Quantum Systems: Dynamics and Control

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Outline

1. Quantum measurement
   - Projective measurement
   - Positive Operator Valued Measurement (POVM)
   - Stochastic process attached to a POVM

2. Markov chains, martingales and convergence theorems

3. Quantum non-demolition measurements and asymptotic behavior

4. Quantum feedback and stabilization of photon number states
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4 Quantum feedback and stabilization of photon number states
Projective measurement

For the system defined on Hilbert space $\mathcal{H}$, take

- an observable $O$ (Hermitian operator) defined on $\mathcal{H}$:

$$O = \sum_{\nu} \lambda_{\nu} P_{\nu},$$

where $\lambda_{\nu}$'s are the eigenvalues of $O$ and $P_{\nu}$ is the projection operator over the associated eigenspace.

- a quantum state given by the wave function $|\psi\rangle$ in $\mathcal{H}$.

**Projective measurement** of the physical observable $O = \sum_{\nu} \lambda_{\nu} P_{\nu}$ for the quantum state $|\psi\rangle$:

1. The probability of obtaining the value $\lambda_{\nu}$ is given by $p_{\nu} = \langle \psi | P_{\nu} | \psi \rangle$; note that $\sum_{\nu} p_{\nu} = 1$ as $\sum_{\nu} P_{\nu} = I_{\mathcal{H}}$ ($I_{\mathcal{H}}$ represents the identity operator of $\mathcal{H}$).

2. After the measurement, the conditional (a posteriori) state $|\psi\rangle_+$ of the system, given the outcome $\lambda_{\nu}$, is

$$|\psi\rangle_+ = \frac{P_{\nu} |\psi\rangle}{\sqrt{p_{\nu}}} \quad \text{(collapse of the wave packet)}.$$
Positive Operator Valued Measurement (POVM) (1)

System $S$ of interest (a quantized electromagnetic field) interacts with the meter $M$ (a probe atom), and the experimenter measures projectively the meter $M$ (the probe atom). Need for a **Composite system**: $\mathcal{H}_S \otimes \mathcal{H}_M$ where $\mathcal{H}_S$ and $\mathcal{H}_M$ are Hilbert spaces of $S$ and $M$. Measurement process in three successive steps:

1. Initially the quantum state is separable
   \[
   \mathcal{H}_S \otimes \mathcal{H}_M \ni |\Psi\rangle = |\psi_S\rangle \otimes |\psi_M\rangle
   \]
   with a well defined and known state $|\psi_M\rangle$ for $M$.

2. Then a **Schrödinger evolution** during a small time (unitary operator $U_{S,M}$) of the composite system from $|\psi_S\rangle \otimes |\psi_M\rangle$ and producing $U_{S,M}(|\psi_S\rangle \otimes |\psi_M\rangle)$, entangled in general.

3. Finally a **projective measurement** of the meter $M$:
   \[
   O_M = I_S \otimes \left( \sum_\nu \lambda_\nu P_\nu \right)
   \]
   the measured observable for the meter. Projection operator $P_\nu$ is a rank-1 projection in $\mathcal{H}_M$ over the eigenstate $|\xi_\nu\rangle \in \mathcal{H}_M$: $P_\nu = |\xi_\nu\rangle \langle \xi_\nu|$. 
Positive Operator Valued Measurement (POVM) (2)

Define the measurement operators $M_\nu$ via

$$\forall |\psi_S\rangle \in \mathcal{H}_S, \quad U_{S,M}(|\psi_S\rangle \otimes |\psi_M\rangle) = \sum_\nu (M_\nu |\psi_S\rangle) \otimes |\xi_\nu\rangle.$$ 

Then $\sum_\nu M_\nu^\dagger M_\nu = I_S$. The set \{\$M_\nu\$\} defines a Positive Operator Valued Measurement (POVM).

In $\mathcal{H}_S \otimes \mathcal{H}_M$, projective measurement of $O_M = I_S \otimes (\sum_\nu \lambda_\nu P_\nu)$ with quantum state $U_{S,M}(|\psi_S\rangle \otimes |\theta_M\rangle)$:

1. The probability of obtaining the value $\lambda_\nu$ is given by $p_\nu = \langle \psi_S | M_\nu^\dagger M_\nu |\psi_S\rangle$

2. After the measurement, the conditional (a posteriori) state of the system, given the outcome $\lambda_\nu$, is

$$|\psi_S\rangle^+ = \frac{M_\nu |\psi_S\rangle}{\sqrt{p_\nu}}.$$
To the POVM $(M_\nu)$ on $\mathcal{H}_S$ is attached a stochastic process of quantum state $|\psi\rangle$

$$|\psi\rangle_+ = \frac{M_\nu|\psi\rangle}{\sqrt{p_\nu}}$$

with probability $p_\nu = \langle\psi|M_\nu^\dagger M_\nu|\psi\rangle$.

For any observable $A$ on $\mathcal{H}_S$, its conditional expectation value after the transition knowing the state $|\psi\rangle$

$$\mathbb{E}\left(\langle\psi|A|\psi\rangle_+ \mid |\psi\rangle\right) = \langle\psi|\left(\sum_\nu M_\nu^\dagger AM_\nu\right)|\psi\rangle.$$

If $\bar{A}$ is a stationary point of the adjoint Kraus map $K^*$, $K^*\bar{A} = \sum_\nu M_\nu^\dagger \bar{A}M_\nu$, then $\langle\psi|\bar{A}|\psi\rangle$ is a martingale:

$$\mathbb{E}\left(\langle\psi|\bar{A}|\psi\rangle_+ \mid |\psi\rangle\right) = \langle\psi|\left(\sum_\nu M_\nu^\dagger \bar{A}M_\nu\right)|\psi\rangle = \langle\psi|\bar{A}|\psi\rangle.$$
Markov process: $|\psi_k\rangle \equiv |\psi\rangle_{t=k\Delta t}, \ k \in \mathbb{N}, \ \Delta t \text{ sampling period},$

$$|\psi_{k+1}\rangle = \begin{cases} 
\frac{M_g|\psi_k\rangle}{\sqrt{\langle \psi_k| M_g^\dagger M_g |\psi_k\rangle}} & \text{with } y_k = g, \ \text{probability } P_g = \langle \psi_k| M_g^\dagger M_g |\psi_k\rangle; \\
\frac{M_e|\psi_k\rangle}{\sqrt{\langle \psi_k| M_e^\dagger M_e |\psi_k\rangle}} & \text{with } y_k = e, \ \text{probability } P_e = \langle \psi_k| M_e^\dagger M_e |\psi_k\rangle,
\end{cases}$$

with

$$M_g = \cos(\varphi_0 + N\vartheta), \quad M_e = \sin(\varphi_0 + N\vartheta).$$
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Consider \((X_n)\) a sequence of random variables defined on the probability space \((\Omega, \mathcal{F}, P)\) and taking values in a metric space \(X\). The random process \(X_n\) is said to,

- converge in probability towards the random variable \(X\) if for all \(\epsilon > 0\),
  \[
  \lim_{n \to \infty} P(|X_n - X| > \epsilon) = \lim_{n \to \infty} P(\omega \in \Omega \mid |X_n(\omega) - X(\omega)| > \epsilon) = 0;
  \]
- converge almost surely towards the random variable \(X\) if
  \[
  P\left(\lim_{n \to \infty} X_n = X\right) = P\left(\omega \in \Omega \mid \lim_{n \to \infty} X_n(\omega) = X(\omega)\right) = 1;
  \]
- converge in mean towards the random variable \(X\) if
  \[
  \lim_{n \to \infty} E(|X_n - X|) = 0.
  \]
Some definitions

**Markov process**

The sequence \((X_n)_{n=1}^{\infty}\) is called a Markov process, if for \(n' > n\) and any measurable function \(f(x)\) with \(\sup_x |f(x)| < \infty\),

\[
\mathbb{E} (f(X_{n'}) \mid X_1, \ldots, X_n) = \mathbb{E} (f(X_{n'}) \mid X_n).
\]

**Martingales**

The sequence \((X_n)_{n=1}^{\infty}\) is called respectively a **supermartingale**, a **submartingale** or a **martingale**, if \(\mathbb{E} (|X_n|) < \infty\) for \(n = 1, 2, \ldots\), and

\[
\mathbb{E} (X_n \mid X_1, \ldots, X_m) \leq X_m \quad (\mathbb{P} \text{ almost surely}), \quad n \geq m,
\]

or

\[
\mathbb{E} (X_n \mid X_1, \ldots, X_m) \geq X_m \quad (\mathbb{P} \text{ almost surely}), \quad n \geq m,
\]

or finally,

\[
\mathbb{E} (X_n \mid X_1, \ldots, X_m) = X_m \quad (\mathbb{P} \text{ almost surely}), \quad n \geq m.
\]
Doob’s first martingale convergence theorem

Let \((X_n)^\infty_{n=1}\) be a submartingale such that \((x^+\) is the positive part of \(x\))

\[
\sup_n \mathbb{E} (X_n^+) < \infty.
\]

Then \(\lim_n X_n (= X_\infty)\) exists with probability 1, and \(\mathbb{E} (X_\infty^+) < \infty\).
Doob’s Inequality

Let \( \{X_n\} \) be a Markov chain on state space \( \mathcal{X} \). Suppose that there is a non-negative function \( V(x) \) satisfying \( \mathbb{E} (V(X_1) \mid X_0 = x) - V(x) = -k(x) \), where \( k(x) \geq 0 \) on the set \( \{x: V(x) < \lambda\} \equiv Q_\lambda \). Then

\[
\mathbb{P} \left( \sup_{\infty > n \geq 0} V(X_n) \geq \lambda \mid X_0 = x \right) \leq \frac{V(x)}{\lambda}.
\]

Corollary

Consider the same assumptions as in above theorem. Assume moreover that there exists \( \bar{x} \in \mathcal{X} \) such that \( V(\bar{x}) = 0 \) and that \( V(x) \neq 0 \) for all \( x \) different from \( \bar{x} \). Then the above theorem implies that the Markov process \( X_n \) is **stable in probability** around \( \bar{x} \), i.e.

\[
\lim_{x \to \bar{x}} \mathbb{P} \left( \sup_n \|X_n - \bar{x}\| \geq \epsilon \mid X_0 = x \right) = 0, \quad \forall \epsilon > 0.
\]
Theorem: H.J. Kushner

Let \( \{X_n\} \) be a Markov chain on the compact state space \( S \). Suppose that there exists a non-negative function \( V(x) \) satisfying
\[
E \left( V(X_{n+1}) \mid X_n = x \right) - V(x) = -k(x),
\]
where \( k(x) \geq 0 \) is a positive continuous function of \( x \). Then the \( \omega \)-limit set (in the sense of almost sure convergence) of \( X_n \) is included in the following set
\[
I = \{ X \mid k(X) = 0 \}.
\]
Trivially, the same result holds true for the case where
\[
E \left( V(X_{n+1}) \mid X_n = x \right) - V(x) = k(x) \quad (V(X_n) \text{ is a submartingale and not a supermartingale}),
\]
with \( k(x) \geq 0 \) and \( V(x) \) bounded from above.
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Markov process: $|\psi_k\rangle \equiv |\psi\rangle_{t=k\Delta t}$, $k \in \mathbb{N}$, $\Delta t$ sampling period,

$$
\rho_{k+1} = \begin{cases} 
\frac{M_g \rho_k M_g^\dagger}{\text{Tr}(M_g \rho_k M_g^\dagger)} & \text{with } y_k = g, \text{ probability } P_g = \text{Tr} \left( M_g \rho_k M_g^\dagger \right); \\
\frac{M_e \rho_k M_e^\dagger}{\text{Tr}(M_e \rho_k M_e^\dagger)} & \text{with } y_k = e, \text{ probability } P_e = \text{Tr} \left( M_e \rho_k M_e^\dagger \right),
\end{cases}
$$

with

$$
M_g = \cos(\varphi_0 + N \vartheta), \quad M_e = \sin(\varphi_0 + N \vartheta).
$$

Quantum Non-Demolition (QND) measurement

The measurement operators $M_{g,e}$ commute with the photon-number observable $N$: photon-number states $|n\rangle\langle n|$ are fixed points of the measurement process. We say that the measurement is QND for the observable $N$. 
Asymptotic behavior

Theorem

Consider the Markov process defined above with an initial density matrix \( \rho_0 \) defined on the subspace \( \text{span}\{|n\rangle \mid n = 0, 1, \cdots, n^{\text{max}}\} \). Also, assume the non-degeneracy assumption

\[
\cos^2(\varphi_m) \neq \cos^2(\varphi_n) \quad \forall n \neq m \in \{0, 1, \cdots, n^{\text{max}}\},
\]

where \( \varphi_n = \varphi_0 + n\vartheta \). Then

- for any \( n \in \{0, \ldots, n^{\text{max}}\} \), \( \text{Tr}(\rho_k|n\rangle\langle n|) = \langle n|\rho_k|n\rangle \) is a martingale
- \( \rho_k \) converges with probability 1 to one of the \( n^{\text{max}} + 1 \) Fock state \( |n\rangle\langle n| \) with \( n \in \{0, \ldots, n^{\text{max}}\} \).
- the probability to converge towards the Fock state \( |n\rangle\langle n| \) is given by \( \text{Tr}(\rho_0|n\rangle\langle n|) = \langle n|\rho_0|n\rangle \).
Asymptotic behavior: numerical simulations

100 Monte-Carlo simulations of $\text{Tr}(\rho_k |3\rangle\langle 3|)$ versus $k$

Fidelity between $\rho_K$ and the Fock state $\xi_3$
Consider the Markov chain $\rho_{k+1} = M_{s_k} \rho_k$ where $s_k = g$ (resp. $s_k = e$) with probability $p_{g,k} = \text{Tr} \left( M_g \rho_k M_g^\dagger \right)$ (resp. $p_{e,k} = \text{Tr} \left( M_e \rho_k M_e^\dagger \right)$). The Kraus operator are given by

$$M_g = \cos \left( \frac{\theta_1}{2} \right) \cos \left( \frac{\Theta}{2} \sqrt{N} \right) - \sin \left( \frac{\theta_1}{2} \right) \left( \frac{\sin \left( \frac{\Theta}{2} \sqrt{N} \right)}{\sqrt{N}} \right) a^\dagger$$

$$M_e = - \sin \left( \frac{\theta_1}{2} \right) \cos \left( \frac{\Theta}{2} \sqrt{N} + 1 \right) - \cos \left( \frac{\theta_1}{2} \right) a \left( \frac{\sin \left( \frac{\Theta}{2} \sqrt{N} \right)}{\sqrt{N}} \right)$$

with $\theta_1 = 0$. Assume the initial state to be defined on the subspace $\{ |n\rangle \}_{n=0}^{n_{\text{max}}}$ and that the cavity state at step $k$ is described by the density operator $\rho_k$.

1. Show that

$$\mathbb{E} \left( \text{Tr} \left( N \rho_{k+1} \right) | \rho_k \right) = \text{Tr} \left( N \rho_k \right) - \text{Tr} \left( \sin^2 \left( \frac{\Theta}{2} \sqrt{N} \right) \rho_k \right).$$

2. Assume that for any integer $n$, $\Theta \sqrt{n}/\pi$ is irrational. Then prove that almost surely $\rho_k$ tends to the vacuum state $|0\rangle \langle 0|$ whatever its initial condition is.

3. When $\Theta \sqrt{n}/\pi$ is rational for some integer $n$, describe the possible $\omega$-limit sets for $\rho_k$. 
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Quantum feedback

**Question:** how to stabilize deterministically a single photon-number state $|\bar{n}\rangle \langle \bar{n}|$?

Controlled Markov chain:

$$
\rho_{k+\frac{1}{2}} = \mathbb{M}_s(\rho_k), \quad \rho_{k+1} = \mathbb{D}_\alpha(\rho_{k+\frac{1}{2}}),
$$

where $\mathbb{D}_\alpha(\rho) = D_\alpha \rho D_\alpha$. 

Control Lyapunov function

Idea: \[
\overline{V}(\rho) = V(\rho) + \sum_{n \geq 0} f(n) \text{Tr}(\rho |n\rangle \langle n|),
\]

Control law:
\[
\alpha_k : = \arg\max_{|\alpha| \leq \bar{\alpha}} \left\{ E \left( \overline{V}(\rho_{k+1}) | \rho_k, \alpha_k = \alpha \right) \right\}
\]
\[
= \arg\max_{|\alpha| \leq \bar{\alpha}} \left\{ \text{Tr} \left( M_g \rho_k M_g \right) \overline{V} \left( D_\alpha \left( M_g(\rho_k) \right) \right) + \text{Tr} \left( M_e \rho_k M_e \right) \overline{V} \left( D_\alpha \left( M_e(\rho_k) \right) \right) \right\}.
\]
Quantum feedback experiment

Stabilization around 3-photon state

$n_t = 3$ photons

With pure state $\rho = |\psi\rangle\langle\psi|$, we have

$$\rho_+ = |\psi_+\rangle\langle\psi_+| = \frac{1}{\text{Tr} \left( M_\mu \rho M_\mu^\dagger \right)} M_\mu \rho M_\mu^\dagger$$

when the atom collapses in $\mu = g$, $e$ with proba. $\text{Tr} \left( M_\mu \rho M_\mu^\dagger \right)$.

Detection error rates: $\mathbb{P}(y = e/\mu = g) = \eta_g \in [0, 1]$ the probability of erroneous assignment to $e$ when the atom collapses in $g$; $\mathbb{P}(y = g/\mu = e) = \eta_e \in [0, 1]$ (given by the contrast of the Ramsey fringes).

Bayes law: expectation $\rho_+$ of $|\psi_+\rangle\langle\psi_+|$ knowing $\rho$ and the imperfect detection $y$.

$$\rho_+ = \begin{cases} 
\frac{(1-\eta_g) M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger}{\text{Tr} \left( (1-\eta_g) M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger \right)} & \text{if } y = g, \text{ proba. } \text{Tr} \left( (1-\eta_g) M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger \right); \\
\eta_g M_g \rho M_g^\dagger + (1-\eta_e) M_e \rho M_e^\dagger}{\text{Tr} \left( \eta_g M_g \rho M_g^\dagger + (1-\eta_e) M_e \rho M_e^\dagger \right)} & \text{if } y = e, \text{ proba. } \text{Tr} \left( \eta_g M_g \rho M_g^\dagger + (1-\eta_e) M_e \rho M_e^\dagger \right). 
\end{cases}$$

$\rho_+$ does not remain pure: the quantum state $\rho_+$ becomes a mixed state; $|\psi_+\rangle$ becomes physically irrelevant.
We get
\[
\rho_+ = \begin{cases} 
\frac{(1-\eta_g)M_g\rho M_g^\dagger + \eta_e M_e\rho M_e^\dagger}{\text{Tr}((1-\eta_g)M_g\rho M_g^\dagger + \eta_e M_e\rho M_e^\dagger)}, & \text{with prob. } \text{Tr}((1-\eta_g)M_g\rho M_g^\dagger + \eta_e M_e\rho M_e^\dagger); \\
\frac{\eta_g M_g\rho M_g^\dagger + (1-\eta_e) M_e\rho M_e^\dagger}{\text{Tr}(\eta_g M_g\rho M_g^\dagger + (1-\eta_e) M_e\rho M_e^\dagger)}, & \text{with prob. } \text{Tr}(\eta_g M_g\rho M_g^\dagger + (1-\eta_e) M_e\rho M_e^\dagger). 
\end{cases}
\]

Key point:

\[
\text{Tr}((1-\eta_g)M_g\rho M_g^\dagger + \eta_e M_e\rho M_e^\dagger) \quad \text{and} \quad \text{Tr}(\eta_g M_g\rho M_g^\dagger + (1-\eta_e) M_e\rho M_e^\dagger)
\]

are the probabilities to detect \(y = g\) and \(e\), knowing \(\rho\).

**Reformulation with quantum maps**: set

\[
K_g(\rho) = (1-\eta_g)M_g\rho M_g^\dagger + \eta_e M_e\rho M_e^\dagger, \quad K_e(\rho) = \eta_g M_g\rho M_g^\dagger + (1-\eta_e) M_e\rho M_e^\dagger.
\]

\[
\rho_+ = \frac{K_y(\rho)}{\text{Tr}(K_y(\rho))} \quad \text{when we detect } y
\]

The probability to detect \(y\) knowing \(\rho\) is \(\text{Tr}(K_y(\rho))\).

We have the following Kraus map:

\[
E(\rho_+ | \rho) = K_g(\rho) + K_e(\rho) = K(\rho) = M_g\rho M_g^\dagger + M_e\rho M_e^\dagger.
\]
Cavity decay (decoherence) seen as unread measurements

The cavity mirrors play the role of a detector with two possible outcomes:

- **zero photon annihilation** during $\Delta T$: Kraus operator
  \[ M_0 = I - \frac{\Delta T}{2} L_{-1}^\dagger L_{-1}, \]
  probability $\approx \text{Tr} \left( M_0 \rho M_0^\dagger \right)$ with back action
  \[ \rho_{t+\Delta T} \approx \frac{M_0 \rho_t M_0^\dagger}{\text{Tr}(M_0 \rho M_0^\dagger)}. \]

- **one photon annihilation** during $\Delta T$: Kraus operator
  \[ M_{-1} = \sqrt{\Delta T} L_{-1}, \]
  probability $\approx \text{Tr} \left( M_{-1} \rho M_{-1}^\dagger \right)$ with back action
  \[ \rho_{t+\Delta T} \approx \frac{M_{-1} \rho_t M_{-1}^\dagger}{\text{Tr}(M_{-1} \rho M_{-1}^\dagger)}. \]

where
\[ L_{-1} = \sqrt{\frac{1}{T_{\text{cav}}}} a \]
is the Lindbald operator associated to cavity damping (see the continuous time models) with $T_{\text{cav}}$ the photon life time and $\Delta T \ll T_{\text{cav}}$ the sampling period ($T_{\text{cav}} = 100 \text{ ms}$ and $\Delta T \approx 100 \mu s$ for the LKB photon Box).
Cavity decoherence: cavity decay, thermal photon(s)

Three possible outcomes:

- **zero photon annihilation** during \( \Delta T \): Kraus operator
  
  \[
  M_0 = I - \frac{\Delta T}{2} L_{-1} L_{-1} - \frac{\Delta T}{2} L_1 L_1,
  \]
  
  probability \( \approx \text{Tr} \left( M_0 \rho M_0^\dagger \right) \) with back action
  
  \[
  \rho_{t+\Delta T} \approx \frac{M_0 \rho_t M_0^\dagger}{\text{Tr} \left( M_0 \rho M_0^\dagger \right)}.
  \]

- **one photon annihilation** during \( \Delta T \): Kraus operator
  
  \[
  M_{-1} = \sqrt{\Delta T} L_{-1},
  \]
  
  probability \( \approx \text{Tr} \left( M_{-1} \rho M_{-1}^\dagger \right) \) with back action
  
  \[
  \rho_{t+\Delta T} \approx \frac{M_{-1} \rho_t M_{-1}^\dagger}{\text{Tr} \left( M_{-1} \rho M_{-1}^\dagger \right)}
  \]

- **one photon creation** during \( \Delta T \): Kraus operator
  
  \[
  M_1 = \sqrt{\Delta T} L_1,
  \]
  
  probability \( \approx \text{Tr} \left( M_1 \rho M_1^\dagger \right) \) with back action
  
  \[
  \rho_{t+\Delta T} \approx \frac{M_1 \rho_t M_1^\dagger}{\text{Tr} \left( M_1 \rho M_1^\dagger \right)}
  \]

where

\[
L_{-1} = \sqrt{\frac{1+n_{th}}{T_{cav}}} a,
L_1 = \sqrt{\frac{n_{th}}{T_{cav}}} a^\dagger
\]

are the Lindbald operators associated to cavity decoherence: \( T_{cav} \) the photon life time, \( \Delta T \ll T_{cav} \) the sampling period and \( n_{th} \) is the average of thermal photon(s) (vanishes with the environment temperature) \( (n_{th} \approx 0.05 \text{ for the LKB photon box}) \).