Quantum Systems: Dynamics and Control\(^1\)

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1. Averaging and control: the recipe

2. Averaging and control a qubit

3. Averaging and control of spin/spring systems
   - The Jaynes Cumming model
   - Resonant interaction
   - Dispersive interaction
Outline

1. Averaging and control: the recipe

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   - The Jaynes Cumming model
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   - Dispersive interaction
Un-measured quantum system $\rightarrow$ Bilinear Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi\rangle = (H_0 + u(t)H_1)|\psi\rangle,$$

- $|\psi\rangle \in \mathcal{H}$ the system’s wavefunction with $\left\| |\psi\rangle \right\|_{\mathcal{H}} = 1$;
- the free Hamiltonian, $H_0$, is a Hermitian operator defined on $\mathcal{H}$;
- the control Hamiltonian, $H_1$, is a Hermitian operator defined on $\mathcal{H}$;
- the control $u(t) : \mathbb{R}^+ \mapsto \mathbb{R}$ is a scalar control.

Here we consider the case of finite dimensional $\mathcal{H}$
Almost periodic control

We consider the controls of the form

\[ u(t) = \epsilon \left( \sum_{j=1}^{r} u_j e^{i\omega_j t} + u_j^* e^{-i\omega_j t} \right) \]

- \( \epsilon > 0 \) is a small parameter;
- \( \epsilon u_j \) is the constant complex amplitude associated to the pulsation \( \omega_j \geq 0 \);
- \( r \) stands for the number of independent pulsations (\( \omega_j \neq \omega_k \) for \( j \neq k \)).

We are interested in approximations, for \( \epsilon \) tending to \( 0^+ \), of trajectories \( t \mapsto |\psi_\epsilon\rangle \) of

\[ \frac{d}{dt} |\psi_\epsilon\rangle = \left( A_0 + \epsilon \left( \sum_{j=1}^{r} u_j e^{i\omega_j t} + u_j^* e^{-i\omega_j t} \right) A_1 \right) |\psi_\epsilon\rangle \]

where \( A_0 = -iH_0/\hbar \) and \( A_1 = -iH_1/\hbar \) are skew-Hermitian.
Consider the following change of variables

\[ |\psi_\epsilon\rangle_t = e^{A_0 t} |\phi_\epsilon\rangle_t. \]

The resulting system is said to be in the “interaction frame”

\[ \frac{d}{dt} |\phi_\epsilon\rangle = \epsilon B(t) |\phi_\epsilon\rangle \]

where \( B(t) \) is a skew-Hermitian operator whose time-dependence is almost periodic:

\[
B(t) = \sum_{j=1}^{r} u_j e^{i\omega_j t} e^{-A_0 t} A_1 e^{A_0 t} + u_j^* e^{-i\omega_j t} e^{-A_0 t} A_1 e^{A_0 t}.
\]

Main idea

We can write

\[ B(t) = \bar{B} + \frac{d}{dt} \tilde{B}(t), \]

where \( \bar{B} \) is a constant skew-Hermitian matrix and \( \tilde{B}(t) \) is a bounded almost periodic skew-Hermitian matrix.
Consider the two systems

\[
\frac{d}{dt} |\phi_{\epsilon}\rangle = \epsilon \left( \bar{B} + \frac{d}{dt}\tilde{B}(t) \right) |\phi_{\epsilon}\rangle,
\]

and

\[
\frac{d}{dt} |\phi^{1st}_{\epsilon}\rangle = \epsilon \bar{B} |\phi^{1st}_{\epsilon}\rangle,
\]

initialized at the same state \( |\phi^{1st}_{\epsilon}\rangle_0 = |\phi_{\epsilon}\rangle_0 \).

**Theorem: first order approximation (Rotating Wave Approximation)**

Consider the functions \( |\phi_{\epsilon}\rangle \) and \( |\phi^{1st}_{\epsilon}\rangle \) initialized at the same state and following the above dynamics. Then, there exist \( M > 0 \) and \( \eta > 0 \) such that for all \( \epsilon \in ]0, \eta[ \) we have

\[
\max_{t \in \left[0, \frac{1}{\epsilon}\right]} \left\| |\phi_{\epsilon}\rangle_t - |\phi^{1st}_{\epsilon}\rangle_t \right\| \leq M\epsilon
\]
Proof’s idea

Almost periodic change of variables:

\[ |\chi_\epsilon\rangle = (1 - \epsilon \tilde{B}(t))|\phi_\epsilon\rangle \]

well-defined for \( \epsilon > 0 \) sufficiently small.

The dynamics can be written as

\[ \frac{d}{dt}|\chi_\epsilon\rangle = (\epsilon \bar{B} + \epsilon^2 F(\epsilon, t))|\chi_\epsilon\rangle \]

where \( F(\epsilon, t) \) is uniformly bounded in time.
Multi-frequency averaging: second order

More precisely, the dynamics of $|\chi_\epsilon\rangle$ is given by

$$\frac{d}{dt} |\chi_\epsilon\rangle = \left( \epsilon \bar{B} + \epsilon^2 [\bar{B}, \bar{B}(t)] - \epsilon^2 \bar{B}(t) \frac{d}{dt} \bar{B}(t) + \epsilon^3 E(\epsilon, t) \right) |\chi_\epsilon\rangle$$

- $E(\epsilon, t)$ is still almost periodic but its entries are no more linear combinations of time-exponentials;
- $\tilde{B}(t) \frac{d}{dt} \tilde{B}(t)$ is an almost periodic operator whose entries are linear combinations of oscillating time-exponentials.

We can write

$$\tilde{B}(t) \frac{d}{dt} \tilde{B}(t) = \tilde{D} + \frac{d}{dt} \tilde{D}(t)$$

where $\tilde{D}(t)$ is almost periodic. We have

$$\frac{d}{dt} |\chi_\epsilon\rangle = \left( \epsilon \bar{B} - \epsilon^2 \tilde{D} + \epsilon^2 \frac{d}{dt} \left( [\bar{B}, \tilde{C}(t)] - \tilde{D}(t) \right) + \epsilon^3 E(\epsilon, t) \right) |\chi_\epsilon\rangle$$

where the skew-Hermitian operators $\bar{B}$ and $\tilde{D}$ are constants and the other ones $\tilde{C}$, $\tilde{D}$, and $E$ are almost periodic.
Multi-frequency averaging: second order

Consider the two systems

\[
\frac{d}{dt} |\phi_\epsilon\rangle = \epsilon \left( \bar{B} + \frac{d}{dt} \tilde{B}(t) \right) |\phi_\epsilon\rangle,
\]

and

\[
\frac{d}{dt} |\phi^{2nd}_\epsilon\rangle = (\epsilon \bar{B} - \epsilon^2 \tilde{D}) |\phi^{2nd}_\epsilon\rangle,
\]

initialized at the same state \(|\phi^{2nd}_\epsilon\rangle_0 = |\phi_\epsilon\rangle_0\).

Theorem: second order approximation

Consider the functions \(|\phi_\epsilon\rangle\) and \(|\phi^{2nd}_\epsilon\rangle\) initialized at the same state and following the above dynamics. Then, there exist \(M > 0\) and \(\eta > 0\) such that for all \(\epsilon \in ]0, \eta[\) we have

\[
\max_{t \in \left[0, \frac{1}{\epsilon^2}\right]} \left\| |\phi_\epsilon\rangle_t - |\phi^{2nd}_\epsilon\rangle_t \right\| \leq M \epsilon
\]
Proof’s idea

Another almost periodic change of variables

\[ |\xi_\epsilon\rangle = \left( I - \epsilon^2 \left( [\bar{B}, \tilde{C}(t)] - \tilde{D}(t) \right) \right) |\chi_\epsilon\rangle. \]

The dynamics can be written as

\[ \frac{d}{dt} |\xi_\epsilon\rangle = \left( \epsilon \bar{B} - \epsilon^2 \bar{D} + \epsilon^3 F(\epsilon, t) \right) |\xi_\epsilon\rangle \]

where \( \epsilon \bar{B} - \epsilon^2 \bar{D} \) is skew Hermitian and \( F \) is almost periodic and therefore uniformly bounded in time.
The Rotating Wave Approximation (RWA) recipes

Schrödinger dynamics \(i\hbar \frac{d}{dt} |\psi\rangle = H(t)|\psi\rangle\), with

\[
H(t) = H_0 + \sum_{k=1}^{m} u_k(t) H_k, \quad u_k(t) = \sum_{j=1}^{r} u_{k,j} e^{\omega_j t} + u_{k,j}^* e^{-\omega_j t}.
\]

The Hamiltonian in interaction frame

\[
H_{\text{int}}(t) = \sum_{k,j} (u_{k,j} e^{\omega_j t} + u_{k,j}^* e^{-\omega_j t}) e^{iH_0 t} H_k e^{-iH_0 t}
\]

We define the first order Hamiltonian

\[
H_{\text{rwa}}^{1\text{st}} = \overline{H_{\text{int}}} = \lim_{T \to \infty} \frac{1}{T} \int_0^T H_{\text{int}}(t) dt,
\]

and the second order Hamiltonian

\[
H_{\text{rwa}}^{2\text{nd}} = H_{\text{rwa}}^{1\text{st}} - i(H_{\text{int}} - \overline{H_{\text{int}}}) \left( \int_t (H_{\text{int}} - \overline{H_{\text{int}}}) \right)
\]

Choose the amplitudes \(u_{k,j}\) and the frequencies \(\omega_j\) such that the propagators of \(H_{\text{rwa}}^{1\text{st}}\) or \(H_{\text{rwa}}^{2\text{nd}}\) admit simple explicit forms that are used to find \(t \mapsto u(t)\) steering \(|\psi\rangle\) from one location to another one.
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RWA and resonant control

In \( i \frac{d}{dt} |\psi\rangle = \left( \frac{\omega_{eg}}{2} \sigma_z + \frac{u}{2} \sigma_x \right) |\psi\rangle \), take a resonant control
\( u = u e^{i\omega_{eg} t} + u^* e^{-i\omega_{eg} t} \) with \( u \) slowly varying complex amplitude
\( |\frac{d}{dt} u| \ll \omega_{eg} |u| \). Set \( H_0 = \frac{\omega_{eg}}{2} \sigma_z \) and \( \epsilon H_1 = \frac{u}{2} \sigma_x \) and consider
\( |\psi\rangle = e^{-i\omega_{eg} t \sigma_z} |\phi\rangle \) to eliminate the drift \( H_0 \) and to get the Hamiltonian in the interaction frame:

\[
i \frac{d}{dt} |\phi\rangle = \frac{u}{2} e^{i\omega_{eg} t} \sigma_z \sigma_x e^{-i\omega_{eg} t} \sigma_z |\phi\rangle = H_{\text{int}} |\phi\rangle
\]

with \( H_{\text{int}} = \frac{u}{2} e^{i\omega_{eg} t} \sigma_x + i \sigma_y + \frac{u}{2} e^{-i\omega_{eg} t} \sigma_x - i \sigma_y \)

The RWA consists in neglecting the oscillating terms at frequency \( 2\omega_{eg} \) when \( |u| \ll \Omega \):

\[
H_{\text{int}} = \left( \frac{ue^{2i\omega_{eg} t} + u^*}{2} \right) \sigma_+ + \left( \frac{u + u^* e^{-2i\omega_{eg} t}}{2} \right) \sigma_-. 
\]

Thus

\[
H_{\text{int}} = \frac{u^* \sigma_+ + u \sigma_-}{2}. 
\]
Second order approximation and Bloch-Siegert shift

The decomposition of $H_{\text{int}}$,

$$H_{\text{int}} = \underbrace{\frac{u^*}{2} \sigma_+ + \frac{u}{2} \sigma_-}_{H_{\text{int}}} + \underbrace{\frac{ue^{2i\omega gt}}{2} \sigma_+ + \frac{u^* e^{-2i\omega gt}}{2} \sigma_-}_{H_{\text{int}} - H_{\text{int}}}$$

provides the first order approximation (RWA)

$$H_{\text{rwa}}^{\text{1st}} = \overline{H_{\text{int}}} = \lim_{T \to \infty} \frac{1}{T} \int_0^T H_{\text{int}}(t) dt,$$

and also the second order approximation

$$H_{\text{rwa}}^{\text{2nd}} = H_{\text{rwa}}^{\text{1st}} - i(H_{\text{int}} - \overline{H_{\text{int}}}) \left( \int_t (H_{\text{int}} - \overline{H_{\text{int}}}) \right).$$

Since

$$\int_t H_{\text{int}} - \overline{H_{\text{int}}} = \frac{ue^{2i\omega gt}}{4i\omega g} \sigma_+ - \frac{u^* e^{-2i\omega gt}}{4i\omega g} \sigma_-,$$

we have

$$\left( H_{\text{int}} - \overline{H_{\text{int}}} \right) \left( \int_t (H_{\text{int}} - \overline{H_{\text{int}}}) \right) = -\frac{|u|^2}{8i\omega g} \sigma_z$$

(use $\sigma_+^2 = \sigma_-^2 = 0$ and $\sigma_z = \sigma_+ \sigma_- - \sigma_- \sigma_+$).

The second order approximation reads:

$$H_{\text{rwa}}^{\text{2nd}} = H_{\text{rwa}}^{\text{1st}} + \left( \frac{|u|^2}{8\omega g} \right) \sigma_z = \frac{u^*}{2} \sigma_+ + \frac{u}{2} \sigma_- + \left( \frac{|u|^2}{8\omega g} \right) \sigma_z.$$

The 2nd order correction $\frac{|u|^2}{4\omega g} (\sigma_z/2)$ is called the Bloch-Siegert shift.
Exercise: controllability of the 2-level systems and Rabi oscillation

Take the first order approximation

\[
(\Sigma) \quad i \frac{d}{dt} |\phi\rangle = \left( \frac{u^* \sigma_+ + u \sigma_-}{2} \right) |\phi\rangle = \left( \frac{u^* |e\rangle \langle g| + u |g\rangle \langle e|}{2} \right) |\phi\rangle
\]

with control \( u \in \mathbb{C} \).

1. Take constant control \( u(t) = \Omega_re^{i\theta} \) for \( t \in [0, T], \ T > 0 \). Show that \( i \frac{d}{dt} |\phi\rangle = \frac{\Omega_r(\cos \theta \sigma_x + \sin \theta \sigma_y)}{2} |\phi\rangle \).

2. Set \( \Theta_r = \frac{\Omega_r}{2} T \). Show that the solution at \( T \) of the propagator \( U_t \in SU(2), \ i \frac{d}{dt} U = \frac{\Omega_r(\cos \theta \sigma_x + \sin \theta \sigma_y)}{2} U, \ U_0 = I \) is given by

\[
U_T = \cos \Theta_r I - i \sin \Theta_r (\cos \theta \sigma_x + \sin \theta \sigma_y),
\]

3. Take a wave function \( |\bar{\phi}\rangle \). Show that exist \( \Omega_r \) and \( \theta \) such that \( U_T |g\rangle = e^{i\alpha} |\bar{\phi}\rangle \), where \( \alpha \) is some global phase.

4. Prove that for any given two wave functions \( |\phi_a\rangle \) and \( |\phi_b\rangle \) exists a piece-wise constant control \( [0, 2T] \ni t \mapsto u(t) \in \mathbb{C} \) such that the solution of \( (\Sigma) \) with \( |\phi\rangle_0 = |\phi_a\rangle \) satisfies \( |\phi\rangle_T = e^{i\beta} |\phi_b\rangle \) for some global phase \( \beta \).
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The Schrödinger system

\[ i \frac{d}{dt} |\psi\rangle = \left( \frac{\omega_{eg}}{2} \sigma_z + \omega_c \left( a^\dagger a + \frac{1}{2} \right) + i \frac{\Omega}{2} \sigma_x (a^\dagger - a) \right) |\psi\rangle \]

corresponds to two coupled scalar PDE's:

\[ i \frac{\partial \psi_e}{\partial t} = + \frac{\omega_{eg}}{2} \psi_e + \frac{\omega_c}{2} \left( x^2 - \frac{\partial^2}{\partial x^2} \right) \psi_e - i \frac{\Omega}{\sqrt{2}} \frac{\partial}{\partial x} \psi_g \]

\[ i \frac{\partial \psi_g}{\partial t} = - \frac{\omega_{eg}}{2} \psi_g + \frac{\omega_c}{2} \left( x^2 - \frac{\partial^2}{\partial x^2} \right) \psi_g - i \frac{\Omega}{\sqrt{2}} \frac{\partial}{\partial x} \psi_e \]

since \( a = \frac{1}{\sqrt{2}} \left( x + \frac{\partial}{\partial x} \right) \) and \( |\psi\rangle \) corresponds to \((\psi_e(x, t), \psi_g(x, t))\) where \( \psi_e(., t), \psi_g(., t) \in L^2(\mathbb{R}, \mathbb{C}) \) and \( \|\psi_e\|^2 + \|\psi_g\|^2 = 1 \).
Resonant case: passage to the interaction frame

\[
\frac{H_{JC}}{\hbar} = \frac{\omega_{eg}}{2} \sigma_z + \omega_c \left( a^\dagger a + \frac{1}{2} \right) + i \frac{\Omega}{2} \sigma_x (a^\dagger - a), \quad \omega_{eg} = \omega_c = \omega \text{ with } |\Omega| \ll \omega. \]

Then \( H_{JC} = H_0 + \epsilon H_1 \) where \( \epsilon \) is a small parameter and

\[
\frac{H_0}{\hbar} = \frac{\omega}{2} \sigma_z + \omega \left( a^\dagger a + \frac{1}{2} \right),
\]

\[
\frac{\epsilon H_1}{\hbar} = i \frac{\Omega}{2} \sigma_x (a^\dagger - a).
\]

\( H_{int} \) is obtained by setting \( |\psi\rangle = e^{-i\omega t (a^\dagger a + \frac{1}{2})} e^{-i\frac{\Omega t}{2} \sigma_z} |\phi\rangle \) in

\[
i\hbar \frac{d}{dt} |\psi\rangle = H_{JC} |\psi\rangle \]

to get \( i\hbar \frac{d}{dt} |\phi\rangle = H_{int} |\phi\rangle \) with

\[
\frac{H_{int}}{\hbar} = i \frac{\Omega}{2} \left( e^{-i\omega t} \sigma_- + e^{i\omega t} \sigma_+ \right) \left( e^{i\omega t} a^\dagger - e^{-i\omega t} a \right)
\]

where we used

\[
e^{i\frac{\theta}{2} \sigma_z} \sigma_x e^{-i\frac{\theta}{2} \sigma_z} = e^{-i\theta} \sigma_- + e^{i\theta} \sigma_+,
\]

\[
e^{i\theta (a^\dagger a + \frac{1}{2})} a e^{-i\theta (a^\dagger a + \frac{1}{2})} = e^{-i\theta} a
\]
The secular terms in $H_{\text{int}}$ are given by (RWA, first order approximation) $H_{\text{rwa}}^{1\text{st}}/\hbar = i\frac{\Omega}{2}(\sigma a^\dagger - \sigma^+ a)$. Since quantum state $|\phi\rangle = e^{+i\omega t(a^\dagger a + \frac{1}{2})} e^{+i\frac{\omega t}{2}\sigma_z} |\psi\rangle$ obeys approximately to $i\hbar \frac{d}{dt} |\phi\rangle = H_{\text{rwa}}^{1\text{st}} |\phi\rangle$, the original quantum state $|\psi\rangle$ is governed by

$$i\frac{d}{dt} |\psi\rangle = \left(\frac{\omega}{2}\sigma_z + \omega \left(a^\dagger a + \frac{1}{2}\right) + i\frac{\Omega}{2}(\sigma a^\dagger - \sigma^+ a)\right) |\psi\rangle$$

The Jaynes-Cumming Hamiltonian, in the resonant case ($\omega_{eg} = \omega_c = \omega$), reads:

$$H_{\text{JC}}^{\text{resonant}}/\hbar = \frac{\omega}{2}\sigma_z + \omega \left(a^\dagger a + \frac{1}{2}\right) + i\frac{\Omega}{2}(\sigma a^\dagger - \sigma^+ a)$$

The corresponding PDE is:

$$i\frac{\partial \psi_e}{\partial t} = +\frac{\omega}{2}\psi_e + \frac{\omega}{2}(x^2 - \frac{\partial^2}{\partial x^2})\psi_e - i\frac{\Omega}{2\sqrt{2}} \left(x - \frac{\partial}{\partial x}\right) \psi_g$$

$$i\frac{\partial \psi_g}{\partial t} = -\frac{\omega}{2}\psi_g + \frac{\omega}{2}(x^2 - \frac{\partial^2}{\partial x^2})\psi_g + i\frac{\Omega}{2\sqrt{2}} \left(x + \frac{\partial}{\partial x}\right) \psi_e$$
Dispersive case: passage to the interaction frame

\[
\frac{H_{JC}}{\hbar} = \frac{\omega_{eg}}{2} \sigma_z + \omega_c (a^\dagger a + \frac{1}{2}) + i \frac{\Omega}{2} \sigma_x (a^\dagger - a)
\]

with |\Omega| \ll |\omega_{eg} - \omega_c| \ll \omega_{eg}, \omega_c.

Then \( H_{JC} = H_0 + \epsilon H_1 \) where \( \epsilon \) is a small parameter and

\[
\frac{H_0}{\hbar} = \frac{\omega_{eg}}{2} \sigma_z + \omega_c (a^\dagger a + \frac{1}{2}), \quad \epsilon \frac{H_1}{\hbar} = i \frac{\Omega}{2} \sigma_x (a^\dagger - a).
\]

\( H_{\text{int}} \) is obtained by setting \( |\psi\rangle = e^{-i\omega_c t (a^\dagger a + \frac{1}{2})} e^{-i\omega_{eg} t} \sigma_z |\phi\rangle \) in

\[
i\hbar \frac{d}{dt} |\psi\rangle = H_{JC} |\psi\rangle \text{ to get } i\hbar \frac{d}{dt} |\phi\rangle = H_{\text{int}} |\phi\rangle \text{ with }
\]

\[
\frac{H_{\text{int}}}{\hbar} = i \frac{\Omega}{2} (e^{-i\omega_{eg} t} \sigma_- + e^{i\omega_{eg} t} \sigma_+) (e^{i\omega_c t} a^\dagger - e^{-i\omega_c t} a)
\]

\[
= i \frac{\Omega}{2} \left( e^{i(\omega_c - \omega_{eg}) t} \sigma_- a^\dagger - e^{-i(\omega_c - \omega_{eg}) t} \sigma_+ a + e^{i(\omega_c + \omega_{eg}) t} \sigma_+ a^\dagger - e^{-i(\omega_c + \omega_{eg}) t} \sigma_- a \right)
\]

Thus \( H_1^{\text{rst}} = H_{\text{int}} = 0 \): no secular term. We have to compute \( H_{\text{rwa}}^2 = \frac{H_{\text{int}}^2}{\hbar} \).

\[
H_{\text{rwa}}^2 = \frac{1}{\hbar} (H_{\text{int}} - H_{\text{int}}) \left( \int_t (H_{\text{int}} - H_{\text{int}}) \right) \text{ where } \int_t (H_{\text{int}} - H_{\text{int}})/\hbar
\]

corresponds to

\[
\frac{\Omega}{2} \left( \frac{e^{i(\omega_c - \omega_{eg}) t}}{\omega_c - \omega_{eg}} \sigma_- a^\dagger + \frac{e^{-i(\omega_c - \omega_{eg}) t}}{\omega_c - \omega_{eg}} \sigma_+ a + \frac{e^{i(\omega_c + \omega_{eg}) t}}{\omega_c + \omega_{eg}} \sigma_+ a^\dagger + \frac{e^{-i(\omega_c + \omega_{eg}) t}}{\omega_c + \omega_{eg}} \sigma_- a \right)
\]
The secular terms in $H_{\text{rwa}}^{2\text{nd}}$ are
\[
\frac{\Omega^2}{4(\omega_c-\omega_{eg})}(\sigma_z \sigma_+ a^\dagger a - \sigma_+ \sigma_a a^\dagger a) + \frac{\Omega^2}{4(\omega_c+\omega_{eg})}(-\sigma_+ \sigma a^\dagger a + \sigma_+ \sigma a a^\dagger a)
\]
Since $|\Omega| \ll |\omega_{eg} - \omega_c| \ll \omega_{eg}, \omega_c$, we have
\[
H_{\text{rwa}}^{2\text{nd}} / \hbar \approx -\frac{\Omega^2}{4(\omega_c-\omega_{eg})}(\sigma_z (N + \frac{1}{2}) + \frac{1}{2})
\]
Since quantum state $|\phi\rangle = e^{i\omega_c t(N + \frac{1}{2})} e^{i\omega_{eg} t} \sigma_z |\psi\rangle$ obeys approximatively to $i\hbar \frac{d}{dt} |\phi\rangle = H_{\text{rwa}}^{1\text{st}} |\phi\rangle$, the original quantum state $|\psi\rangle$ is governed by $i \frac{d}{dt} |\psi\rangle = \left(\frac{H_{\text{disp.}}^{\text{JC}}}{\hbar} - \frac{\Omega^2}{8(\omega_c-\omega_{eg})}\right) |\psi\rangle$ with
\[
H_{\text{disp.}}^{\text{JC}} / \hbar = \frac{\omega_{eg}}{2} \sigma_z + \omega_c (N + \frac{1}{2}) - \frac{\chi}{2} \sigma_z (N + \frac{1}{2}) \quad \text{and} \quad \chi = \frac{\Omega^2}{2(\omega_c-\omega_{eg})}
\]
The corresponding PDE is:
\[
i \frac{\partial \psi_e}{\partial t} = + \frac{\omega_{eg}}{2} \psi_e + \frac{1}{2}(\omega_c - \frac{\chi}{2})(\chi^2 - \frac{\partial^2}{\partial x^2}) \psi_e
\]
\[
i \frac{\partial \psi_g}{\partial t} = - \frac{\omega_{eg}}{2} \psi_g + \frac{1}{2}(\omega_c + \frac{\chi}{2})(\chi^2 - \frac{\partial^2}{\partial x^2}) \psi_g
\]
Consider the JC model in the resonant case ($\Omega \ll |\omega|$),

$$H_{JC} = \frac{\omega}{2} \sigma_z + \omega \left( a^\dagger a + \frac{1}{2} \right) + i \frac{\Omega}{2} \sigma_x (a^\dagger - a) + u (a + a^\dagger)$$

with a real control input $u(t) \in \mathbb{R}$:

1. Show that with the resonant control $u(t) = u e^{i \omega t} + u^* e^{-i \omega t}$ with complex amplitude $u$ such that $|u| \ll \omega$, the first order RWA approximation yields to the following dynamics in the interaction frame:

$$i \frac{d}{dt} |\psi\rangle = \left( i \frac{\Omega}{2} (\sigma a^\dagger - \sigma^+ a) + u a^\dagger + u^* a \right) |\psi\rangle$$

2. Set $v \in \mathbb{C}$ solution of $\frac{d}{dt} v = -i u$ and consider the following change of frame

$$|\phi\rangle = D_{-v} |\psi\rangle$$

with the displacement operator $D_{-v} = e^{-va^\dagger + v^* a}$. Show that, up to a global phase change, we have, with $\tilde{u} = i \frac{\Omega}{2} v$,

$$i \frac{d}{dt} |\phi\rangle = \left( i \frac{\Omega}{2} (\sigma a^\dagger - \sigma^+ a) + (\tilde{u}\sigma_+ + \tilde{u}^* \sigma_-) \right) |\phi\rangle$$

3. Take the orthonormal basis $\{|g, n\rangle, |e, n\rangle\}$ with $n \in \mathbb{N}$ being the photon number and where for instance $|g, n\rangle$ stands for the tensor product $|g\rangle \otimes |n\rangle$. Set

$$|\phi\rangle = \sum_n \phi_{g, n} |g, n\rangle + \phi_{e, n} |e, n\rangle$$

with $\phi_{g, n}, \phi_{e, n} \in \mathbb{C}$ depending on $t$ and

$$\sum_n |\phi_{g, n}|^2 + |\phi_{e, n}|^2 = 1.$$ Show that, for $n \geq 0$

$$i \frac{d}{dt} \phi_{g, n+1} = i \frac{\Omega}{2} \sqrt{n+1} \phi_{e, n} + \tilde{u}^* \phi_{e, n+1}, \quad i \frac{d}{dt} \phi_{e, n} = -i \frac{\Omega}{2} \sqrt{n+1} \phi_{g, n+1} + \tilde{u} \phi_{g, n}$$

and $i \frac{d}{dt} \phi_{g, 0} = \tilde{u}^* \phi_{e, 0}$.

4. Assume that $|\phi\rangle_0 = |g, 0\rangle$. Construct an open-loop control $[0, T] \ni t \mapsto \tilde{u}(t)$ such that $|\phi\rangle_T = |g, 1\rangle$.

5. Generalize the above open-loop control when the goal state $|\phi\rangle_T$ is $|g, n\rangle$ with any arbitrary photon number $n$. 