Quantum Systems: Dynamics and Control

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Outline

1. Photon Box: a key example

2. Open quantum systems in discrete-time
Models of open quantum systems are based on three features\textsuperscript{4}

1. **Schrödinger**: wave funct. $|\psi\rangle \in \mathcal{H}$ or density op. $\rho \sim |\psi\rangle \langle \psi|$

\[
\frac{d}{dt} |\psi\rangle = -\frac{i}{\hbar} H |\psi\rangle, \quad \frac{d}{dt} \rho = -\frac{i}{\hbar} [H, \rho], \quad H = H_0 + uH_1
\]

2. **Entanglement and tensor product** for composite systems $(S, M)$:
   - Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M$
   - Hamiltonian $H = H_S \otimes I_M + H_{int} + I_S \otimes H_M$
   - observable on sub-system $M$ only: $O = I_S \otimes O_M$.

3. **Randomness and irreversibility** induced by the measurement of observable $O$ with spectral decomp. $\sum_\mu \lambda_\mu P_\mu$:
   - measurement outcome $\mu$ with proba.
     \[P_\mu = \langle \psi | P_\mu | \psi \rangle = \text{Tr}(\rho P_\mu)\] depending on $|\psi\rangle$, $\rho$ just before the measurement
   - measurement back-action if outcome $\mu = y$:
     \[|\psi\rangle \mapsto |\psi\rangle_+ = \frac{P_y |\psi\rangle}{\sqrt{\langle \psi | P_y | \psi \rangle}}, \quad \rho \mapsto \rho_+ = \frac{P_y \rho P_y}{\text{Tr}(\rho P_y)}\]

System $S$ corresponds to a quantized harmonic oscillator:

$$H_S = H_c = \left\{ \sum_{n=0}^{\infty} c_n |n\rangle \right\},$$

where $|n\rangle$ represents the Fock state associated to exactly $n$ photons inside the cavity.

Meter $M$ is a qubit, a 2-level system: $H_M = H_a = \mathbb{C}^2$, each atom admits two energy levels and is described by a wave function $c_g |g\rangle + c_e |e\rangle$ with $|c_g|^2 + |c_e|^2 = 1$.

State of the full system $|\psi\rangle \in H_S \otimes H_M = H_c \otimes H_a$:

$$|\psi\rangle = \sum_{n=0}^{+\infty} c_{ng} |n\rangle \otimes |g\rangle + c_{ne} |n\rangle \otimes |e\rangle,$$

Ortho-normal basis: $(|n\rangle \otimes |g\rangle, |n\rangle \otimes |e\rangle)_{n\in \mathbb{N}}$. 
When atom comes out $B$, $|\psi_B\rangle$ of the full system is **separable**

$$|\psi_B\rangle = |\psi\rangle \otimes |g\rangle.$$  

Just before the measurement in $D$, the state is in general **entangled** (not separable):

$$|\psi_{R_2}\rangle = U_{SM}(|\psi\rangle \otimes |g\rangle) = (M_g |\psi\rangle \otimes |g\rangle) + (M_e |\psi\rangle \otimes |e\rangle)$$

where $U_{SM}$ is a unitary transformation (Schrödinger propagator) defining the linear measurement operators $M_g$ and $M_e$ on $\mathcal{H}_S$. Since $U_{SM}$ is unitary, $M_g^\dagger M_g + M_e^\dagger M_e = I$. 
Markov model (2)

Just before $D$, the field/atom state is entangled:

$$M_g |\psi\rangle \otimes |g\rangle + M_e |\psi\rangle \otimes |e\rangle$$

Denote by $\mu \in \{g, e\}$ the measurement outcome in detector $D$: with probability $P_\mu = \langle \psi | M_\mu^\dagger M_\mu |\psi\rangle$ we get $\mu$. Just after the measurement outcome $\mu = y$, the state becomes separable:

$$|\psi\rangle_D = \frac{1}{\sqrt{P_y}} (M_y |\psi\rangle) \otimes |y\rangle = \left( \frac{M_y}{\sqrt{\langle \psi | M_y^\dagger M_y |\psi\rangle}} |\psi\rangle \right) \otimes |y\rangle.$$

Markov process: $|\psi_k\rangle \equiv |\psi\rangle_{t=k\Delta t}$, $k \in \mathbb{N}$, $\Delta t$ sampling period,

$$|\psi_{k+1}\rangle = \begin{cases} \frac{M_g |\psi_k\rangle}{\sqrt{\langle \psi_k | M_g^\dagger M_g |\psi_k\rangle}} & \text{with } y_k = g, \text{ probability } P_g = \langle \psi_k | M_g^\dagger M_g |\psi_k\rangle; \\ \frac{M_e |\psi_k\rangle}{\sqrt{\langle \psi_k | M_e^\dagger M_e |\psi_k\rangle}} & \text{with } y_k = e, \text{ probability } P_e = \langle \psi_k | M_e^\dagger M_e |\psi_k\rangle. \end{cases}$$
Dispersive case

\[
U_{R_1} = \frac{1}{\sqrt{2}} (I + |g\rangle\langle e| - |e\rangle\langle g|)
\]

\[
U_{R_2} = \frac{1}{\sqrt{2}} \left( I + e^{i\eta} |g\rangle\langle e| - e^{-i\eta} |e\rangle\langle g| \right)
\]

\[
U_C = |g\rangle\langle g| e^{-i\phi(N)} + |e\rangle\langle e| e^{i\phi(N+I)}
\]

where \( \phi(N) = \vartheta_0 + \vartheta N \).

With \( \eta = 2(\varphi_0 - \vartheta_0) - \vartheta - \pi \), the measurement operators \( M_g \) and \( M_e \) are the following bounded operators:

\[
M_g = \cos(\varphi_0 + N\vartheta), \quad M_e = \sin(\varphi_0 + N\vartheta)
\]

up to irrelevant global phases.

**Exercise:** Show that \( M_g^\dagger M_g + M_e^\dagger M_e = I \).
Resonant case: $U_{SM} = U_{R_2} U_C U_{R_1}$

$$U_{R_1} = e^{-i \frac{\theta_1}{2} \sigma_y} = \cos \left( \frac{\theta_1}{2} \right) + \sin \left( \frac{\theta_1}{2} \right) (|g\rangle\langle e| - |e\rangle\langle g|) \quad \text{and} \quad U_{R_2} = I$$

and

$$U_C = |g\rangle\langle g| \cos \left( \frac{\Theta_2}{2} \sqrt{N} \right) + |e\rangle\langle e| \cos \left( \frac{\Theta_2}{2} \sqrt{N + 1} \right)$$

$$+ |g\rangle\langle e| \left( \frac{\sin \left( \frac{\Theta_2}{2} \sqrt{N} \right)}{\sqrt{N}} \right) a^\dagger - |e\rangle\langle g| a \left( \frac{\sin \left( \frac{\Theta_2}{2} \sqrt{N} \right)}{\sqrt{N}} \right)$$

The measurement operators $M_g$ and $M_e$ are the following bounded operators:

$$M_g = \cos \left( \frac{\theta_1}{2} \right) \cos \left( \frac{\Theta_2}{2} \sqrt{N} \right) - \sin \left( \frac{\theta_1}{2} \right) \left( \frac{\sin \left( \frac{\Theta_2}{2} \sqrt{N} \right)}{\sqrt{N}} \right) a^\dagger$$

$$M_e = - \sin \left( \frac{\theta_1}{2} \right) \cos \left( \frac{\Theta_2}{2} \sqrt{N + 1} \right) - \cos \left( \frac{\theta_1}{2} \right) a \left( \frac{\sin \left( \frac{\Theta_2}{2} \sqrt{N} \right)}{\sqrt{N}} \right)$$

Exercise: Show that $M_g^\dagger M_g + M_e^\dagger M_e = I$. 
Markov process with detection inefficiency

- With pure state $\rho = |\psi\rangle\langle\psi|$, we have

$$\rho_+ = |\psi_+\rangle\langle\psi_+| = \frac{1}{\operatorname{Tr}(M_\mu \rho M_\mu^\dagger)} M_\mu \rho M_\mu^\dagger$$

when the atom collapses in $\mu = g, e$ with proba. $\operatorname{Tr}(M_\mu \rho M_\mu^\dagger)$.

- Detection efficiency: the probability to detect the atom is $\eta \in [0, 1]$. Three possible outcomes for $y$: $y = g$ if detection in $g$, $y = e$ if detection in $e$ and $y = 0$ if no detection.

The only possible update is based on $\rho$: expectation $\rho_+$ of $|\psi_+\rangle\langle\psi_+|$ knowing $\rho$ and the outcome $y \in \{g, e, 0\}$.

$$\rho_+ = \begin{cases} \frac{M_g \rho M_g^\dagger}{\operatorname{Tr}(M_g \rho M_g)} & \text{if } y = g, \text{ probability } \eta \operatorname{Tr}(M_g \rho M_g) \\ \frac{M_e \rho M_e^\dagger}{\operatorname{Tr}(M_e \rho M_e)} & \text{if } y = e, \text{ probability } \eta \operatorname{Tr}(M_e \rho M_e) \\ M_g \rho M_g^\dagger + M_e \rho M_e^\dagger & \text{if } y = 0, \text{ probability } 1 - \eta \end{cases}$$

For $\eta = 0$: $\rho_+ = M_g \rho M_g^\dagger + M_e \rho M_e^\dagger = \mathbb{K}(\rho) = \mathbb{E} \left( \rho_+ \mid \rho \right)$ defines a Kraus map.
With pure state $\rho = |\psi\rangle\langle\psi|$, we have

$$\rho_+ = |\psi_+\rangle\langle\psi_+| = \frac{1}{\text{Tr}(M_\mu \rho M_\mu^\dagger)} M_\mu \rho M_\mu^\dagger$$

when the atom collapses in $\mu = g, e$ with proba. $\text{Tr}(M_\mu \rho M_\mu^\dagger)$.

Detection error rates: $\mathbb{P}(y = e/\mu = g) = \eta_g \in [0, 1]$ the probability of erroneous assignation to $e$ when the atom collapses in $g$; $\mathbb{P}(y = g/\mu = e) = \eta_e \in [0, 1]$ (given by the contrast of the Ramsey fringes).

Bayesian law: expectation $\rho_+$ of $|\psi_+\rangle\langle\psi_+|$ knowing $\rho$ and the imperfect detection $y$.

$$\rho_+ = \begin{cases} 
\frac{(1-\eta_g)M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger}{\text{Tr}\left((1-\eta_g)M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger\right)} & \text{if } y = g, \text{ prob. } \text{Tr}\left((1 - \eta_g)M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger\right); \\
\frac{\eta_g M_g \rho M_g^\dagger + (1-\eta_e) M_e \rho M_e^\dagger}{\text{Tr}\left(\eta_g M_g \rho M_g^\dagger + (1-\eta_e) M_e \rho M_e^\dagger\right)} & \text{if } y = e, \text{ prob. } \text{Tr}\left(\eta_g M_g \rho M_g^\dagger + (1 - \eta_e) M_e \rho M_e^\dagger\right)
\end{cases}$$

$\rho_+$ does not remain pure: the quantum state $\rho_+$ becomes a mixed state; $|\psi_+\rangle$ becomes physically irrelevant.
We get
\[
\rho_+ = \begin{cases} 
\frac{(1-\eta_g)M_g\rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger}{\text{Tr}((1-\eta_g)M_g\rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger)}, & \text{with prob. } \text{Tr}\left((1 - \eta_g)M_g\rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger\right); \\
\frac{\eta_g M_g \rho M_g^\dagger + (1-\eta_e) M_e \rho M_e^\dagger}{\text{Tr}(\eta_g M_g \rho M_g^\dagger + (1-\eta_e) M_e \rho M_e^\dagger)}, & \text{with prob. } \text{Tr}\left(\eta_g M_g \rho M_g^\dagger + (1 - \eta_e) M_e \rho M_e^\dagger\right). 
\end{cases}
\]

Key point:
\[
\text{Tr}\left((1 - \eta_g)M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger\right) \text{ and } \text{Tr}\left(\eta_g M_g \rho M_g^\dagger + (1 - \eta_e) M_e \rho M_e^\dagger\right)
\]
are the probabilities to detect \( y = g \) and \( e \), knowing \( \rho \).

**Reformulation with quantum maps** : set
\[
K_g(\rho) = (1-\eta_g)M_g\rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger, \quad K_e(\rho) = \eta_g M_g \rho M_g^\dagger + (1-\eta_e) M_e \rho M_e^\dagger.
\]

\[
\rho_+ = \frac{K_y(\rho)}{\text{Tr}(K_y(\rho))} \quad \text{when we detect } y
\]

The probability to detect \( y \) knowing \( \rho \) is \( \text{Tr}(K_y(\rho)) \).

We have the following Kraus map:
\[
E(\rho_+ \mid \rho) = K_g(\rho) + K_e(\rho) = K(\rho) = M_g\rho M_g^\dagger + M_e \rho M_e^\dagger.
\]
1 Photon Box: a key example

2 Open quantum systems in discrete-time
Any open model of quantum system in discrete time is governed by a Markov chain of the form

\[ \rho_{k+1} = \frac{\mathbb{K}_{y_k}(\rho_k)}{\text{Tr}(\mathbb{K}_{y_k}(\rho_k))}, \]

with the probability \( \text{Tr}(\mathbb{K}_{y_k}(\rho_k)) \) to have the measurement outcome \( y_k \) knowing \( \rho_{k-1} \).

The structure of the super-operators \( \mathbb{K}_y \) is as follows. Each \( \mathbb{K}_y \) is a linear completely positive map (a quantum operation, a partial Kraus map\(^5\)) and \( \sum_y \mathbb{K}_y(\rho) = \mathbb{K}(\rho) \) is a Kraus map, i.e. \( \mathbb{K}(\rho) = \sum_\mu K_\mu \rho K_\mu^\dagger \) with \( \sum_\mu K_\mu^\dagger K_\mu = I \).

\(^{5}\)Each \( \mathbb{K}_y \) admits the expression

\[ \mathbb{K}_y(\rho) = \sum_\mu K_{y,\mu} \rho K_{y,\mu}^\dagger \]

where \((K_{y,\mu})\) are bounded operators on \( \mathcal{H} \).
Without measurement record, the quantum state $\rho_k$ obeys to the master equation

$$\rho_{k+1} = K(\rho_k).$$

since $E(\rho_{k+1} | \rho_k) = K(\rho_k)$ (ensemble average).

$K$ is always a contraction (not strict in general) for the following two such metrics. For any density operators $\rho$ and $\rho'$ we have

$$\|K(\rho) - K(\rho')\|_1 \leq \|\rho - \rho'\|_1 \quad \text{and} \quad F(K(\rho), K(\rho')) \geq F(\rho, \rho')$$

where the trace norm $\|\cdot\|_1$ and fidelity $F$ are given by

$$\|\rho - \rho'\|_1 \triangleq \text{Tr} (|\rho - \rho'|) \quad \text{and} \quad F(\rho, \rho') \triangleq \text{Tr} \left( \sqrt{\sqrt{\rho} \rho' \sqrt{\rho}} \right).$$