Quantum Systems: Dynamics and Control

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1See the web page: http://cas.ensmp.fr/~rouchon/MasterUPMC/index.html

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Outline

1. Photon Box: a key example
2. Open quantum systems in discrete-time
The first experimental realization of a quantum state feedback


Stabilization of a quantum state with exactly \( n = 0, 1, 2, 3, \ldots \) photon(s).


4Courtesy of Igor Dotsenko. Sampling period \( \Delta t \approx 80 \mu s \).
Experimental closed-loop data


Decoherence due to finite photon life time around 70 ms)

Detection efficiency 40%
Detection error rate 10%
Delay 4 sampling periods

Model includes cavity decoherence, measurement imperfections, delays (Bayes law).

Truncation to 9 photons
Models of open quantum systems are based on three features\(^5\)

1. **Schrödinger**: wave funct. \(|\psi\rangle \in \mathcal{H}\) or density op. \(\rho \sim |\psi\rangle \langle \psi|\)

\[
\frac{d}{dt} |\psi\rangle = -\frac{i}{\hbar} H |\psi\rangle, \quad \frac{d}{dt} \rho = -\frac{i}{\hbar} [H, \rho], \quad H = H_0 + uH_1
\]

2. **Entanglement and tensor product** for composite systems \((S, M)\):
   - Hilbert space \(\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M\)
   - Hamiltonian \(H = H_S \otimes I_M + H_{\text{int}} + I_S \otimes H_M\)
   - observable on sub-system \(M\) only: \(O = I_S \otimes O_M\).

3. **Randomness and irreversibility** induced by the measurement of observable \(O\) with spectral decomp. \(\sum_\mu \lambda_\mu P_\mu\):
   - measurement outcome \(\mu\) with proba. \(P_\mu = \langle \psi | P_\mu | \psi \rangle = \text{Tr}(\rho P_\mu)\) depending on \(|\psi\rangle\), \(\rho\) just before the measurement
   - measurement back-action if outcome \(\mu = y\):
     \[
     |\psi\rangle \mapsto |\psi\rangle_+ = \frac{P_y |\psi\rangle}{\sqrt{\langle \psi | P_y | \psi \rangle}}, \quad \rho \mapsto \rho_+ = \frac{P_y \rho P_y}{\text{Tr}(\rho P_y)}
     \]

Composite system built with an harmonic oscillator and a qubit.

- **System** $S$ corresponds to a quantized harmonic oscillator:

$$\mathcal{H}_S = \mathcal{H}_c = \left\{ \sum_{n=0}^{\infty} c_n |n\rangle \left| (c_n)_{n=0}^{\infty} \in l^2(\mathbb{C}) \right. \right\},$$

where $|n\rangle$ represents the Fock state associated to exactly $n$ photons inside the cavity.

- **Meter** $M$ is a qu-bit, a 2-level system (idem 1/2 spin system): $\mathcal{H}_M = \mathcal{H}_a = \mathbb{C}^2$, each atom admits two energy levels and is described by a wave function $c_g |g\rangle + c_e |e\rangle$ with $|c_g|^2 + |c_e|^2 = 1$; atoms leaving $B$ are all in state $|g\rangle$.

- **State of the full system** $|\Psi\rangle \in \mathcal{H}_S \otimes \mathcal{H}_M = \mathcal{H}_c \otimes \mathcal{H}_a$:

$$|\Psi\rangle = \sum_{n=0}^{+\infty} c_{ng} |n\rangle \otimes |g\rangle + c_{ne} |n\rangle \otimes |e\rangle, \quad c_{ne}, c_{ng} \in \mathbb{C}.$$

Ortho-normal basis: $(|n\rangle \otimes |g\rangle, |n\rangle \otimes |e\rangle)_{n \in \mathbb{N}}$. 
When atom comes out $B$, $|\psi\rangle_B$ of the full system is **separable** $|\psi\rangle_B = |\psi\rangle \otimes |g\rangle$.

Just before the measurement in $D$, the state is in general **entangled** (not separable):

$$|\psi\rangle_{R_2} = U_{SM} (|\psi\rangle \otimes |g\rangle) = (M_g|\psi\rangle) \otimes |g\rangle + (M_e|\psi\rangle) \otimes |e\rangle$$

where $U_{SM}$ is a unitary transformation (Schrödinger propagator) defining the linear measurement operators $M_g$ and $M_e$ on $\mathcal{H}_S$. Since $U_{SM}$ is unitary, $M_g^\dagger M_g + M_e^\dagger M_e = I$. 

The Markov model (2)

Just before $D$, the field/atom state is **entangled**:

$$ M_g |\psi\rangle \otimes |g\rangle + M_e |\psi\rangle \otimes |e\rangle $$

Denote by $\mu \in \{g, e\}$ the measurement outcome in detector $D$: with probability $P_\mu = \langle \psi | M_\mu^\dagger M_\mu |\psi\rangle$ we get $\mu$. Just after the measurement outcome $\mu = y$, the state becomes separable:

$$ |\psi\rangle_D = \frac{1}{\sqrt{P_y}} \left( M_y |\psi\rangle \right) \otimes |y\rangle = \left( \frac{M_y}{\sqrt{\langle \psi | M_y^\dagger M_y |\psi\rangle}} |\psi\rangle \right) \otimes |y\rangle. $$

Markov process: $|\psi_k\rangle \equiv |\psi\rangle_{t=k\Delta t}, k \in \mathbb{N}$, $\Delta t$ sampling period,

$$ |\psi_{k+1}\rangle = \begin{cases} 
\frac{M_g |\psi_k\rangle}{\sqrt{\langle \psi_k | M_g^\dagger M_g |\psi_k\rangle}} & \text{with } y_k = g, \text{ probability } P_g = \langle \psi_k | M_g^\dagger M_g |\psi_k\rangle; \\
\frac{M_e |\psi_k\rangle}{\sqrt{\langle \psi_k | M_e^\dagger M_e |\psi_k\rangle}} & \text{with } y_k = e, \text{ probability } P_e = \langle \psi_k | M_e^\dagger M_e |\psi_k\rangle.
\end{cases} $$
The dispersive case

\[ U_{R_1} = \frac{1}{\sqrt{2}} (I + |g\rangle\langle e| - |e\rangle\langle g|) \]
\[ U_{R_2} = \frac{1}{\sqrt{2}} \left( I + e^{i\eta} |g\rangle\langle e| - e^{-i\eta} |e\rangle\langle g| \right) \]
\[ U_C = |g\rangle\langle g| e^{-i\phi(N)} + |e\rangle\langle e| e^{i\phi(N+1)} \]

where \( \phi(N) = \vartheta_0 + \vartheta N \).

With \( \eta = 2(\varphi_0 - \vartheta_0) - \vartheta - \pi \), the measurement operators \( M_g \)
and \( M_e \) are the following bounded operators:

\[ M_g = \cos(\varphi_0 + N\vartheta), \quad M_e = \sin(\varphi_0 + N\vartheta) \]

up to irrelevant global phases.

**Exercise:** Show that \( M_g^\dagger M_g + M_e^\dagger M_e = I \).
The resonant case: \( U_{SM} = U_{R2} U_{C} U_{R1} \)

\[
U_{R1} = e^{-i \frac{\theta_1}{2} \sigma_y} = \cos \left( \frac{\theta_1}{2} \right) + \sin \left( \frac{\theta_1}{2} \right) (|g\rangle \langle e| - |e\rangle \langle g|) \quad \text{and} \quad U_{R2} = I
\]

and

\[
U_{C} = |g\rangle \langle g| \cos \left( \frac{\Theta}{2} \sqrt{N} \right) + |e\rangle \langle e| \cos \left( \frac{\Theta}{2} \sqrt{N + 1} \right)
\]

\[
+ |g\rangle \langle e| \left( \frac{\sin \left( \frac{\Theta}{2} \sqrt{N} \right)}{\sqrt{N}} \right) a^\dagger - |e\rangle \langle g| a \left( \frac{\sin \left( \frac{\Theta}{2} \sqrt{N} \right)}{\sqrt{N}} \right)
\]

The measurement operators \( M_g \) and \( M_e \) are the following bounded operators:

\[
M_g = \cos \left( \frac{\theta_1}{2} \right) \cos \left( \frac{\Theta}{2} \sqrt{N} \right) - \sin \left( \frac{\theta_1}{2} \right) \left( \frac{\sin \left( \frac{\Theta}{2} \sqrt{N} \right)}{\sqrt{N}} \right) a^\dagger
\]

\[
M_e = -\sin \left( \frac{\theta_1}{2} \right) \cos \left( \frac{\Theta}{2} \sqrt{N + 1} \right) - \cos \left( \frac{\theta_1}{2} \right) a \left( \frac{\sin \left( \frac{\Theta}{2} \sqrt{N} \right)}{\sqrt{N}} \right)
\]

**Exercise:** Show that \( M_g^\dagger M_g + M_e^\dagger M_e = I \).
With pure state $\rho = |\psi\rangle\langle \psi|$, we have

$$\rho_+ = |\psi_+\rangle\langle \psi_+| = \frac{1}{\text{Tr} (M_\mu \rho M_{\mu}^\dagger)} M_\mu \rho M_{\mu}^\dagger$$

when the atom collapses in $\mu = g, e$ with proba. $\text{Tr} (M_\mu \rho M_{\mu}^\dagger)$.

**Detection efficiency:** the probability to detect the atom is $\eta \in [0, 1]$. Three possible outcomes for $y$: $y = g$ if detection in $g$, $y = e$ if detection in $e$ and $y = 0$ if no detection.

The only possible update is based on $\rho$: expectation $\rho_+$ of $|\psi_+\rangle\langle \psi_+|$ knowing $\rho$ and the outcome $y \in \{g, e, 0\}$.

$$\rho_+ = \begin{cases} 
\frac{M_g \rho M_g^\dagger}{\text{Tr}(M_g \rho M_g)} & \text{if } y = g, \text{ probability } \eta \text{ Tr} (M_g \rho M_g) \\
\frac{M_e \rho M_e^\dagger}{\text{Tr}(M_e \rho M_e)} & \text{if } y = e, \text{ probability } \eta \text{ Tr} (M_e \rho M_e) \\
M_g \rho M_g^\dagger + M_e \rho M_e^\dagger & \text{if } y = 0, \text{ probability } 1 - \eta
\end{cases}$$

For $\eta = 0$: $\rho_+ = M_g \rho M_g^\dagger + M_e \rho M_e^\dagger = \mathbb{K}(\rho) = \mathbb{E} (\rho_+ | \rho)$ defines a Kraus map.
Several operator spaces

- $\mathcal{H}$ separable Hilbert space. Pure states $|\psi\rangle$ are unitary vectors of $\mathcal{H}$ also called (probability amplitude) wave functions.

- $\mathcal{L}(\mathcal{H})$ is the space of linear operators from $\mathcal{H}$ to $\mathcal{H}$: it contains the spaces of
  - bounded operators (Banach space $\mathcal{B}(\mathcal{H})$ with sup-norm)
  - compact operators (space $\mathcal{K}^c(\mathcal{H})$)
  - Hilbert-Schmidt operators (Hilbert space $\mathcal{K}^2(\mathcal{H})$ with the Frobenius norm)
  - trace class operators (Banach space $\mathcal{K}^1(\mathcal{H})$ with the trace norm).

- the most general quantum state $\rho$ is non negative Hermitian trace class operator of trace one. $\rho$ live in a closed convex subset of $\mathcal{K}^1(\mathcal{H})$.
  If $\text{Tr}(\rho^2) = 1$ then $\rho = |\psi\rangle\langle\psi|$ where $|\psi\rangle$ is pure state.

For $\mathcal{H}$ of finite dimension, these operator spaces coincide. For $\mathcal{H}$ of infinite dimension, they are all different:

$$\dim \mathcal{H} = \infty \quad \Rightarrow \quad \mathcal{K}^1(\mathcal{H}) \subsetneq \mathcal{K}^2(\mathcal{H}) \subsetneq \mathcal{K}^c(\mathcal{H}) \subsetneq \mathcal{B}(\mathcal{H}) \subsetneq \mathcal{L}(\mathcal{H}).$$
With pure state $\rho = |\psi\rangle\langle\psi|$, we have

$$\rho_+ = |\psi_+\rangle\langle\psi_+| = \frac{1}{\text{Tr}(M_\mu \rho M_\mu^\dagger)} M_\mu \rho M_\mu^\dagger$$

when the atom collapses in $\mu = g, e$ with proba. $\text{Tr}(M_\mu \rho M_\mu^\dagger)$.

Detection error rates: $\mathbb{P}(y = e/\mu = g) = \eta_g \in [0, 1]$ the probability of erroneous assignation to $e$ when the atom collapses in $g$; $\mathbb{P}(y = g/\mu = e) = \eta_e \in [0, 1]$ (given by the contrast of the Ramsey fringes).

Bayes law: expectation $\rho_+$ of $|\psi_+\rangle\langle\psi_+|$ knowing $\rho$ and the imperfect detection $y$.

$$\rho_+ = \begin{cases} 
\frac{(1-\eta_g) M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger}{\text{Tr}((1-\eta_g) M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger)} & \text{if } y = g, \text{ prob. } \text{Tr}((1-\eta_g) M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger); \\
\frac{\eta_g M_g \rho M_g^\dagger + (1-\eta_e) M_e \rho M_e^\dagger}{\text{Tr}(\eta_g M_g \rho M_g^\dagger + (1-\eta_e) M_e \rho M_e^\dagger)} & \text{if } y = e, \text{ prob. } \text{Tr}(\eta_g M_g \rho M_g^\dagger + (1-\eta_e) M_e \rho M_e^\dagger). 
\end{cases}$$

$\rho_+$ does not remain pure: the quantum state $\rho_+$ becomes a mixed state; $|\psi_+\rangle$ becomes physically irrelevant.
We get

\[ \rho_+ = \begin{cases} 
\frac{(1-\eta_g)M_g\rho M_g^\dagger + \eta_e M_e\rho M_e^\dagger}{\text{Tr}\left((1-\eta_g)M_g\rho M_g^\dagger + \eta_e M_e\rho M_e^\dagger\right)}, & \text{with prob. } \text{Tr}\left((1-\eta_g)M_g\rho M_g^\dagger + \eta_e M_e\rho M_e^\dagger\right); \\
\frac{\eta_g M_g\rho M_g^\dagger + (1-\eta_e) M_e\rho M_e^\dagger}{\text{Tr}\left(\eta_g M_g\rho M_g^\dagger + (1-\eta_e) M_e\rho M_e^\dagger\right)}, & \text{with prob. } \text{Tr}\left(\eta_g M_g\rho M_g^\dagger + (1-\eta_e) M_e\rho M_e^\dagger\right). 
\end{cases} \]

Key point:

\[ \text{Tr}\left((1-\eta_g)M_g\rho M_g^\dagger + \eta_e M_e\rho M_e^\dagger\right) \text{ and } \text{Tr}\left(\eta_g M_g\rho M_g^\dagger + (1-\eta_e) M_e\rho M_e^\dagger\right) \]

are the probabilities to detect \( y = g \) and \( e \), knowing \( \rho \).

Reformulation with quantum maps: set

\[ K_g(\rho) = (1-\eta_g)M_g\rho M_g^\dagger + \eta_e M_e\rho M_e^\dagger, \quad K_e(\rho) = \eta_g M_g\rho M_g^\dagger + (1-\eta_e) M_e\rho M_e^\dagger. \]

\[ \rho_+ = \frac{K_y(\rho)}{\text{Tr}(K_y(\rho))} \quad \text{when we detect } y \]

The probability to detect \( y \) knowing \( \rho \) is \( \text{Tr}(K_y(\rho)) \).

We have the following Kraus map:

\[ \mathbb{E}(\rho_+ | \rho) = K_g(\rho) + K_e(\rho) = K(\rho) = M_g\rho M_g^\dagger + M_e\rho M_e^\dagger. \]
Cavity decay (decoherence) seen as unread measurements

The cavity mirrors play the role of a detector with two possible outcomes:

- **zero photon annihilation** during $\Delta T$: Kraus operator
  \[ M_0 = I - \frac{\Delta T}{2} L_{-1}^\dagger L_{-1}, \]
  probability $\approx \text{Tr} \left( M_0 \rho M_0^\dagger \right)$ with back action
  \[ \rho_{t+\Delta T} \approx \frac{M_0 \rho_t M_0^\dagger}{\text{Tr} \left( M_0 \rho M_0^\dagger \right)}. \]

- **one photon annihilation** during $\Delta T$: Kraus operator
  \[ M_{-1} = \sqrt{\Delta T} L_{-1}, \]
  probability $\approx \text{Tr} \left( M_{-1} \rho M_{-1}^\dagger \right)$ with back action
  \[ \rho_{t+\Delta T} \approx \frac{M_{-1} \rho_t M_{-1}^\dagger}{\text{Tr} \left( M_{-1} \rho M_{-1}^\dagger \right)}. \]

where

\[ L_{-1} = \sqrt{\frac{1}{T_{cav}}} a \]

is the Lindbald operator associated to cavity damping (see the continuous time models) with $T_{cav}$ the photon life time and $\Delta T \ll T_{cav}$ the sampling period ($T_{cav} = 100 \text{ ms}$ and $\Delta T \approx 100 \mu s$ for the LKB photon Box).
Cavity decoherence: cavity decay, thermal photon(s)

Three possible outcomes:

- **zero photon annihilation** during $\Delta T$: Kraus operator
  
  \[ M_0 = I - \frac{\Delta T}{2} L_{-1} L_{-1} - \frac{\Delta T}{2} L_1 L_1, \]
  
  probability $\approx \text{Tr} \left( M_0 \rho M_0^\dagger \right)$ with back action $\rho_{t+\Delta T} \approx \frac{M_0 \rho_t M_0^\dagger}{\text{Tr} \left( M_0 \rho M_0^\dagger \right)}$.

- **one photon annihilation** during $\Delta T$: Kraus operator
  
  \[ M_{-1} = \sqrt{\Delta T} L_{-1}, \]
  
  probability $\approx \text{Tr} \left( M_{-1} \rho M_{-1}^\dagger \right)$ with back action $\rho_{t+\Delta T} \approx \frac{M_{-1} \rho_t M_{-1}^\dagger}{\text{Tr} \left( M_{-1} \rho M_{-1}^\dagger \right)}$.

- **one photon creation** during $\Delta T$: Kraus operator
  
  \[ M_1 = \sqrt{\Delta T} L_1, \]
  
  probability $\approx \text{Tr} \left( M_1 \rho M_1^\dagger \right)$ with back action $\rho_{t+\Delta T} \approx \frac{M_1 \rho_t M_1^\dagger}{\text{Tr} \left( M_1 \rho M_1^\dagger \right)}$.

where

\[ L_{-1} = \sqrt{\frac{1+n_{th}}{T_{cav}}} a, \quad L_1 = \sqrt{\frac{n_{th}}{T_{cav}}} a^\dagger \]

are the Lindbald operators associated to cavity decoherence: $T_{cav}$ the photon life time, $\Delta T \ll T_{cav}$ the sampling period and $n_{th}$ is the average of thermal photon(s) (vanishes with the environment temperature) ($n_{th} \approx 0.05$ for the LKB photon box).
Valeur moyenne du nombre de photons le long d'une longue séquence de mesure: observation d'une trajectoire stochastique

A partir de la probabilité $P_i(n)$ inférée après chaque atome, on déduit le nombre moyen de photons:

$$\langle n \rangle = \sum_n n P_i(n) \quad (6 \sim 10)$$

See the quantum Monte Carlo simulations of the Matlab script: RealisticModelPhotonBox.m.

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Outline

1. Photon Box: a key example

2. Open quantum systems in discrete-time
Any open model of quantum system in discrete time is governed by a Markov chain of the form

\[
\rho_{k+1} = \frac{\mathbb{K}_{y_k}(\rho_k)}{\text{Tr}(\mathbb{K}_{y_k}(\rho_k))},
\]

with the probability \( \text{Tr}(\mathbb{K}_{y_k}(\rho_k)) \) to have the measurement outcome \( y_k \) knowing \( \rho_{k-1} \).

The structure of the super-operators \( \mathbb{K}_y \) is as follows. Each \( \mathbb{K}_y \) is a linear completely positive map (a quantum operation, a partial Kraus map\(^7\)) and \( \sum_y \mathbb{K}_y(\rho) = \mathbb{K}(\rho) \) is a Kraus map, i.e. \( \mathbb{K}(\rho) = \sum_\mu K_\mu \rho K_\mu^\dagger \) with \( \sum_\mu K_\mu^\dagger K_\mu = I \).

\(^7\)Each \( \mathbb{K}_y \) admits the expression

\[
\mathbb{K}_y(\rho) = \sum_\mu K_{y,\mu} \rho K_{y,\mu}^\dagger
\]

where \( (K_{y,\mu}) \) are bounded operators on \( \mathcal{H} \).
Without measurement record, the quantum state $\rho_k$ obeys to the master equation

$$\rho_{k+1} = K(\rho_k).$$

since $E(\rho_{k+1} \mid \rho_k) = K(\rho_k)$ (ensemble average).

$K$ is always a contraction (not strict in general) for the following two such metrics. For any density operators $\rho$ and $\rho'$ we have

$$\|K(\rho) - K(\rho')\|_1 \leq \|\rho - \rho'\|_1 \quad \text{and} \quad F(K(\rho), K(\rho')) \geq F(\rho, \rho')$$

where the trace norm $\| \cdot \|_1$ and fidelity $F$ are given by

$$\|\rho - \rho'\|_1 \triangleq \text{Tr}(|\rho - \rho'|) \quad \text{and} \quad F(\rho, \rho') \triangleq \text{Tr} \left( \sqrt{\sqrt{\rho} \rho' \sqrt{\rho}} \right).$$
The "Heisenberg description" is given by iterates $A_{k+1} = K^*(A_k)$ from an initial bounded Hermitian operator $A_0$ of the the dual map $K^*$ characterized as follows: $\text{Tr}(A K(\rho)) = \text{Tr}(K^*(A) \rho)$ for any bounded operator $A$ on $\mathcal{H}$. Thus

$$K^*(A) = \sum_{\mu} K_{\mu}^\dagger A K_{\mu} \quad \text{when} \quad K(\rho) = \sum_{\mu} K_{\mu} \rho K_{\mu}^\dagger.$$ 

$K^*$ is an unital map, i.e., $K^*(I) = I$, and the image via $K^*$ of any bounded operator is a bounded operator.

When $\mathcal{H}$ is of finite dimension, we have, for any Hermitian operator $A$:

$$\lambda_{\text{min}}(A) \leq \lambda_{\text{min}}(K^*(A)) \leq \lambda_{\text{max}}(K^*(A)) \leq \lambda_{\text{max}}(A)$$

where $\lambda_{\text{min}}$ and $\lambda_{\text{max}}$ correspond to the smallest and largest eigenvalues.

If $\overline{A} = K^*(\overline{A})$, then $\text{Tr}(\rho_k \overline{A}) = \text{Tr}(\rho_0 \overline{A})$ is a constant of the motion of $\rho$. 
Take a Kraus map $K$ and its adjoint unital map $K^*$. When $\mathcal{H}$ is of finite dimension, the following two statements are equivalent:

- Global convergence towards the fixed point $\rho = K(\rho)$ (pointer-state) of $\rho_{k+1} = K(\rho_k)$: for any initial density operator $\rho_0$, $\lim_{k \to +\infty} \rho_k = \rho$ for the trace norm $\| \cdot \|_1$.

- Global convergence of $A_{k+1} = K^*(A_k)$: exists a unique density operator $\rho$ such that, for any initial bounded operator $A_0$, $\lim_{k \to +\infty} A_k = \text{Tr}(A_0\rho) I$ for the sup norm on the bounded operators on $\mathcal{H}$. 