2 \right)
$$
First order stationary conditions

For two-points problem, the first order stationary conditions read:

\[
\begin{align*}
i \frac{d}{dt} |\psi\rangle &= (H_0 + \sum_{k=1}^{m} u_k H_k) |\psi\rangle, \quad t \in (0, T) \\
i \frac{d}{dt} |p\rangle &= (H_0 + \sum_{k=1}^{m} u_k H_k) |p\rangle, \quad t \in (0, T) \\
u_k &= -\Re \left( \langle p | H_k | \psi \rangle \right), \quad k = 1, \ldots, m, \quad t \in (0, T) \\
|\psi\rangle_{t=0} &= |\psi_a\rangle, \quad |\langle \psi_b | \psi \rangle|_{t=T}^2 = 1
\end{align*}
\]

For the relaxed problem, the first order stationary conditions read:

\[
\begin{align*}
i \frac{d}{dt} |\psi\rangle &= (H_0 + \sum_{k=1}^{m} u_k H_k) |\psi\rangle, \quad t \in (0, T) \\
i \frac{d}{dt} |p\rangle &= (H_0 + \sum_{k=1}^{m} u_k H_k) |p\rangle, \quad t \in (0, T) \\
u_k &= -\Re \left( \langle p | H_k | \psi \rangle \right), \quad k = 1, \ldots, m, \quad t \in (0, T) \\
|\psi\rangle_{t=0} &= |\psi_a\rangle, \quad |p\rangle_{t=T} = -\alpha \langle \psi_b | \psi \rangle_{t=T} |\psi_b\rangle.
\end{align*}
\]
The underlying classical Hamiltonian dynamics

The dynamical system

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{d}{dt} |\psi\rangle &= \left( H_0 + \sum_{k=1}^{m} u_k H_k \right) |\psi\rangle, \ t \in (0, T) \\
\frac{d}{dt} |p\rangle &= \left( H_0 + \sum_{k=1}^{m} u_k H_k \right) |p\rangle, \ t \in (0, T) \\
u_k &= -\Im \left( \langle p | H_k |\psi\rangle \right), \ k = 1, \ldots, m, \ t \in (0, T)
\end{array} \right.
\end{align*}
\]

is Hamiltonian with \(|\psi\rangle\) and \(|p\rangle\) being the conjugate variables. The underlying Hamiltonian function is given by (Pontryagin Maximum Principle): \(\overline{H}(|\psi\rangle, |p\rangle) = \min_{u \in \mathbb{R}^m} H(|\psi\rangle, |p\rangle, u)\) where

\[
\begin{align*}
H(|\psi\rangle, |p\rangle, u) &= \frac{1}{2} \left( \sum_{k=1}^{m} u_k^2 \right) + \Im \left( \langle p | H_0 + \sum_{k=1}^{m} u_k H_k |\psi\rangle \right).
\end{align*}
\]

Thus for any solutions \((|\psi\rangle, |p\rangle)\) of \((\Sigma)\),

\[
\overline{H}(|\psi\rangle, |p\rangle) = \Im \left( \langle p | H_0 |\psi\rangle \right) - \frac{1}{2} \left( \sum_{k=1}^{m} \Im \left( \langle p | H_k |\psi\rangle \right)^2 \right).
\]

is independent of \(t\).

Main difficulty: such systems are not, in general, integrable in the Arnold-Liouville sense.
Take an $L^2$ control $[0, T] \ni t \mapsto u(t)$ ($\dim(u) = 1$ here) and denote by

- $|\psi_u\rangle$ the solution of the forward system $i \frac{d}{dt} |\psi\rangle = (H_0 + uH_1) |\psi\rangle$ starting from $|\psi_a\rangle$.

- $|p_u\rangle$ the adjoint associated to $u$, i.e. the solution of the backward system $i \frac{d}{dt} |p_u\rangle = (H_0 + uH_1) |p_u\rangle$ with $|p_u\rangle_T = -\alpha P |\psi_u\rangle_T$, $P$ projector on $|\psi_b\rangle$, $P |\phi\rangle \equiv \langle \psi_b | \phi \rangle |\psi_b\rangle$.

- $J(u) = \frac{1}{2} \int_0^T u^2 + \frac{\alpha}{2} (1 - |\langle \psi_b | \psi_u \rangle|^2_T)$. 

Starting from an initial guess $u^0 \in L^2([0, T], \mathbb{R})$, the monotone scheme generates a sequence of controls $u^\nu \in L^2([0, T], \mathbb{R})$, $\nu = 1, 2, \ldots$, such that the cost $J(u^\nu)$ is decreasing, $J(u^{\nu+1}) \leq J(u^\nu)$.

---

Monotone numerical scheme for the relaxed problem (2)

Assume that, at step $\nu$, we have computed the control $u^\nu$, the associated quantum state $|\psi^\nu\rangle = |\psi_{u^\nu}\rangle$ and its adjoint $|p^\nu\rangle = |p_{u^\nu}\rangle$. We get their new time values $u^{\nu+1}$, $|\psi^{\nu+1}\rangle$ and $|p^{\nu+1}\rangle$ in two steps:

1. **Imposing** $u^{\nu+1} = -\Re \left( \langle p^\nu | H_1 | \psi^{\nu+1}\rangle \right)$ is just a feedback; one get $u^{\nu+1}$ just by a **forward integration** of the nonlinear Schrödinger equation,

$$i \frac{d}{dt} |\psi\rangle = (H_0 - \Re (\langle p^\nu | H_1 | \psi\rangle) H_1) |\psi\rangle, \quad |\psi\rangle_0 = |\psi_a\rangle,$$

that provides $[0, T] \ni t \mapsto |\psi^{\nu+1}\rangle$ and the new control $u^{\nu+1}$.

2. **Backward integration** from $t = T$ to $t = 0$ of

$$i \frac{d}{dt} |p\rangle = \left( H_0 + u^{\nu+1}(t) H_1 \right) |p\rangle, \quad |p\rangle_T = -\alpha \langle \psi_b | \psi^{\nu+1}\rangle_T |\psi_b\rangle$$

yields to the new adjoint trajectory $[0, T] \ni t \mapsto |p^{\nu+1}\rangle$. 
Monotone numerical scheme for the relaxed problem (3)

Why $J(u^{\nu+1}) \leq J(u^\nu)$?

- Because we have the identity for any open-loop controls $u$ and $v$.

\[
J(u) - J(v) = -\frac{\alpha}{2} \left( \langle \psi_u - \psi_v | P | \psi_u - \psi_v \rangle \right)_T \\
+ \frac{1}{2} \left( \int_0^T (u - v) (u + v + 2 \Im (\langle p_v | H_1 | \psi_u \rangle)) \right).
\]

- If $u = -\Im (\langle p_v | H_1 | \psi_u \rangle)$ for all $t \in [0, T)$, we have

\[
J(u) - J(v) = -\frac{\alpha}{2} \left( \langle \psi_u - \psi_v | P | \psi_u - \psi_v \rangle \right)_T - \frac{1}{2} \left( \int_0^T (u - v)^2 \right)
\]

and thus $J(u) \leq J(v)$.

- Take $v = u^\nu$, $u = u^{\nu+1}$: then $|p_v\rangle = |p^\nu\rangle$, $|\psi_v\rangle = |\psi^\nu\rangle$, $|p_u\rangle = |p^{\nu+1}\rangle$ and $|\psi_u\rangle = |\psi^{\nu+1}\rangle$. 
Proof of

\[ J(u) - J(v) = -\frac{\alpha}{2} (\langle \psi_u - \psi_v | P | \psi_u - \psi_v \rangle)_T + \frac{1}{2} \left( \int_0^T (u - v) (u + v + 2 \Im (\langle p_v | H_1 | \psi_u \rangle)) \right). \]

Start with

\[ J(u) - J(v) = -\alpha \left( \langle \psi_u - \psi_v | P | \psi_u - \psi_v \rangle_T + \langle \psi_u - \psi_v | P | \psi_v \rangle_T + \langle \psi_v | P | \psi_u - \psi_v \rangle_T \right) + \int_0^T (u - v) (u + v). \]

Hermitian product of \( i \frac{d}{dt} (|\psi_u \rangle - |\psi_v \rangle) = (H_0 + vH_1) (|\psi_u \rangle - |\psi_v \rangle) + (u - v)H_1 |\psi_u \rangle \) with \(|p_v \rangle\):

\[ \langle p_v | \frac{d(|\psi_u \rangle - |\psi_v \rangle)}{dt} \rangle = \langle p_v | \frac{H_0 + vH_1}{i} | \psi_u - \psi_v \rangle + \langle p_v | \frac{(u-v)H_1}{i} | \psi_u \rangle. \]

Integration by parts (use \(|\psi_v \rangle_0 = |\psi_u \rangle_0, |p_v \rangle_T = -\alpha P |\psi_v \rangle_T \) and \( \frac{d}{dt} \langle p_v \rangle = -\langle p_v | \left( \frac{H_0 + vH_1}{i} \right) \rangle):$

\[ \int_0^T \langle p_v | \frac{d(|\psi_u \rangle - |\psi_v \rangle)}{dt} \rangle = \langle p_v | \psi_u - \psi_v \rangle_T - \langle p_v | \psi_u - \psi_v \rangle_0 - \int_0^T \langle \frac{dp_v}{dt} | \psi_u - \psi_v \rangle 
\]

\[ = -\alpha \langle \psi_v | P | \psi_u - \psi_v \rangle_T + \int_0^T \langle p_v | \frac{H_0 + vH_1}{i} | \psi_u - \psi_v \rangle. \]

Thus \( -\alpha \langle \psi_v | P | \psi_u - \psi_v \rangle_T = \int_0^T \langle p_v | \frac{(u-v)H_1}{i} | \psi_u \rangle \) and

\( \alpha \Re (\langle \psi_v | P | \psi_u - \psi_v \rangle_T) = -\int_0^T \Im (\langle p_v | (u - v)H_1 | \psi_u \rangle). \) Finally we have

\[ J(u) - J(v) = -\frac{\alpha}{2} (\langle \psi_u - \psi_v | P | \psi_u - \psi_v \rangle)_T + \frac{1}{2} \left( \int_0^T (u - v) (u + v + 2 \Im (\langle p_v | H_1 | \psi_u \rangle)) \right). \]
For given $T$, $a_k \geq 0$ and $b_k \geq 0$ ($\sum_{k=1}^{n} a_k^2 = \sum_{k=1}^{n} b_k^2 = 1$),

\[
\min_{u_{k,l} \in L^2([0, T], \mathbb{C}), \ (k, l) \in I} \frac{1}{2} \int_0^T \left( \sum_{(k,l) \in I} |u_{kl}|^2 \right) dt
\]

\[
i \frac{d}{dt} |\psi\rangle = \left( \sum_{(k,l) \in I} \mu_{kl} u_{kl} |k\rangle \langle l| \right) |\psi\rangle,
\]

\[
|\langle k|\psi\rangle|^2_{t=0} = a_k^2, \ |\langle k|\psi\rangle|^2_{t=T} = b_k^2, \ k = 1, \ldots, n
\]

admits the same minimal cost as the following reduced problem

\[
\min_{v_{k,l} \in L^2([0, T], \mathbb{R}), \ v_{kl} = -v_{l,k}, \ (k, l) \in I} \frac{1}{2} \int_0^T \left( \sum_{(k,l) \in I} |v_{kl}|^2 \right) dt
\]

\[
\frac{d}{dt} |\phi\rangle = \left( \sum_{(k,l) \in I} \mu_{kl} v_{kl} |k\rangle \langle l| \right) |\phi\rangle
\]

\[
|\langle k|\phi\rangle|^2_{t=0} = a_k, \ |\langle k|\phi\rangle|^2_{t=T} = b_k, \ k = 1, \ldots, n
\]

where the components of $|\psi\rangle = |\phi\rangle$ remain real, the $u_{kl}$'s are purely imaginary, $u_{kl} = iv_{kl}$ ($v_{kl} \in \mathbb{R}$ with $v_{kl} = -v_{lk}$).

---

Go back to resulting optimal physical controls \( u_{kl} = iv_{kl} \):

\[
u_{kl}(t)e^{i(\omega_k - \omega_l)t} + u^*_{kl}(t)e^{-i(\omega_k - \omega_l)t} = -2v_{kl}(t)\sin((\omega_k - \omega_l)t).
\]

They are in resonance with the frequency transition between \(|k\rangle\) and \(|l\rangle\). They contain only amplitude modulations (up to a \(\pi\) phase-shift since \(v_{kl}\) can pass through zero).

For drift-less quantum systems

\[
i\frac{d}{dt}|\psi\rangle = \left( \sum_{(k,l) \in I} \mu_{kl} u_{kl} |k\rangle \langle l| \right) |\psi\rangle
\]

population transfer minimizing the \(L^2\) control norm is achieved by resonant controls \(u_{kl} = iv_{kl}\) with \(v_{kl} \in \mathbb{R}\) (the reduction of the problem to a real case of half dimension).
Associated to any $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ consider

$$|\psi\rangle \mapsto |\psi^\theta\rangle = \left( \sum_{k=1}^{n} e^{i\theta_k} |k\rangle \langle k| \right) |\psi\rangle,$$

$$u_{kl} \mapsto u_{kl}^\theta = e^{i(\theta_k - \theta_l)} u_{kl}.$$ 

These transformations leave unchanged cost and constraints of

$$\min_{u_{k,l} \in L^2([0, T], \mathbb{C}), \ (k, l) \in I} \left( \frac{1}{2} \int_{0}^{T} \left( \sum_{(k,l) \in I} |u_{kl}|^2 \right) dt \right).$$

$$i \frac{d}{dt} |\psi\rangle = \left( \sum_{(k,l) \in I} \mu_{kl} u_{kl} |k\rangle \langle l| \right) |\psi\rangle,$$

$$|\langle k|\psi\rangle|_{t=0}^2 = a_k^2, \ |\langle k|\psi\rangle|_{t=T}^2 = b_k^2, \ k = 1, \ldots, n$$

that coincides with

$$\min_{u_{k,l} \in L^2([0, T], \mathbb{C}), \ (k, l) \in I} \left( \frac{1}{2} \int_{0}^{T} \left( \sum_{(k,l) \in I} |u_{kl}|^2 \right) dt \right).$$

$$\langle k|\psi\rangle_{t=0} = a_k, \ |\langle k|\psi\rangle|_{t=T}^2 = b_k^2, \ k = 1, \ldots, n.$$
Set $\psi_k = \langle k | \psi \rangle$ and $z_{kl} = \psi_k \psi_l^*$. Evolution of the direction of $\psi_k$ in the complex plane is governed by

$$
\psi_k^* \frac{d}{dt} \psi_k - \psi_k \frac{d}{dt} \psi^* = \sum_{l \mid (k, l) \in I} \mu_{kl} \frac{u_{kl} z_{kl}^* - u_{kl}^* z_{kl}}{i}.
$$

For $(k, l) \in I$ set $v_{kl}(t) = \begin{cases} 0, & \text{if } z_{kl}(t) = 0; \\ \frac{u_{kl}(t) z_{kl}^*(t) - u_{kl}^*(t) z_{kl}(t)}{2i|z_{kl}(t)|}, & \text{if } z_{kl}(t) \neq 0. \end{cases}$

We have $v_{kl} = -v_{lk}$ since $u_{kl}^* = u_{lk}$ and $z_{kl}^* = z_{lk}$. Moreover $|v_{kl}| \leq |u_{kl}|$. Thus each $v_{kl}$ belongs to $L^2([0, T], \mathbb{R})$ and the solution $|\phi\rangle$ of

$$
\frac{d}{dt} \phi_k = \sum_{l \mid (k, l) \in I} \mu_{kl} v_{kl} \phi_l, \quad \phi_k(0) = a_k, \quad k = 1, \ldots, n
$$

coincides with $\phi_k = |\psi_k|$.

To summarize: starting from complex controls $u_{kl} \in L^2([0, T], \mathbb{C})$ satisfying the constraints of the full problem, we have constructed real controls $v_{kl} \in L^2([0, T], \mathbb{C})$ satisfying the constraints of the reduced problem; the cost associated to $u_{kl}$ is larger than the cost associated to $v_{kl}$ since $|v_{kl}| \leq |u_{kl}|$. 
Outline of the 8 lectures

Lect. 1 (Oct. 4) Introduction on LKB Photon-Box: control issues for classical and quantum oscillators (creation/annihilation operator, coherent state).

Part 1, open-loop control of Schrödinger systems:

Lect. 2 (Oct. 11) RWA and multi-frequency averaging; 2-level system (half spin) and Jaynes-Cummings model (spin-spring)

Lect. 3 (Oct. 25) Law-Eberly method for trapped ions; adiabatic invariance and control.

Lect. 4 (Nov. 22) Controllability, Lyapounov control and optimal control

Part 2, closed-loop control of open quantum systems:

Lect. 5 (Nov. 29) Measurement and quantum trajectories (discrete time, Kraus operators, LKB-photon box)

Lect. 6 (Dec. 6) Feedback stabilization (Photon-box, quantum filter, Lyapunov, separation principle, delay compensation)

Lect. 7 (Dec. 13) Quantum trajectories (continuous time with Poisson process, Lindblad operators, time-scale reduction, synchronization loop on a Λ-system)

Lect. 8 (Dec. 14) Quantum trajectories (continuous time with Wiener process, homodyn detection, Lyapunov feedback stabilization of entangled states).