Quantum Systems: Dynamics and Control[1]

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Outline

1 Reservoir engineering
- A classical example
- The quantum case
- Analysis and Design: general guiding principles
- Other examples of quantum reservoir engineering

2 Protecting Quantum Information
- Encoding quantum information towards its protection
- encoding in redundant qubits
- encoding in a harmonic oscillator

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1. Reservoir engineering
   - A classical example
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2. Protecting Quantum Information
   - Encoding quantum information towards its protection
   - encoding in redundant qubits
   - encoding in a harmonic oscillator

The reservoir engineering idea (1)

- An open quantum system, without measurement nor real-time feedback control, follows a Lindblad differential equation (cont.time) or a Kraus map (discr.time), which usually stabilizes the system in some state: see decoherence. Reservoir engineering smartly adjusts some parameters such that this decoherence goes in a "beneficial direction".
- Given a Hilbert space $\mathcal{H}_C$ and any target state $\bar{\rho}$ on $\mathcal{H}_C$, it is not hard to build mathematically a Kraus map or Lindblad diff.eq. that asymptotically stabilizes $\bar{\rho}$.

  Exercise: make this construction for $\bar{\rho}$ pure state. You can use the following hint: if the set of Kraus operators $\{\boldsymbol{M}_\mu\}$ or of Lindblad operators $\{\boldsymbol{L}_\nu\}$ stabilizes $\tilde{\boldsymbol{\rho}}$ (e.g. $\tilde{\boldsymbol{\rho}}$ diagonal in some canonical basis), then the set $\{\boldsymbol{U}\boldsymbol{M}_\mu\boldsymbol{U}^\dagger\}$ or $\{\boldsymbol{U}\boldsymbol{L}_\nu\boldsymbol{U}^\dagger\}$ stabilizes $\boldsymbol{U}\rho\boldsymbol{U}^\dagger$.

  However, it would not be realistic to postulate: we can physically construct arbitrarily chosen Lindblad operators or Kraus maps on $\mathcal{H}_C$.
- Feedback control can be viewed as adjusting a Kraus or Lindblad superoperator thanks to feedback actions, e.g. memoryless feedback controller in discrete-time:

  (physical design): measurement backaction with $\{\boldsymbol{M}_{\mu,y}\}_\mu$ if detection result y
  (feedback control): apply unitary $\boldsymbol{U}_y$ if detection result y
  $\Rightarrow$ Kraus map $\mathbb{K}(\boldsymbol{\rho}) = \sum_{\mu,y}(\boldsymbol{U}_y\boldsymbol{M}_{\mu,y})\boldsymbol{\rho}(\boldsymbol{U}_y\boldsymbol{M}_{\mu,y})^\dagger$

  Limitation 1: this does not span all $\bar{\rho}$ for fixed $\{\boldsymbol{M}_{\mu,y}\}_\mu$
  Limitation 2: interaction of fragile quantum system with a feedback computer (external intervention) in real time

The reservoir engineering idea (2)

- Reservoir engineering takes a step back and re-considers the coupling between the target system and the meter. The goal is that the meter should not be measured anymore: it is just reset, and thereby becomes a dissipative auxiliary system $\mathcal{H}_A$. The reset operation is something simple, that we cannot tune much. The coupled system on $\mathcal{H}_C \otimes \mathcal{H}_A$ will follow a Kraus map or Lindblad diff.equation. The behavior of target system $\mathcal{H}_C$ in this coupled system is controlled by smartly engineering the coupling Hamiltonian between $\mathcal{H}_A$ and $\mathcal{H}_C$.

Key features:

- Stabilization based purely on physical coupling of a system of interest to an auxiliary system (the “reservoir”). No informatics intervention: no measurement, no feedback computation during system operation
- Reservoir must be dissipative ($\neq$ Hamiltonian) in order to allow stabilization (=allow different initial conditions of the system of interest to converge towards each other when $\mathcal{H}_C$ is coupled to the reservoir) and to be, on the long run, independent of reservoir initial condition.
- Control knob: ideally, tuning the Hamiltonians (turning on/off some drives, modifying their amplitude) allows to select different objectives on $\mathcal{H}_C$

Possible objectives:

- Drive the system to some particularly interesting state $\bar{\rho}$
- Make a system more robust to parameter uncertainties
- Make a system more robust to perturbing dynamics (Information Protection)

The reservoir engineering principle: Continuous-time

Given:

- A system of interest, Hilbert space $\mathcal{H}_C$, that we want to control. Isolated dynamics
$$\frac{d}{dt}\rho_C = 0 \quad \text{up to perturbations}$$
- A target behavior for the system on $\mathcal{H}_C$, e.g. robust exponential convergence towards $\bar{\rho}_{\text{target}}$
- A "simple" dissipative auxiliary system, Hilbert space $\mathcal{H}_A$, isolated dynamics
$$\frac{d}{dt}\rho_A = -\tfrac{i}{\hbar}[\boldsymbol{H}_A, \rho_A] + \sum_\nu \boldsymbol{L}_{\nu,A}\rho_A \boldsymbol{L}_{\nu,A}^\dagger - \tfrac{1}{2}(\boldsymbol{L}_{\nu_A}^\dagger \boldsymbol{L}_{\nu,A}\rho_A + \rho_A \boldsymbol{L}_{\nu,A}^\dagger \boldsymbol{L}_{\nu,A})$$
(e.g. a damped harmonic oscillator)
- Capability to implement a variety of Hamiltonians $\boldsymbol{H}_C,\ \boldsymbol{H}_A$ on the individual subsystems and $\boldsymbol{H}_{int}$ coupling them

Task:

- Design a particular tuning of $\boldsymbol{H}_C,\ \boldsymbol{H}_A,\ \boldsymbol{H}_{int}$ such that the target behavior on $\mathcal{H}_C$ is achieved by the coupled system on $\mathcal{H}_C \otimes \mathcal{H}_A$

The reservoir engineering principle: Discrete-time

Given:

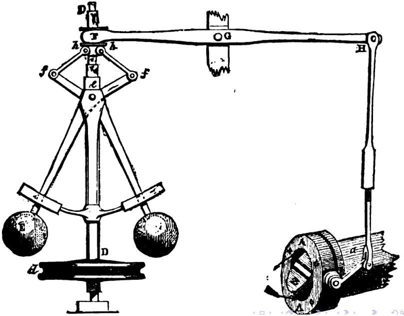
- A system of interest, Hilbert space $\mathcal{H}_C$, that we want to control. Isolated dynamics
$$\rho_C(t+1) = \rho_C(t) \quad \text{up to perturbations}$$
- A target behavior for the system on $\mathcal{H}_C$, e.g. robust exponential convergence towards $\bar{\rho}_{\text{target}}$
- A "simple" dissipative auxiliary system, Hilbert space $\mathcal{H}_A$, isolated dynamics
$$\rho_A(t+1) = \mathbb{K}_A(\rho_A(t))$$
e.g. a reset operation $\mathbb{K}_A(\rho_A(t)) = \bar{\rho}_A$ some simple fixed state (ground state).
- Capability to implement a variety of unitaries $\boldsymbol{U}_C$, $\boldsymbol{U}_A$ and $\boldsymbol{U}_{int}$

Task:

- Design $\boldsymbol{U}_C$, $\boldsymbol{U}_A$, $\boldsymbol{U}_{int}$ such that the dynamics
$$\rho(t+1) = (\mathbb{I} \otimes \mathbb{K}_A) \left(\boldsymbol{U}_{int} \, (\boldsymbol{U}_C \otimes \boldsymbol{U}_A) \rho(t) (\boldsymbol{U}_C \otimes \boldsymbol{U}_A)^\dagger \, \boldsymbol{U}_{int}^\dagger \right)$$
of the joint state ρ on $\mathcal{H}_C \otimes \mathcal{H}_A$ achieves target behavior on $\mathcal{H}_C$.
- Particular case $\mathbb{K}(\rho_A(t)) = \bar{\rho}_A$:
$$\rho_C(t+1) = Tr_A \left(\boldsymbol{U}_{int} \, (\boldsymbol{U}_C \rho_C(t) \boldsymbol{U}_C^\dagger \otimes \boldsymbol{U}_A \bar{\rho}_A \boldsymbol{U}_A^\dagger) \, \boldsymbol{U}_{int}^\dagger \right)$$

A classical analogue: the Watt regulator (1)

Flyball governor to stabilize the rotation speed of steam engines[5]
The flyballs are mounted on the (vertical) rotation axis of the machine, like two pendulums. The equilibrium angle of the pendulum depends on rotation speed: the faster the axis turns, the closer to horizontal the flyballs want to be. A mechanism closes the steam engine valve when flyballs get horizontal, slowing down the rotation. One expects that this should stabilize the rotation speed.



The linearization around equilibrium of rotation speed $\delta\omega$ and flyball angle $\delta\theta$ follow:

$$\frac{d}{dt}\delta\omega = -a\delta\theta - \Gamma_r \delta\omega$$
$$\frac{d^2}{dt^2}\delta\theta = -\Gamma_p \frac{d}{dt}\delta\theta - \Omega^2(\delta\theta - b\delta\omega)$$

positive parameters $\Gamma_r \ll 1$, Γ_p, a, b, Ω

[5]Implemented by James Watt, 1788. Analysis see J.C. Maxwell: On governors. Proc. of the Royal Society, No.100, 1868

A classical analogue: the Watt regulator (2)

Reservoir features:

- Isolated steam engine (target system), with perturbations:

$$\frac{d}{dt}\delta\omega = \delta F_{\text{steam}} - \delta F_{\text{load}} - \Gamma_r \delta\omega$$

  Steady state rotation speed depends on Γ_r (not robust).
- Flyball governor (auxiliary system): damped harmonic oscillator, damping necessary to ensure convergence
- Coupling the two (coupled equations of previous slide):
  Third-order linear system stable iff $\Gamma_p(\Omega^2 + \Gamma_p\Gamma_r + \Gamma_r^2) > ab\Omega^2$, essentially $\Gamma_p > ab$.
  Shifted steady state $\delta\omega = \frac{\delta F_{\text{steam}} - \delta F_{\text{load}}}{ab + \Gamma_r}$, essentially depends only on coupling parameters ab, dominating the poorly robust friction Γ_r.
- Small dependence on $\delta F_{\text{steam}} - \delta F_{\text{load}}$ is obtained for $ab \gg 1$ (coupling), which requires for stability $\Gamma_p \gg 1$ (dissipation), but nothing on Γ_r.
  In particular, we can in principle take $\Gamma_r = 0$ and consume no energy by friction except for stabilization, since in steady state $\Gamma_p \frac{d}{dt}\theta = 0$ (while $\omega \neq 0$).

Simple quantum example: two coupled harmonic oscillators

- Reservoir subsystem: damped driven harmonic oscillator, annihilation operator $\boldsymbol{a}$, in rotating frame, real drive amplitude u:

$$\frac{d}{dt}\rho = -iu[\boldsymbol{a} + \boldsymbol{a}^\dagger, \rho] + \kappa_a(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\rho - \tfrac{1}{2}\rho\boldsymbol{a}^\dagger\boldsymbol{a})$$

- Target subsystem: harmonic oscillator, annihilation operator $\boldsymbol{c}$, resonantly coupled to reservoir:

$$\begin{aligned}\frac{d}{dt}\rho &= -ig[\boldsymbol{c}^\dagger\boldsymbol{a} + \boldsymbol{c}\boldsymbol{a}^\dagger, \rho] - iu[\boldsymbol{a} + \boldsymbol{a}^\dagger, \rho] + \kappa_a(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\rho - \tfrac{1}{2}\rho\boldsymbol{a}^\dagger\boldsymbol{a}) \\ &= -ig[(\boldsymbol{c}^\dagger + \tfrac{u}{g})\boldsymbol{a} + (\boldsymbol{c} + \tfrac{u}{g})\boldsymbol{a}^\dagger, \rho] + \kappa_a(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\rho - \tfrac{1}{2}\rho\boldsymbol{a}^\dagger\boldsymbol{a})\end{aligned}$$

- The joint system features a steady state $\rho = |\alpha\rangle\langle\alpha|_C \otimes |0\rangle\langle 0|_A$ where the target subsystem is in the coherent state $|\alpha\rangle$ with $\alpha = -\frac{u}{g}$ and the reservoir subsystem is in vacuum.
- The steady state amplitude $\alpha = -\frac{u}{g}$ can be adjusted by Hamiltonian u and does not depend on the loss parameter κ_a. In steady state, the Hamiltonian coupling amounts to 0 and no energy is dissipated through κ_a. In contrast, the steady state of a single damped driven harmonic oscillator (take the reservoir alone) depends on κ_a and relies on a dynamical equilibrium between drive and dissipation.

Simple quantum example: convergence

Consider Lyapunov function

$$V(\boldsymbol{\rho}) = \mathrm{Tr}\left(\boldsymbol{\rho}\left((\boldsymbol{c}^\dagger - \alpha)(\boldsymbol{c} - \alpha) + \boldsymbol{a}^\dagger \boldsymbol{a}\right)\right) .$$

We have

$$\frac{d}{dt} V = -\kappa_a \,\mathrm{Tr}\left(\boldsymbol{\rho}\, \boldsymbol{a}^\dagger \boldsymbol{a}\right) \leq 0$$

The set where $\frac{d}{dt} V(\boldsymbol{\rho}) = 0$ is $\mathcal{M} = \{\boldsymbol{\rho}_C \otimes |0\rangle\langle 0|_A\}$. On this set we have

$$\frac{d}{dt}\boldsymbol{\rho} = -ig(\boldsymbol{c} - \alpha)\boldsymbol{\rho}_C \otimes |1\rangle\langle 0|_A + ig\boldsymbol{\rho}_C(\boldsymbol{c}^\dagger - \alpha) \otimes |0\rangle\langle 1|_A$$

which leaves the set $\mathcal{M}$ unless $(\boldsymbol{c} - \alpha)\boldsymbol{\rho}_C = \boldsymbol{\rho}_C(\boldsymbol{c}^\dagger - \alpha) = 0$.

By the LaSalle Invariance Principle[6], $\boldsymbol{\rho}$ converges towards the identified steady state $|\alpha\rangle\langle\alpha|_C \otimes |0\rangle\langle 0|_A$.

[6]We here skip the possible difficulties related to infinite-dimensional spaces; in Fock state coordinates $|n\rangle$, all states involved decay exponentially for large n.

Turning on/off the quantum reservoir built from RWA

The model presented above results from a RWA with resonant drives and two resonant harmonic oscillators. Taking different frequencies on $\mathcal{H}_A$ and $\mathcal{H}_C$, allows to turn on or off the reservoir coupling with more drives.

- Model before rotating frame[7]:

$$\boldsymbol{H} = \omega_a \boldsymbol{a}^\dagger \boldsymbol{a} + \omega_c \boldsymbol{c}^\dagger \boldsymbol{c} + \chi u_p(t)(\boldsymbol{a}^\dagger + \boldsymbol{a})(\boldsymbol{c}^\dagger + \boldsymbol{c}) + u_a(t)(\boldsymbol{a}^\dagger + \boldsymbol{a})$$

with $u_p(t) = v(e^{i(\omega_a - \omega_c)t} + e^{-i(\omega_a - \omega_c)t})$ and $u_a(t) = u(e^{i\omega_a t} + e^{-i\omega_a t})$

with dissipation $\kappa_a(\boldsymbol{a}\boldsymbol{\rho}\boldsymbol{a}^\dagger - \frac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\rho} - \frac{1}{2}\boldsymbol{\rho}\boldsymbol{a}^\dagger\boldsymbol{a})$.

- Going to rotating frame $\boldsymbol{\rho} = U(t)\xi U(t)^\dagger$ with $U(t) = e^{-i(\omega_a \boldsymbol{a}^\dagger \boldsymbol{a} + \omega_c \boldsymbol{c}^\dagger \boldsymbol{c})t}$ and keeping only constant terms (first-order RWA), the dissipation does not change while the Hamiltonian becomes:

$$\bar{\boldsymbol{H}} = u(\boldsymbol{a} + \boldsymbol{a}^\dagger) + \chi v(\boldsymbol{c}^\dagger \boldsymbol{a} + \boldsymbol{c}\boldsymbol{a}^\dagger) \ .$$

Thus the coupling strength $g = \chi\, v$ is mediated by the amplitude of drive $u_p(t)$.

In particular, for $u = 0$, we can turn on ($v \neq 0$) or off ($v = 0$) a process that resets the harmonic oscillator C towards its vacuum ground state $\rho_C = |0\rangle\langle 0|$.
(In practice, there are more efficient reservoir constructions for fast reset to $|0\rangle\langle 0|$.)

[7] The way to physically implement a drive on the coupling amplitude is out of scope here, but it is possible; this is called "parametric driving".

Effect of perturbations on the quantum reservoir setting

Stabilization is meant to protect against perturbations. We can illustrate how the reservoir protects the target state $\boldsymbol{\rho}_C = |\alpha\rangle\langle\alpha|$ from spurious photon annihilation. Denote $\mathbb{D}_{\boldsymbol{L}}(\boldsymbol{\rho}) = \boldsymbol{L}\boldsymbol{\rho}\boldsymbol{L}^\dagger - \frac{1}{2}\boldsymbol{L}^\dagger\boldsymbol{L}\boldsymbol{\rho} - \frac{1}{2}\boldsymbol{\rho}\boldsymbol{L}^\dagger\boldsymbol{L}$.

- We can solve for parameters $\alpha_1, \alpha_2 \in \mathbb{C}$ such that

  $$\begin{aligned}\frac{d}{dt}\boldsymbol{\rho} &= -ig[(\boldsymbol{c}^\dagger - \alpha)\boldsymbol{a} + (\boldsymbol{c} - \alpha)\boldsymbol{a}^\dagger, \boldsymbol{\rho}] + \kappa_a\mathbb{D}_{\boldsymbol{a}}(\boldsymbol{\rho}) + \kappa_c\mathbb{D}_{\boldsymbol{c}}(\boldsymbol{\rho}) \\ &= -ig[(\boldsymbol{c} - \alpha_1)(\boldsymbol{a} - \alpha_2)^\dagger + (\boldsymbol{c} - \alpha_1)^\dagger(\boldsymbol{a} - \alpha_2),\ \boldsymbol{\rho}] + \kappa_a\mathbb{D}_{\boldsymbol{a}-\alpha_2}(\boldsymbol{\rho}) + \kappa_c\mathbb{D}_{\boldsymbol{c}-\alpha_1}(\boldsymbol{\rho})\ .\end{aligned}$$

  This yields $\alpha_1 = \dfrac{\alpha}{1 + \frac{\kappa_a\kappa_c}{g^2}}$ and $\alpha_2 = \dfrac{\frac{i\kappa_c}{g}\alpha}{1 + \frac{\kappa_a\kappa_c}{g^2}}$.
  We thus have steady state $\boldsymbol{\rho} = |\psi\rangle\langle\psi|$ with $|\psi\rangle = |\alpha_1\rangle_C \otimes |\alpha_2\rangle_A$ product of coherent states.
- $|\alpha_1\rangle_C$ is nearly insensitive to any $\kappa_c \ll \frac{g^2}{\kappa_a}$. We can say that thanks to the reservoir, the stabilized state is protected against photon loss with a strength $\frac{g^2}{\kappa_a}$. This may appear counterintuitive e.g. for $\kappa_a \to 0$. Do not forget that it is only a very partial viewpoint: value of the steady state; and for a particular perturbation.

For other perturbations, such simple exact analysis will in general not be possible. Also for stabilizing other target states, exact solutions may be too hard to implement. We will have to rely on approximate analysis, based on timescale separations like $\kappa_a \ll g \ll \kappa_c$.

General guiding principles

Given:

- target system $\mathcal{H}_C$, target state $|\bar{\psi}\rangle \in \mathcal{H}_C$ to stabilize
  (this can be generalized to stabilizing subspaces, with more properties on reservoir behavior)
- a dissipative harmonic oscillator $\mathcal{H}_A$, annihilation operator $\boldsymbol{a}$
  (this is just by far the most common example in current hardware)

Design:

1. Construct an operator $\boldsymbol{R}$ on $\mathcal{H}_C$, acting nontrivially (e.g. not just $\boldsymbol{R} \equiv 0$) and such that $\boldsymbol{R}|\bar{\psi}\rangle = 0$. This is just a mathematical construction. See Exercise on slide 4, use degrees of freedom to facilitate the next design steps.
2. Construct a coupling Hamiltonian between target and auxiliary systems:
   $$\boldsymbol{H}_{int} = \boldsymbol{R}^\dagger \boldsymbol{a} + \boldsymbol{R}\boldsymbol{a}^\dagger \quad \text{on } \mathcal{H}_C \otimes \mathcal{H}_A \, .$$
   Building this with available components may require tricks, for instance RWA.
3. [8] Steps 1 and 2 ensure that $|\bar{\psi}\rangle_C \otimes |0\rangle_A$ is an invariant state of the joint system; and, dynamics elsewhere should be nontrivial.
   Now, if necessary, “add elements” to make sure that $|\bar{\psi}\rangle_C \otimes |0\rangle_A$ is globally attractive. This can involve Hamiltonians on $\mathcal{H}_A \setminus \{\lambda|0\rangle : \lambda \in \mathbb{R}\}$ and on $\mathcal{H}_C \setminus \{\lambda|\bar{\psi}\rangle : \lambda \in \mathbb{R}\}$, or also some dissipation operators.

The level of protection against perturbations (see $\frac{g^2}{\kappa_a}$ on previous slide) will have to be approached with approximate methods, see further.

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[8] This step follows a series of papers by F.Ticozzi and L.Viola.

Timescale separation analysis: adiabatic elimination

This analysis is useful for (i) examining convergence rates, protection against perturbations; or (ii) designing reservoirs which implement the guiding principles of previous slide *approximately*.

- Timescale separation: usually the single-system processes have much faster timescales than the ones involving interactions among components. Consider on $\mathcal{H}_C \otimes \mathcal{H}_A$ the reservoir design

  $$\frac{d}{dt}\rho = -ig[\boldsymbol{H}_{int},\ \rho] + \kappa \mathbb{I} \otimes \mathcal{L}_A(\rho)$$

  where $\mathcal{L}_A$ represents the fast dissipation process and typically $g \ll \kappa$ (assuming the super-operator norms of $[\boldsymbol{H}_{int}, .]$ and $\mathcal{L}_A$ are of order 1).

- This takes the form of a linear system with linear perturbation:

  $\dot{x} = (A_0 + \epsilon A_1)\, x$ where $\epsilon = \frac{g}{\kappa} \ll 1$ and A_0 has a degenerate 0-eigenspace (namely $\otimes \mathcal{H}_C$). We are interested in how this 0-eigenspace is modified for $\epsilon \neq 0$.

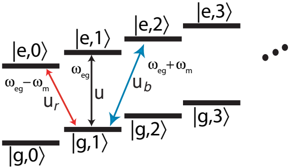
- Linear perturbation theory states that:

  - the 0-dynamics of A_0 on $\mathcal{H}_C$ is perturbed towards dynamics analytic in ϵ (thus this dynamics is the reservoir convergence effect on $\mathcal{H}_C$)
  - the 0-eigenspace (essentially $\mathcal{H}_C$) is displaced analytically in ϵ (thus the true steady state can become slightly entangled on $\mathcal{H}_C \otimes \mathcal{H}_A$)

- Both effects can be computed efficiently with a series expansion. Quantum specific: the dynamics on $\mathcal{H}_C$ can be expressed with a Lindblad diff.eq., and its entanglement by a Kraus map, at least up to $O(\epsilon^2)$ included.[9]

[9] see Rémi Azouit PhD thesis.

The Cirac-Zoller reservoir for trapped ion[10] (1)



Cfr. Lectures 5 & 9. After RWA:

$$\boldsymbol{H}_{int} = u|g\rangle\langle e| + u^*|e\rangle\langle g| + \bar{u}_b|g\rangle\langle e|\boldsymbol{a} + \bar{u}_b^*|e\rangle\langle g|\boldsymbol{a}^\dagger + \bar{u}_r|g\rangle\langle e|\boldsymbol{a}^\dagger + \bar{u}_r^*|e\rangle\langle g|\boldsymbol{a}$$

- Target system $\mathcal{H}_c$: harmonic oscillator, annihilation operator $\boldsymbol{a}$
- Auxiliary reservoir $\mathcal{H}_q$: qubit degree of freedom $|g\rangle, |e\rangle$ with dissipation:

$$\frac{d}{dt}\boldsymbol{\rho}_q = \kappa\mathbb{D}_{\sigma_-}(\boldsymbol{\rho}_q) := \kappa\left(\sigma_-\boldsymbol{\rho}_q\sigma_+ - \frac{1}{2}\boldsymbol{\rho}_q\sigma_+\sigma_- - \frac{1}{2}\sigma_+\sigma_-\boldsymbol{\rho}_q\right) .$$

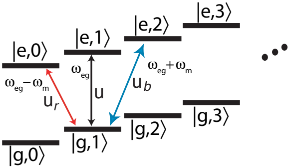
- Various target states can be stabilized by tuning the lasers u, $\bar{u}_b$, $\bar{u}_r$ in $\boldsymbol{H}_{int}$:
  - Take $\bar{u}_b = 0$, constant $\bar{u}_r \neq 0$ and $\bar{u}$ to get, for $\alpha = -\bar{u}^*/\bar{u}_r^*$:

    $$\frac{d}{dt}\boldsymbol{\rho} = -i[\,\bar{u}_r|g\rangle\langle e|(\boldsymbol{a}-\alpha)^\dagger + \bar{u}_r^*|e\rangle\langle g|(\boldsymbol{a}-\alpha)\,,\ \boldsymbol{\rho}] + \kappa\mathbb{D}_{\sigma_-}(\boldsymbol{\rho}_q)\ .$$

    Clearly $|\bar{\psi}\rangle = |g\rangle_q \otimes |\alpha\rangle_C$ is invariant. It is also attractive (adapt Lect.9). This stabilizes a coherent target state.

[10] cited as first reservoir. Poyatos, Cirac & Zoller (1996), Phys.Rev.Lett. 77.23.

The Cirac-Zoller reservoir for trapped ion[11] (2)



Cfr. Lectures 5 & 9. After RWA:

$$\boldsymbol{H}_{int} = u|g\rangle\langle e| + u^*|e\rangle\langle g| + \bar{u}_b|g\rangle\langle e|\boldsymbol{a} + \bar{u}_b^*|e\rangle\langle g|\boldsymbol{a}^\dagger + \bar{u}_r|g\rangle\langle e|\boldsymbol{a}^\dagger + \bar{u}_r^*|e\rangle\langle g|\boldsymbol{a}$$

- Various target states can be stabilized by tuning the lasers u, $\bar{u}_b$, $\bar{u}_r$ in $\boldsymbol{H}_{int}$:
  - Take ($\bar{u} = 0$ to simplify and) constant $\bar{u}_b \neq 0$, $\bar{u}_r \neq 0$ to get

    $$\frac{d}{dt}\boldsymbol{\rho} = -i[\,|g\rangle\langle e|(\bar{u}_r^*\boldsymbol{a} + \bar{u}_b^*\boldsymbol{a}^\dagger)^\dagger + |e\rangle\langle g|(\bar{u}_r^*\boldsymbol{a} + \bar{u}_b^*\boldsymbol{a}^\dagger)\,,\ \boldsymbol{\rho}] + \kappa\mathbb{D}_{\sigma_-}(\boldsymbol{\rho}_q)$$
    $$= -i[\,|g\rangle\langle e|v^*\boldsymbol{s}^\dagger + |e\rangle\langle g|v\boldsymbol{s}\,,\ \boldsymbol{\rho}] + \kappa\mathbb{D}_{\sigma_-}(\boldsymbol{\rho}_q) \text{ where}$$

    $\boldsymbol{s} = \boldsymbol{S}^\dagger(r\,e^{i\phi_r})\,\boldsymbol{a}\,\boldsymbol{S}(r\,e^{i\phi_r})$ with $\bar{u}_r = \cosh(r)$, $\bar{u}_b = -e^{i\phi_r}\sinh(r)$ and unitary op. $\boldsymbol{S}(\zeta) = exp(\frac{1}{2}(\zeta^*\boldsymbol{a}^2 - \zeta(\boldsymbol{a}^\dagger)^2))$ the *squeezing* coord.transf.

    Clearly (coord.transf.) this stabilizes, the squeezed vacuum state

    $$|\bar{\psi}\rangle = |g\rangle_q \otimes (\boldsymbol{S}(r\,e^{i\phi_r})|0\rangle)_C$$

    for which $\Delta\boldsymbol{X}_{\frac{\phi_r}{2}} = \frac{1}{2}e^{-r} < \frac{1}{2}$ and $\Delta\boldsymbol{X}_{\frac{\phi_r}{2}+\frac{\pi}{2}} = \frac{1}{2}e^{r} > \frac{1}{2}$, satisfying the Heisenberg unc.pr. $\Delta\boldsymbol{X}_\lambda\,\Delta\boldsymbol{X}_{\lambda+\frac{\pi}{2}} \geq \frac{1}{4}$ $\forall\lambda$ but with very low uncertainty on coordinate $\Delta\boldsymbol{X}_{\frac{\phi_r}{2}}$.

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[11] cited as first reservoir. Poyatos, Cirac & Zoller (1996), Phys.Rev.Lett. 77.23.

Fast reset of a qubit[12]

- Target system: qubit, $\mathcal{H}_c = \text{span}\{|g\rangle, |e\rangle\}$, to be stabilized in $|g\rangle$
- Auxiliary reservoir $\mathcal{H}_a$: harmonic oscillator with dissipation
  $\frac{d}{dt}\boldsymbol{\rho} = \kappa(\boldsymbol{a}\boldsymbol{\rho}\boldsymbol{a}^\dagger - \boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\rho}/2 - \boldsymbol{\rho}\boldsymbol{a}^\dagger\boldsymbol{a}/2)$
- Dispersive interaction Hamiltonian, two drives $u_q(t) = \bar{u}_q\, e^{-i(\omega_q - \bar{n}\chi)t}$ and $u_c(t) = \bar{u}_c\, e^{-i(\omega_c - \chi/2)t}$ for $\bar{n} \in \mathbb{N}$,

$$\boldsymbol{H}_{int} = \omega_c \boldsymbol{a}^\dagger \boldsymbol{a} + \frac{\omega_q}{2}\boldsymbol{\sigma_z} - \frac{\chi}{2}\boldsymbol{\sigma_z}\boldsymbol{a}^\dagger\boldsymbol{a} + (u_q(t)\boldsymbol{\sigma_+} u_q^*(t)\boldsymbol{\sigma_-}) + (u_c(t)\boldsymbol{a}^\dagger + u_c^*(t)\boldsymbol{a})$$

- After RWA in rotating frame ($\mathbb{D}_{\boldsymbol{L}}(\boldsymbol{\rho}) = \boldsymbol{L}\boldsymbol{\rho}\boldsymbol{L}^\dagger - \frac{1}{2}\boldsymbol{L}^\dagger\boldsymbol{L}\boldsymbol{\rho} - \frac{1}{2}\boldsymbol{\rho}\boldsymbol{L}^\dagger\boldsymbol{L}$):

$$\frac{d}{dt}\boldsymbol{\rho} = -i\bar{u}_q[\boldsymbol{\sigma_x} \otimes |\bar{n}\rangle\langle\bar{n}|,\, \boldsymbol{\rho}] - i\bar{u}_c[|e\rangle\langle e| \otimes (\boldsymbol{a} + \boldsymbol{a}^\dagger),\, \boldsymbol{\rho}] + \kappa\mathbb{D}_{|g\rangle\langle g|\boldsymbol{a}}(\boldsymbol{\rho}) + \kappa\mathbb{D}_{|e\rangle\langle e|\boldsymbol{a}}(\boldsymbol{\rho})$$

  For qubit in $|e\rangle$: u_c drives harm.osc. to overlap with $|\bar{n}\rangle$. For harm.osc. on $|\bar{n}\rangle$: u_q makes qubit oscillate between $|e\rangle$ and $|g\rangle$. For qubit on $|g\rangle$: just dissipation, harm.osc. moves away from $|\bar{n}\rangle$ so qubit stays in $|g\rangle$

- Clearly, $|\bar{\psi}\rangle = |g\rangle_c|0\rangle_A$ is steady state. It is also attractive, so with $u_c(t)$, $u_q(t)$ we can turn on or off a process that resets the qubit to $|g\rangle$.

For details on RWA and on the convergence towards $|\bar{\psi}\rangle$, see Exam question 2015.

[12] K.Geerlings, Z.Leghtas et al (2013), Phys.Rev.Lett. 110.12.

Exercise: discrete-time reservoir engineering

- Target system $\mathcal{H}_c$: harmonic oscillator, annihilation operator $\boldsymbol{a}$
- Auxiliary reservoir $\mathcal{H}_q$: qubit degree of freedom $|g\rangle, |e\rangle$ with dissipation:
  $\boldsymbol{\rho}_q(t+1) = \mathbb{K}(\boldsymbol{\rho}_q(t)) = \bar{\boldsymbol{\rho}}_q$, where $\bar{\boldsymbol{\rho}}_q$ is fixed according to needs.
- Target behavior (see last item of slide 7):
  $\boldsymbol{\rho}_C(t+1) = \mathbb{K}_C(\boldsymbol{\rho}_C(t))$ such that $\boldsymbol{\rho}_C$ converges towards a Fock state $|\bar{n}\rangle\langle\bar{n}|$.

- Resonant Jaynes-Cummings Interaction (see Lectures 2 and 4).
- Exercise in detail: see the second exercise of Exam 2018

Outline

1 Reservoir engineering
- A classical example
- The quantum case
- Analysis and Design: general guiding principles
- Other examples of quantum reservoir engineering

2 Protecting Quantum Information
- Encoding quantum information towards its protection
- encoding in redundant qubits
- encoding in a harmonic oscillator

Digital quantum information processing[13]

- Classical systems typically move continuously in some vector space or manifold. Nevertheless, an efficient way to make computations, communicate,... i.e. process information is by discretizing it into a sequence of elementary chunks: bits can only take values in $\{0,1\}$. A digital classical machine is an abstract object acting on the state space $\{0,1\}^d$, involving 2^d possible so-called logical states.
- Similarly, quantum states can be efficiently represented by combining qubits. A qubit is a two-level quantum system, $\mathcal{H}_q \equiv \mathbb{C}^2$. The logical state space of a digital quantum machine on d qubits is

$$\mathcal{H}_q \otimes \mathcal{H}_q \otimes ...(d \text{ times}) \;\; =: \;\; \mathcal{H}_q^{\otimes d} \;\; \equiv \;\; \mathbb{C}^{2^d}$$

- Classical processing takes its efficiency from the existence of a finite set of universal gates, i.e. operations on 1, 2 or 3 logical bits from which all maps on the logical state space can be built.
- Quantum processing too can be built from such a finite set of universal gates:
  a few unitary operations on 1,2,3 qubits;
  preparation of 1 qubit in a fixed initial state;
  measurement of 1 qubit in a fixed basis.

  (This is not as trivial as for classical processing, since the set of possible quantum operations (unitary operations on $\mathcal{H}_q^{\otimes d}$) forms a continuum, but we will not further discuss logical operations here.)

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[13] Michael A. Nielsen and Isaac L. Chuang (2000). Quantum Computation and Quantum Information. Cambridge University Press. Chapter 10 covers Error Correction

Protecting logical information: general

- As a side-effect of this digital viewpoint, we get a simple roadmap for making logical states robust to perturbations. Indeed, the logical state space will in general be physically implemented by particular states ("the code space") inside a larger physical state space.
  This allows to engineer systems in which physical perturbations are counteracted, inside the larger physical state space, before they have any effect on the logical state space.
- Classical example[14]: in physical space $x \in \mathbb{R}$, select as the code space of one logical bit the positions $x = -1$ (logical 0) and $x = +1$ (logical 0).

  Setting up a potential $V(x) = \cos(\pi x)$, the code space lies at the minima of $V(x)$. Stabilizing these minima against perturbations allows to avoid that $x(0) \simeq -1$ ever drifts to $x(t) \simeq +1$ or vice versa.

  The strength of this protection can be increased by
  - implementing a stronger potential e.g. $10\, V(x)$;
  - selecting another code space, e.g. $x = -7$ (logical 0), $x = +7$ (logical 1);
  - taking several copies of this system, coupled to favor $x_1 = x_2 = \ldots$
- Note the difference between 'information protection' and 'stabilization of a state': the protection here must be built without knowing which state will be used inside the code space
  $\Rightarrow$ We must stabilize a subspace instead of a state, and in fact robustify the identity action on this subspace.

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[14] Stabilization of classical bits resembles this picture, with electromagnetic degrees of freedom

Protecting logical information: quantum model

For the quantum case:

- The logical qubit is implemented by selecting two orthonormal vectors $|0_L\rangle, |1_L\rangle$ inside the physical Hilbert space $\mathcal{H}$.
- Invariance: in absence of perturbations, the code subspace $\mathcal{H}_c = \text{span}(|0_L\rangle, |1_L\rangle)$ should undergo no dynamics (in an appropriate rotating frame).
- Error correction: this is most often formulated in discrete-time. Consider

$$\boldsymbol{\rho}_+ = \mathbb{K}_E(\boldsymbol{\rho}) \quad \text{(Kraus map)}$$

  modeling, for each E, a possible relevant physical perturbation on $\mathcal{H}$ over time dt. Ideally, there should exist an error recovery map

$$\boldsymbol{\rho}_+ = \mathbb{K}_R(\boldsymbol{\rho}) \quad \text{(Kraus map) such that}$$

$$\mathbb{K}_R(\mathbb{K}_E(\boldsymbol{\rho})) = \boldsymbol{\rho} \quad \text{for all } E \text{ and for all } \boldsymbol{\rho} \text{ with support in } \mathcal{H}_c \text{ (code space).}$$

  The existence of such perfect $\mathbb{K}_R$ is not realistic. Most often the recovery will be approximate: $\mathbb{K}_R(\mathbb{K}_E(\boldsymbol{\rho})) \simeq \boldsymbol{\rho}$ for $\boldsymbol{\rho}$ in the code space; the goal is to reduce the probability of logical errors as much as possible, for a realistic set of $\mathbb{K}_E$.
- $\mathbb{K}_R$ can be implemented through some physical process (reservoir engineering), or via measurement and feedback action. We will see one example of each.

(A last point to consider is how to perform controlled logical operations (gates) when the error protection is precisely designed to counteract any (spurious) logical operations. This has been actively treated, yet it goes beyond the scope of this course.)

The 9-qubit Bacon-Shor code[15]

This encoding starts from physical qubits. The main idea is to copy the information of one logical qubit among several physical qubits, and correct errors occurring on a single qubit thanks to majority vote.

Classical equivalent: if $|0_L\rangle = |0\rangle|0\rangle|0\rangle|0\rangle|0\rangle\ , \quad |1_L\rangle = |1\rangle|1\rangle|1\rangle|1\rangle|1\rangle$ and we see $|\psi\rangle = |0\rangle|1\rangle|0\rangle|0\rangle|0\rangle \in \mathcal{H}$, then we correct it back towards $|0_L\rangle$.

Quantum information comprises $|0_L\rangle, |1_L\rangle$ but also all their linear combinations. To protect them too with majority vote, a slightly larger code is needed.

- Physical state space: 9 qubits, $\mathcal{H} = (\mathbb{C}^2)^{\otimes 9}$.
- Errors to counter: any perturbation of a single qubit, i.e. ($\mathbb{I}$ identity superoperator)

$$\begin{aligned}\{\mathbb{K}_E\} =& \{\mathbb{K} \otimes \mathbb{I}_{2^8} \text{ for any Kraus map } \mathbb{K} \text{ on the first qubit}\} \cup \\ & \{\mathbb{I}_2 \otimes \mathbb{K} \otimes \mathbb{I}_{2^7} \text{ for any Kraus map } \mathbb{K} \text{ on the 2nd qubit}\} \cup \ldots\end{aligned}$$

- Code space:

$$|0_L\rangle = \frac{|\tilde{+}\rangle|\tilde{+}\rangle|\tilde{+}\rangle + |\tilde{-}\rangle|\tilde{-}\rangle|\tilde{-}\rangle}{\sqrt{2}} \quad \text{and} \quad |1_L\rangle = \frac{|\tilde{+}\rangle|\tilde{+}\rangle|\tilde{+}\rangle - |\tilde{-}\rangle|\tilde{-}\rangle|\tilde{-}\rangle}{\sqrt{2}} \quad \text{(second stage)}$$

$$\text{where} \quad |\tilde{+}\rangle = \frac{|0\rangle|0\rangle|0\rangle + |1\rangle|1\rangle|1\rangle}{\sqrt{2}} \quad \text{and} \quad |\tilde{-}\rangle = \frac{|0\rangle|0\rangle|0\rangle - |1\rangle|1\rangle|1\rangle}{\sqrt{2}} \quad \text{(first stage)}$$

on each group of three qubits (1,2,3), (4,5,6) and (7,8,9).

[15] Peter W.Shor, (1995). "Scheme for reducing decoherence in quantum computer memory". Physical Review A, 52(4).

The Kraus map providing protection against single-qubit errors (1)

We build $\mathbb{K}_R$ progressively, and illustrating how it can work with unitary feedback conditioned on measurement of some commuting observables ("error syndromes").

First consider a bit-flip perturbation, e.g. $\mathbb{K}_E(\rho) = (\sigma_x \otimes \boldsymbol{I}_{2^8})\rho(\sigma_x \otimes \boldsymbol{I}_{2^8})$ on first qubit. This can be corrected by majority vote on the first stage: on the qubit group (1,2,3)

- Projectively measure the observables $S_1 = \sigma_z \otimes \sigma_z \otimes \boldsymbol{I}_2$ and $S_2 = \boldsymbol{I}_2 \otimes \sigma_z \otimes \sigma_z$.

  This gives $\mathbb{K}_{(s_1,s_2)}(\rho) = \frac{\boldsymbol{P}_{s_1,s_2}\rho\boldsymbol{P}^\dagger_{s_1,s_2}}{\mathrm{Tr}\left(\boldsymbol{P}_{s_1,s_2}\rho\boldsymbol{P}^\dagger_{s_1,s_2}\right)}$ with proba $\mathrm{Tr}\left(\boldsymbol{P}_{s_1,s_2}\rho\boldsymbol{P}^\dagger_{s_1,s_2}\right)$ where

$$\begin{aligned}
\boldsymbol{P}_{s_1,s_2} = & |000\rangle\langle 000| + |111\rangle\langle 111| \quad \text{if } s_1 = s_2 = 1 \;; \\
& |100\rangle\langle 100| + |011\rangle\langle 011| \quad \text{if } s_1 = -1,\; s_2 = 1 \;; \\
& |001\rangle\langle 001| + |110\rangle\langle 110| \quad \text{if } s_1 = 1,\; s_2 = -1 \;; \\
& |010\rangle\langle 010| + |101\rangle\langle 101| \quad \text{if } s_1 = s_2 = -1 \;.
\end{aligned}$$

- Conditional on the measurement result, apply the unitary feedback action:

$$\begin{aligned}
\boldsymbol{U}_{s_1,s_2} &= \boldsymbol{I}_8 \text{ if } s_1 = s_2 = 1 \;; & &= \sigma_x \otimes \boldsymbol{I}_4 \text{ if } s_1 = -1,\; s_2 = 1 \;; \\
&= \boldsymbol{I}_4 \otimes \sigma_x \text{ if } s_1 = 1,\; s_2 = -1 \;; & &= \boldsymbol{I}_2 \otimes \sigma_x \otimes \boldsymbol{I}_2 \text{ if } s_1 = s_2 = -1 \;.
\end{aligned}$$

  The resulting Kraus map $\mathbb{K}_1(\rho) = \sum_{s_1,s_2} \boldsymbol{U}_{s_1,s_2}\boldsymbol{P}_{s_1,s_2}\rho\boldsymbol{P}^\dagger_{s_1,s_2}\boldsymbol{U}^\dagger_{s_1,s_2}$ satisfies $\mathbb{K}_1(\mathbb{K}_E(\rho)) = \rho$ for any ρ in the code space and $\mathbb{K}_E$ either identity, or a bit flip on one single of the qubits (1,2,3). (Note that σ_x on two qubits makes this fail.)

Proceeding similarly in parallel on qubit groups (4,5,6) and (7,8,9) yields $\mathbb{K}_1(\rho) = \sum_{s_1,s_2,s_3,s_4,s_5,s_6} \boldsymbol{U} \ldots$, offering recovery from a single σ_x on any qubit.

The Kraus map providing protection against single-bit errors (2)

Next consider phase-flip perturbation, e.g. $\mathbb{K}_E(\rho) = (\sigma_z \otimes \boldsymbol{I}_{2^8})\rho(\sigma_z \otimes \boldsymbol{I}_{2^8})$ on 1st qubit.

In fact this corresponds to a unitary perturbation $|\tilde{+}\rangle\langle\tilde{-}| + |\tilde{-}\rangle\langle\tilde{+}|$. We can address it by a similar majority vote, on the components $|\tilde{+}\rangle, |\tilde{-}\rangle$ of the second stage. We denote $\tilde{\sigma}_x = |\tilde{+}\rangle\langle\tilde{+}| - |\tilde{-}\rangle\langle\tilde{-}|$.

- Projectively measure observables $S_7 = \tilde{\sigma}_x \otimes \tilde{\sigma}_x \otimes \boldsymbol{I}_{2^6}$ and $S_8 = \boldsymbol{I}_{2^6} \otimes \tilde{\sigma}_x \otimes \tilde{\sigma}_x$.
- The rules for the measurement results and conditional feedback action are analogue to the previous slide. For the feedback action, we can choose e.g.

$$\boldsymbol{U}_{(s_7=-1,s_8=1)} \quad = \sigma_z \otimes \boldsymbol{I}_{2^8} \quad \text{or} \quad = \boldsymbol{I}_2 \otimes \sigma_z \otimes \boldsymbol{I}_{2^7} \quad \text{or} \quad = \boldsymbol{I}_4 \otimes \sigma_z \otimes \boldsymbol{I}_{2^6} .$$

  Indeed, they all have the same effect on subspace resulting from $\boldsymbol{P}_{(s_7=-1,s_8=1)}$.
- Note that each measurement now involves 6 qubits at once, each feedback action involves a single qubit. We call the resulting Kraus map $\mathbb{K}_2$.

The total Kraus map $\mathbb{K}_R = \mathbb{K}_2 \circ \mathbb{K}_1$ satisfies $\mathbb{K}_R(\mathbb{K}_E(\rho)) = \rho$ for any ρ in the code space and $\mathbb{K}_E$ either identity, or a σ_x, or a σ_z perturbation, on one single of the qubits.

In fact, this holds for any Kraus map acting on a single qubit [Error Discretization Thm.].

The efficiency of this protection depends on proba. of perturbation on a single qubit, vs. proba. that several qubits are affected before each iteration of $\mathbb{K}_R$. Better efficiency can be obtained with more qubits, and with different encodings ensuring that each measurement and feedback operation only involves a small number of qubits.[16]

[16] for more on this qubit-network-based approach, see Chapter 10 of Nielsen and Chuang (2000).

1. a. Analyze how the code and recovery Kraus map provide quantum information protection against an error σ_y acting on one of the qubits.

   b. Idem for a possible error σ_- acting on one of the qubits. NB: Since σ_- is not unitary, you should consider an error Kraus map with $\boldsymbol{M}_1 = \sqrt{p}\sigma_-$ and $\boldsymbol{M}_2 = ...$, for an arbitrarily fixed error probability parameter $p > 0$.

   c. Show how the code and recovery Kraus map allow to correct the effect of *any* error Kraus map acting on one of the qubits. [Error Discretization Theorem]
   Hint: Observe how your analysis of a. and b. is directly related to protection against σ_x and σ_z errors.

2. The Bacon-Shor recovery Kraus map has been built with measurement operators which only distinguish whether qubit states are equal or different. Check what goes wrong if one would start by measuring the state of each individual qubit.
   NB: You may reduce this analysis to the 3-qubit repetition code of the first stage of the code.

The “cat-qubit” encoding[17]

This encoding starts from a harmonic oscillator, it is a so-called bosonic code. The main idea is that

- Coherent states (see Lecture 1) are close to classical states, i.e. it should be feasible to avoid confusing two coherent states $|0_L\rangle \simeq |\alpha\rangle$ and $|1_L\rangle \simeq |-\alpha\rangle$: like a dead or alive cat.
- The key implementation is with reservoir engineering:

  Target system $\mathcal{H}_C$ and auxiliary system $\mathcal{H}_A$ are harmonic oscillators, with annihilation operators $\boldsymbol{c}$ and $\boldsymbol{a}$ respectively. Let

$$\frac{d}{dt}\boldsymbol{\rho} = -ig[(\boldsymbol{c}^2 - \alpha^2)\boldsymbol{a}^\dagger + (\boldsymbol{c}^2 - \alpha^2)^\dagger \boldsymbol{a},\ \rho] + \kappa_a(\boldsymbol{a}\boldsymbol{\rho}\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}\boldsymbol{a}^\dagger\boldsymbol{a})$$

  This reservoir features an invariant subspace $\text{span}\{|\alpha\rangle, |-\alpha\rangle\}_C \otimes |0\rangle_A$.

  Adiabatic elimination: dynamics on $\mathcal{H}_C$ induced by this reservoir, for $g \ll \kappa_a$, is

$$\frac{d}{dt}\boldsymbol{\rho} = \frac{g^2}{\kappa_a}\left((\boldsymbol{c}^2-\alpha^2)\boldsymbol{\rho}(\boldsymbol{c}^2-\alpha^2)^\dagger - \tfrac{1}{2}(\boldsymbol{c}^2-\alpha^2)^\dagger(\boldsymbol{c}^2-\alpha^2)\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}(\boldsymbol{c}^2-\alpha^2)^\dagger(\boldsymbol{c}^2-\alpha^2)\right).$$

  This dissipative dynamics of strength $\frac{g^2}{\kappa_a}$ protects the cat subspace against perturbations weaker than $\frac{g^2}{\kappa_a}$ and which tend to confuse $|\pm\alpha\rangle$.
- There remains to protect information against perturbations transforming $\frac{|\alpha\rangle+|-\alpha\rangle}{\sqrt{2}}$ into $\frac{|\alpha\rangle-|-\alpha\rangle}{\sqrt{2}}$ (phase-flips), which must be done with other means e.g. majority vote.

[17] see M.Mirrahimi, Z.Leghtas et al (2014), New Journal of Physics, 16(4) + series of articles building on this one.