Mathematical methods for modeling and control of open quantum systems[1]

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[1] Lecture-notes, slides and Matlab simulation scripts available at: http://cas.ensmp.fr/~rouchon/LIASFMA/index.html

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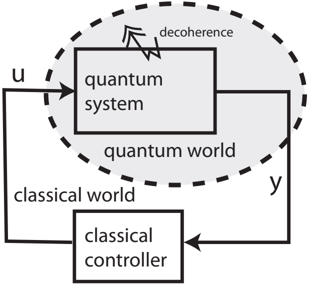
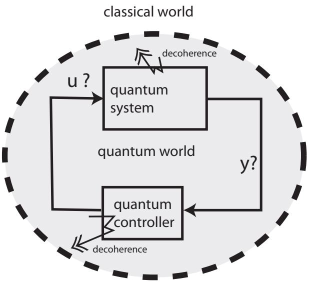
Outline

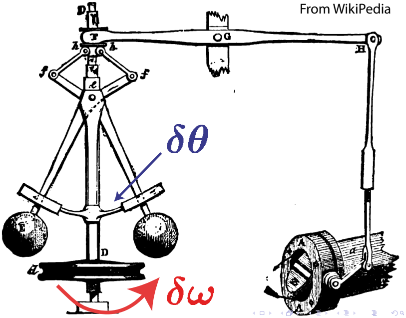
Outline



Measurement-based feedback: controller is classical; measurement back-action on the quantum system of Hilbert space $\mathcal{H}$ is stochastic (collapse of the wave-packet); the measured output y is a classical signal; the control input u is a classical variable appearing in some controlled Schrödinger equation; $u(t)$ depends on the past measurements $y(\tau)$, $\tau \leq t$.



Coherent/autonomous feedback and reservoir engineering: the **system of Hilbert space** $\mathcal{H}$ is coupled to **the controller, another quantum system**; the composite system of Hilbert space $\mathcal{H}_{controller} \otimes \mathcal{H}$, is an open-quantum system relaxing to some target (separable) state. Origin of such relaxation behaviors in open quantum systems: optical pumping of Alfred Kastler, physics Nobel prize 1966.



The first variations of speed $\delta\omega$ and governor angle $\delta\theta$ obey to

$$\frac{d}{dt}\delta\omega = -a\delta\theta$$
$$\frac{d^2}{dt^2}\delta\theta = -\Lambda\frac{d}{dt}\delta\theta - \Omega^2(\delta\theta - b\delta\omega)$$

with (a, b, Λ, Ω) positive parameters.

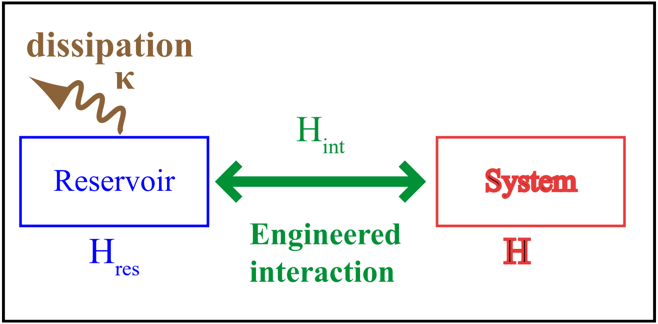
Third order system

$$\frac{d^3}{dt^3}\delta\omega + \Lambda\frac{d^2}{dt^2}\delta\omega + \Omega^2\frac{d}{dt}\delta\omega + ab\Omega^2\delta\omega = 0.$$

Characteristic polynomial $P(s) = s^3 + \Lambda s^2 + \Omega^2 s + ab\Omega^2$ with roots having negative real parts iff $\Lambda > ab$: governor damping must be strong enough to ensure asymptotic stability.

Key issues: asymptotic stability and convergence rates.

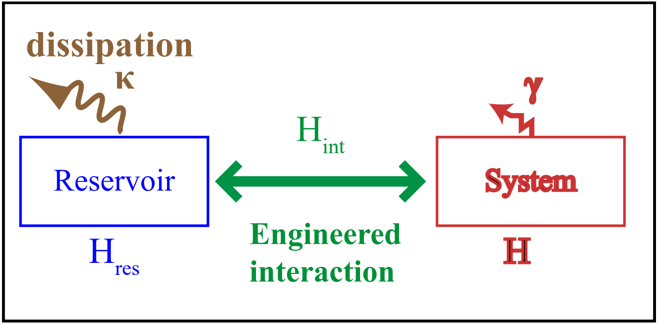
[5] J.C. Maxwell: On governors. Proc. of the Royal Society, No.100, 1868.



$$H = \boldsymbol{H}_{\text{res}} + \boldsymbol{H}_{\text{int}} + \boldsymbol{H}$$

If $\rho \underset{t\to\infty}{\longrightarrow} \rho_{res} \otimes |\bar{\psi}\rangle\langle\bar{\psi}|$ exponentially on a time scale of $\tau > 0$ then . . . . . .

[6] See, e.g., the lectures of H. Mabuchi delivered at the "Ecole de physique des Houches", July 2011.



$$H = H_{res} + H_{int} + H$$

...... $\rho \underset{t\to\infty}{\to} \rho_{res} \otimes |\bar{\psi}\rangle\langle\bar{\psi}| + \overline{\delta\rho}$, if $\tau\gamma \ll 1$ then $|\overline{\delta\rho}| \ll 1$

Convergence issues of open-quantum systems

Continuous-time models: Lindbald master eq. (quantum Fokker-Planck eq.):

$$\frac{d}{dt}\rho = -\mathcal{A}(\rho) \triangleq -\tfrac{i}{\hbar}[\boldsymbol{H}, \rho] + \sum_\nu \left(\boldsymbol{L}_\nu \rho \boldsymbol{L}_\nu^\dagger - (\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \rho + \rho \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)/2 \right),$$

of state ρ a density operator (Hermitian, non negative, trace-class, trace one) with $\boldsymbol{H}$ Hermitian operator and $\boldsymbol{L}_\nu$ arbitrary operators (usually unbounded).

When $\mathcal{H}$ is of finite dimension, $(e^{-t\mathcal{A}})_{t\geq 0}$ is a contraction semi-group for many metrics ($\text{Tr}\,(|\rho - \sigma|)$, $\text{Tr}\left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)$, see the work of D. Petz).
Open issues motivated by robust quantum information processing:

1. characterization of the Ω-limit support of ρ: decoherence free spaces are affine spaces where the dynamics are of Schrödinger types; they can be reduced to a point (pointer-state);
2. Estimation of convergence rate and robustness.
3. Reservoir engineering: design of realistic H and L_ν to achieve rapid convergence towards prescribed affine spaces (protection against decoherence).

Lecture goal: **cat-qubits** and autonomous QEC of bit-flips[7]

[7] R. Lescanne, ..., M. Mirrahimi, M. and Z. Leghtas: Exponential suppression of bit-flips in a qubit encoded in an oscillator. 2020, Nat. Phys. , Vol. 16, p. 509-513.

Outline

The driven and damped classical oscillator

Dynamics in the (x', p') phase plane with $\omega \gg \kappa, \sqrt{u_1^2 + u_2^2}$:

$$\frac{d}{dt}x' = \omega p', \quad \frac{d}{dt}p' = -\omega x' - \kappa p' - 2u_1 \sin(\omega t) + 2u_2 \cos(\omega t)$$

Define the frame rotating at ω by $(x', p') \mapsto (x, p)$ with

$$x' = \cos(\omega t)x + \sin(\omega t)p, \quad p' = -\sin(\omega t)x + \cos(\omega t)p.$$

Removing highly oscillating terms (rotating wave approximation), from

$$\frac{d}{dt}x = -\kappa \sin^2(\omega t)x + 2u_1 \sin^2(\omega t) + (\kappa p - 2u_2) \sin(\omega t) \cos(\omega t)$$
$$\frac{d}{dt}p = -\kappa \cos^2(\omega t)p + 2u_2 \cos^2(\omega t) + (\kappa x - 2u_1) \sin(\omega t) \cos(\omega t)$$

we get, with $\alpha = x + ip$ and $u = u_1 + iu_2$:

$$\frac{d}{dt}\alpha = -\tfrac{\kappa}{2}\alpha + u.$$

With $x' + ip' = \alpha' = e^{-i\omega t}\alpha$, we have $\frac{d}{dt}\alpha' = -(\frac{\kappa}{2} + i\omega)\alpha' + ue^{-i\omega t}$

- The Lindblad master equation (quantum analogue of $\frac{d}{dt}\alpha = -\frac{\kappa}{2}\alpha + u$ with $\alpha = \text{Tr}(\boldsymbol{a}\rho)$):

$$\frac{d}{dt}\boldsymbol{\rho} = [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \boldsymbol{\rho}] + \kappa\left(\boldsymbol{a}\boldsymbol{\rho}\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}\boldsymbol{a}^\dagger\boldsymbol{a}\right).$$

- Consider $\boldsymbol{\rho} = \boldsymbol{D}_{\overline{\alpha}}\boldsymbol{\xi}\boldsymbol{D}_{-\overline{\alpha}}$ with $\overline{\alpha} = 2u/\kappa$ and $\boldsymbol{D}_{\overline{\alpha}} = e^{\overline{\alpha}\boldsymbol{a}^\dagger - \overline{\alpha}^*\boldsymbol{a}}$. We get

$$\frac{d}{dt}\boldsymbol{\xi} = \kappa\left(\boldsymbol{a}\boldsymbol{\xi}\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\xi} - \tfrac{1}{2}\boldsymbol{\xi}\boldsymbol{a}^\dagger\boldsymbol{a}\right)$$

since $\boldsymbol{D}_{-\overline{\alpha}}\boldsymbol{a}\boldsymbol{D}_{\overline{\alpha}} = \boldsymbol{a} + \overline{\alpha}$.

- Informal convergence proof with the strict Lyapunov function $V(\boldsymbol{\xi}) = \text{Tr}(\boldsymbol{\xi}\boldsymbol{N})$:

$$\frac{d}{dt}V(\boldsymbol{\xi}) = -\kappa V(\boldsymbol{\xi}) \Rightarrow V(\boldsymbol{\xi}(t)) = V(\boldsymbol{\xi}_0)e^{-\kappa t}.$$

Since $\boldsymbol{\xi}(t)$ is Hermitian and non-negative, $\boldsymbol{\xi}(t)$ tends to $|0\rangle\langle 0|$ when $t \mapsto +\infty$.

The rigorous underlying convergence result

Theorem

Consider with $u \in \mathbb{C}$, $\kappa > 0$, the following Cauchy problem

$$\frac{d}{dt}\rho = [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \rho] + \kappa\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\rho - \tfrac{1}{2}\rho\boldsymbol{a}^\dagger\boldsymbol{a}\right), \quad \rho(0) = \rho_0.$$

Assume that the initial state ρ_0 is a density operator with finite energy $\mathrm{Tr}\left(\rho_0 \boldsymbol{N}\right) < +\infty$*. Then exists a unique solution to the Cauchy problem in the Banach space $\mathcal{K}^1(\mathcal{H})$, the set of trace class operators on $\mathcal{H}$. It is defined for all $t > 0$ with $\rho(t)$ a density operator (Hermitian, non-negative and trace-class) that remains in the domain of the Lindblad super-operator*

$$\rho \mapsto [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \rho] + \kappa\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\rho - \tfrac{1}{2}\rho\boldsymbol{a}^\dagger\boldsymbol{a}\right).$$

This means that $t \mapsto \rho(t)$ is differentiable in the Banach space $\mathcal{K}^1(\mathcal{H})$. Moreover $\rho(t)$ converges for the trace-norm towards $|\overline{\alpha}\rangle\langle\overline{\alpha}|$ when t tends to $+\infty$, where $|\overline{\alpha}\rangle$ is the coherent state of complex amplitude $\overline{\alpha} = \frac{2u}{\kappa}$.

Link with the classical oscillator

Lemma

Consider with $u \in \mathbb{C}$, $\kappa > 0$, the following Cauchy problem

$$\frac{d}{dt}\rho = [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \rho] + \kappa\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\rho - \tfrac{1}{2}\rho\boldsymbol{a}^\dagger\boldsymbol{a}\right), \quad \rho(0) = \rho_0.$$

1. *for any initial density operator ρ_0 with $\mathrm{Tr}\left(\rho_0 \boldsymbol{N}\right) < +\infty$, we have $\frac{d}{dt}\alpha = -\frac{\kappa}{2}(\alpha - \overline{\alpha})$ where $\alpha = \mathrm{Tr}\left(\rho\boldsymbol{a}\right)$ and $\overline{\alpha} = \frac{2u}{\kappa}$.*
2. *Assume that $\rho_0 = |\beta_0\rangle\langle\beta_0|$ where β_0 is some complex amplitude. Then for all $t \geq 0$, $\rho(t) = |\beta(t)\rangle\langle\beta(t)|$ remains a coherent state of amplitude $\beta(t)$ solution of the following equation: $\frac{d}{dt}\beta = -\frac{\kappa}{2}(\beta - \overline{\alpha})$ with $\beta(0) = \beta_0$.*

Statement 2 relies on:

$$\boldsymbol{a}|\beta\rangle = \beta|\beta\rangle, \quad |\beta\rangle = e^{-\frac{\beta\beta^*}{2}} e^{\beta\boldsymbol{a}^\dagger}|0\rangle \quad \frac{d}{dt}|\beta\rangle = \left(-\tfrac{1}{2}(\beta^*\dot{\beta} + \beta\dot{\beta}^*) + \dot{\beta}\boldsymbol{a}^\dagger\right)|\beta\rangle.$$

Driven and damped quantum oscillator with thermal photon

Parameters $\omega \gg \kappa, |u|$ and $n_{\text{th}} \geq 0$:

$$\frac{d}{dt}\boldsymbol{\rho} = [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \rho] + (1+n_{\text{th}})\kappa\left(\boldsymbol{a}\boldsymbol{\rho}\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}\boldsymbol{a}^\dagger\boldsymbol{a}\right) + n_{\text{th}}\kappa\left(\boldsymbol{a}^\dagger\boldsymbol{\rho}\boldsymbol{a} - \tfrac{1}{2}\boldsymbol{a}\boldsymbol{a}^\dagger\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}\boldsymbol{a}\boldsymbol{a}^\dagger\right).$$

Key issue: $\lim_{t\mapsto+\infty} \boldsymbol{\rho}(t) = ?$.
The passage to another representation via the Wigner function:

- Since $\boldsymbol{D}_\alpha e^{i\pi\boldsymbol{N}}\boldsymbol{D}_{-\alpha}$ bounded and Hermitian operator (the dual of $\mathcal{K}^1(\mathcal{H})$ is $\mathcal{B}(\mathcal{H})$),

$$W^{\{\boldsymbol{\rho}\}}(x,p) = \tfrac{2}{\pi}\operatorname{Tr}\left(\boldsymbol{\rho}\boldsymbol{D}_\alpha e^{i\pi\boldsymbol{N}}\boldsymbol{D}_{-\alpha}\right) \quad \text{with} \quad \alpha = x + ip \in \mathbb{C},$$

  defines a real and bounded function $|W^{\{\boldsymbol{\rho}\}}(x,p)| \leq \frac{2}{\pi}$.
- For a coherent state $\boldsymbol{\rho} = |\beta\rangle\langle\beta|$ with $\beta \in \mathbb{C}$:

$$W^{\{|\beta\rangle\langle\beta|\}}(x,p) = \tfrac{2}{\pi} e^{-2|\beta-(x+ip)|^2}.$$

The partial differential equation satisfied by the Wigner function (1)[8]

With $\boldsymbol{D}_\alpha = e^{\alpha \boldsymbol{a}^\dagger} e^{-\alpha^* \boldsymbol{a}} e^{-\alpha\alpha^*/2} = e^{-\alpha^* \boldsymbol{a}} e^{\alpha \boldsymbol{a}^\dagger} e^{\alpha\alpha^*/2}$ we have:

$$\tfrac{\pi}{2} W^{\{\rho\}}(\alpha, \alpha^*) = \mathrm{Tr}\left(\rho e^{\alpha \boldsymbol{a}^\dagger} e^{-\alpha^* \boldsymbol{a}} e^{i\pi \boldsymbol{N}} e^{\alpha^* \boldsymbol{a}} e^{-\alpha \boldsymbol{a}^\dagger}\right)$$

where α and α^* are seen as independent variables:

$$\frac{\partial}{\partial \alpha} = \tfrac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial p}\right), \quad \frac{\partial}{\partial \alpha^*} = \tfrac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial p}\right)$$

We have $\frac{\pi}{2}\frac{\partial}{\partial\alpha} W^{\{\rho\}}(\alpha, \alpha^*) = \mathrm{Tr}\left((\rho \boldsymbol{a}^\dagger - \boldsymbol{a}^\dagger \rho)\boldsymbol{D}_\alpha e^{i\pi \boldsymbol{N}} \boldsymbol{D}_{-\alpha}\right)$ Since $\boldsymbol{a}^\dagger \boldsymbol{D}_\alpha e^{i\pi \boldsymbol{N}} \boldsymbol{D}_{-\alpha} = \boldsymbol{D}_\alpha e^{i\pi \boldsymbol{N}} \boldsymbol{D}_{-\alpha}(2\alpha^* - \boldsymbol{a}^\dagger)$, we get

$$\frac{\partial}{\partial \alpha} W^{\{\rho\}}(\alpha, \alpha^*) = 2\alpha^* W^{\{\rho\}}(\alpha, \alpha^*) - 2W^{\{\boldsymbol{a}^\dagger \rho\}}(\alpha, \alpha^*).$$

Thus $W^{\{\boldsymbol{a}^\dagger \rho\}}(\alpha, \alpha^*) = \alpha^* W^{\{\rho\}}(\alpha, \alpha^*) - \frac{1}{2}\frac{\partial}{\partial\alpha} W^{\{\rho\}}(\alpha, \alpha^*)$, i.e.

$$W^{\{\boldsymbol{a}^\dagger \rho\}} = \left(\alpha^* - \tfrac{1}{2}\frac{\partial}{\partial \alpha}\right) W^{\{\rho\}}.$$

[8] See the excellent Wikipedia article:
https://en.wikipedia.org/wiki/Wigner_quasiprobability_distribution

The partial differential equation satisfied by the Wigner function (2)

Similar computations yield to the following correspondence rules:

$$W^{\{\rho \boldsymbol{a}\}} = \left(\alpha - \tfrac{1}{2}\frac{\partial}{\partial \alpha^*}\right) W^{\{\rho\}}, \quad W^{\{\boldsymbol{a}\rho\}} = \left(\alpha + \tfrac{1}{2}\frac{\partial}{\partial \alpha^*}\right) W^{\{\rho\}}$$

$$W^{\{\rho \boldsymbol{a}^\dagger\}} = \left(\alpha^* + \tfrac{1}{2}\frac{\partial}{\partial \alpha}\right) W^{\{\rho\}}, \quad W^{\{\boldsymbol{a}^\dagger\rho\}} = \left(\alpha^* - \tfrac{1}{2}\frac{\partial}{\partial \alpha}\right) W^{\{\rho\}}.$$

Thus

$$\frac{d}{dt}\rho = [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \rho] + (1+n_{\text{th}})\kappa\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\rho - \tfrac{1}{2}\rho\boldsymbol{a}^\dagger\boldsymbol{a}\right) + n_{\text{th}}\kappa\left(\boldsymbol{a}^\dagger\rho\boldsymbol{a} - \tfrac{1}{2}\boldsymbol{a}\boldsymbol{a}^\dagger\rho - \tfrac{1}{2}\rho\boldsymbol{a}\boldsymbol{a}^\dagger\right).$$

becomes

$$\frac{\partial}{\partial t}W^{\{\rho\}} = \frac{\kappa}{2}\left(\frac{\partial}{\partial\alpha}(\alpha - \overline{\alpha}) + \frac{\partial}{\partial\alpha^*}(\alpha^* - \overline{\alpha}^*) + (1+2n_{\text{th}})\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right) W^{\{\rho\}}$$

Solutions of the quantum Fokker-Planck equation

Since the Green function of

$$\frac{\partial}{\partial t} W^{\{\rho\}} = \frac{\kappa}{2}\Big(\frac{\partial}{\partial x}\Big((x-\overline{x})W^{\{\rho\}}\Big) + \frac{\partial}{\partial p}\Big((p-\overline{p})W^{\{\rho\}}\Big) + \tfrac{1+2n_{\text{th}}}{4}\Big(\frac{\partial^2 W^{\{\rho\}}}{\partial x^2} + \frac{\partial^2 W^{\{\rho\}}}{\partial p^2}\Big)\Big)$$

is the following time-varying Gaussian function

$$G(x,p,t,x_0,p_0) = \frac{\exp\left(-\frac{\left(x-\overline{x}-(x_0-\overline{x})e^{-\frac{\kappa t}{2}}\right)^2 + \left(p-\overline{p}-(p_0-\overline{p})e^{-\frac{\kappa t}{2}}\right)^2}{(n_{\text{th}}+\frac{1}{2})(1-e^{-\kappa t})}\right)}{\pi(n_{\text{th}}+\frac{1}{2})(1-e^{-\kappa t})}$$

we can compute $W_t^{\{\rho\}}$ from $W_0^{\{\rho\}}$ for all $t > 0$:

$$W_t^{\{\rho\}}(x,p) = \int_{\mathbb{R}^2} W_0^{\{\rho\}}(x',p')G(x,p,t,x',p')\,dx'dp'.$$

Combining

- $W_t^{\{\rho\}}(x,p) = \int_{\mathbb{R}^2} W_0^{\{\rho\}}(x',p') G(x,p,t,x',p')\, dx'dp'$.
- G uniformly bounded and $\lim_{t\mapsto+\infty} G(x,p,t,x',p') = \frac{1}{\pi(n_{\text{th}}+\frac{1}{2})} \exp\left(-\frac{(x-\overline{x})^2+(p-\overline{p})^2}{(n_{\text{th}}+\frac{1}{2})}\right)$
- $W_0^{\{\rho\}}$ in L^1 with $\iint_{\mathbb{R}^2} W_0^{\{\rho\}} = 1$
- dominate convergence theorem

shows that all the solutions converge to a unique steady-state Gaussian density function, centered in $(\overline{x},\overline{p})$ with variance $\frac{1}{2}+n_{\text{th}}$:

$$\forall (x,p)\in\mathbb{R}^2, \quad \lim_{t\mapsto+\infty} W_t^{\{\rho\}}(x,p) = \frac{1}{\pi(n_{\text{th}}+\frac{1}{2})} \exp\left(-\frac{(x-\overline{x})^2+(p-\overline{p})^2}{(n_{\text{th}}+\frac{1}{2})}\right).$$

Diffusion along x and p of Wigner function $W^{\rho}(x,p)$

With correspondence rules:

$$W^{\{\rho a\}} = \left(\alpha - \tfrac{1}{2}\frac{\partial}{\partial\alpha^*}\right) W^{\{\rho\}}, \quad W^{\{a\rho\}} = \left(\alpha + \tfrac{1}{2}\frac{\partial}{\partial\alpha^*}\right) W^{\{\rho\}}$$

$$W^{\{\rho a^\dagger\}} = \left(\alpha^* + \tfrac{1}{2}\frac{\partial}{\partial\alpha}\right) W^{\{\rho\}}, \quad W^{\{a^\dagger\rho\}} = \left(\alpha^* - \tfrac{1}{2}\frac{\partial}{\partial\alpha}\right) W^{\{\rho\}}$$

the super-operator

$$\rho \mapsto (a+a^\dagger)\rho(a+a^\dagger) - \frac{1}{2}\left((a+a^\dagger)^2\rho + \rho(a+a^\dagger)^2\right)$$

becomes in Wigner representation[9]

$$W^{\{\rho\}} \mapsto \tfrac{-1}{2}\left(\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\alpha^*}\right)^2 W^{\{\rho\}}(\alpha,\alpha^*) \equiv \tfrac{1}{2}\frac{\partial^2}{\partial p^2} W^{\{\rho\}}(x,p).$$

Similarly, the super-operator

$$\rho \mapsto (a-a^\dagger)\rho(a-a^\dagger) - \frac{1}{2}\left((a-a^\dagger)^2\rho + \rho(a-a^\dagger)^2\right)$$

becomes in Wigner representation

$$W^{\{\rho\}} \mapsto \tfrac{-1}{2}\left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\alpha^*}\right)^2 W^{\{\rho\}}(\alpha,\alpha^*) \equiv \tfrac{1}{2}\frac{\partial^2}{\partial x^2} W^{\{\rho\}}(x,p).$$

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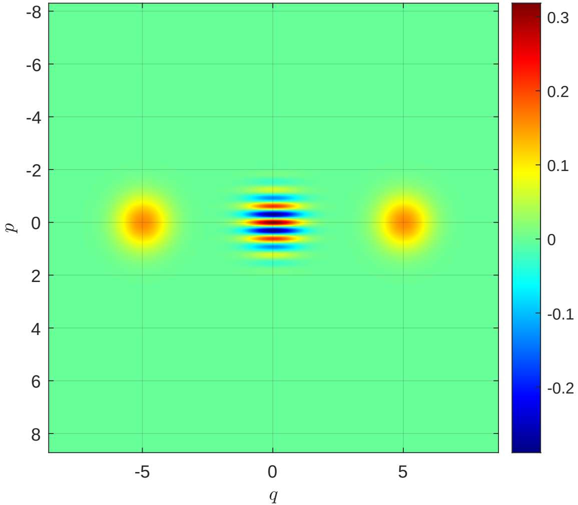
[9] Use the fact that $\frac{\partial}{\partial\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial p}\right)$, $\frac{\partial}{\partial\alpha^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial p}\right)$ and $\alpha = x + ip$.

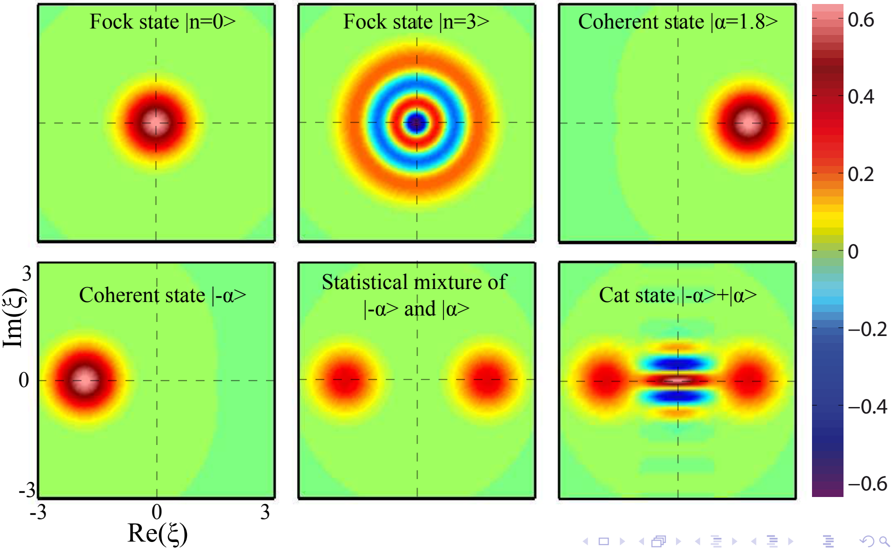
Wigner function[10] of $|\alpha\rangle + |\text{-}\alpha\rangle$ ("Schrödinger cat" with $\alpha = 5$)

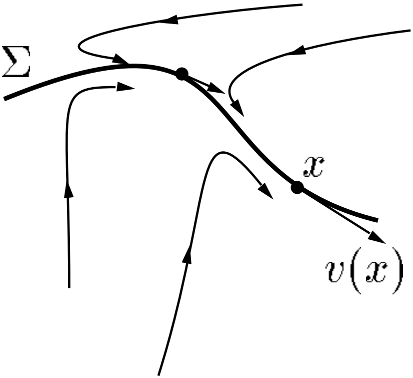


[10] For $\psi \in L^2(\mathbb{R}, \mathbb{C})$: $W(q,p) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \psi^*(q - \frac{u}{2})\psi(q + \frac{u}{2}) e^{-2ipu} du$.

Wigner function W^ρ for different values of the density operator ρ

$$W^\rho : \mathbb{C} \ni \xi \to \frac{2}{\pi} \operatorname{Tr}\left(\left(\boldsymbol{D}_\xi e^{i\pi \boldsymbol{N}} \boldsymbol{D}_\xi^\dagger \right) \ \rho \right) \in [-2/\pi, 2/\pi]$$

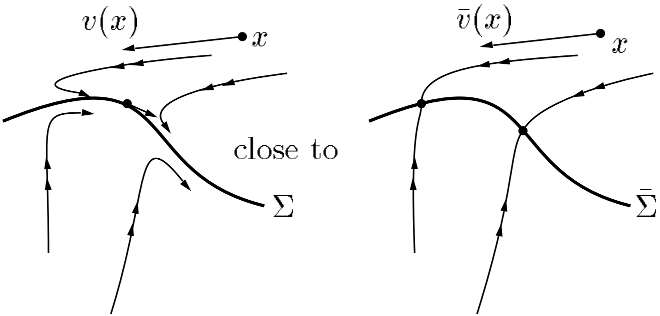




A possible answer:

restriction to an attractive invariant manifold Σ.

Slow/fast systems (coordinate free setting)

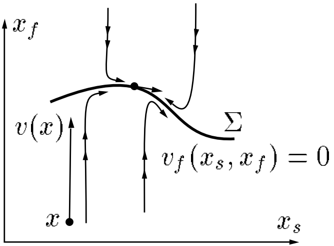


Geometric definition independent of coordinates due to Fenichel[11]:

- $x \mapsto v(x)$ close to $x \mapsto \overline{v}(x)$.
- $\overline{v}(x) = 0$ define a manifold $\overline{\Sigma}$ of dimension $n_s < n = \dim(x)$ of steady-states for $\overline{v}(x)$.
- $n_f = n - n_s$ eigenvalues of $\left.\frac{\partial \overline{v}}{\partial x}\right|_{\overline{\Sigma}}$ are stable (negative real parts).

[11] N. Fenichel: Geometric singular perturbation theory for ordinary differential equations. J. Diff. Equations, 1979, 31, 53-98.

Tikhonov normal form [12] and model reduction



Any slow/fast system, can be put, after a suitable change of coordinates, in to a **quasi-vertical vector field** v:

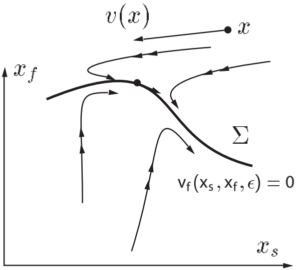
$$\frac{d}{dt}x_s = v_s(x_s, x_f) = \epsilon \tilde{v}_s(x_s, x_f, \epsilon)$$
$$\frac{d}{dt}x_f = v_f(x_s, x_f)$$

with $0 < \epsilon \ll 1$.

The reduced system $\frac{d}{dt}x_s = v_s(x_s, x_f)$ with $0 = v_f(x_s, x_f)$ is correct if $\frac{d}{dt}\xi_f = v_f(x_s, \xi_f)$ hyperbolically stable for any fixed x_s.

In general, modeling variables x are **not** Tikhonov variables.

[12] See, e.g., F. Verhulst: Methods and Applications of Singular Perturbations: Boundary Layers and Multiple Timescale Dynamics. Springer, 2005



Example with the heuristic method:

$$\frac{d}{dt}x_s = 2(x_f - x_s) + \epsilon x_f \quad \frac{d}{dt}x_f = x_s - x_f$$

1- compute x_f versus x_s from $\frac{d}{dt}x_f = 0$;
2- plug $x_f = x_s$ into $\frac{d}{dt}x_s$ to obtain
$\boldsymbol{\frac{d}{dt}x_s = \epsilon x_s}$ (wrong slow model !)

The reduced model of $\frac{d}{dt}x_s = v_s(x_s, x_f, \epsilon)$, $\frac{d}{dt}x_f = v_f(x_s, x_f, \epsilon)$ is[13]

$$\frac{d}{dt}x_s = \left(1 + \frac{\partial v_s}{\partial x_f}\left(\frac{\partial v_f}{\partial x_f}\right)^{-2}\frac{\partial v_f}{\partial x_s}\right)^{-1} v_s(x_s, x_f, \epsilon) + O(\epsilon^2), \quad v_f(x_s, x_f, \epsilon) = 0.$$

Same example with the correct method: with $\frac{\partial v_s}{\partial x_f} = 2$, $\frac{\partial v_f}{\partial x_s} = 1 = -\frac{\partial v_f}{\partial x_f}$,
we get the correct slow model , $\boldsymbol{\frac{d}{dt}x_s = \epsilon x_s / 3}$.

[13] J. Carr: Application of Center Manifold Theory. Springer, 1981.
P. Duchêne, P.R. : Kinetic scheme reduction via geometric singular perturbation techniques. Chem. Eng. Science, 1996, 51, 4661-4672.

Slow/fast composite quantum systems

Take $0 < \epsilon \ll 1$ and composite system made of subsystem A with Hilbert space $\mathcal{H}_A$ and subsystem B with Hilbert space $\mathcal{H}_B$:

$$\frac{d}{dt}\boldsymbol{\rho} = \mathcal{L}_B(\boldsymbol{\rho}) + \epsilon\Big(- i[\boldsymbol{H}_{\text{int}}, \rho] + \mathcal{L}_A(\boldsymbol{\rho})\Big)$$

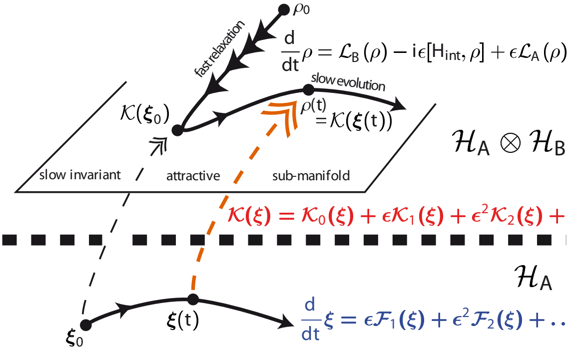
where

- $\mathcal{L}_B(\boldsymbol{\rho})$ is a Lindbladian dynamics on $\mathcal{H}_B$ converging towards a unique steady-state density operator $\overline{\rho}_B$ on $\mathcal{H}_B$.
- $\mathcal{L}_A(\boldsymbol{\rho})$ is a Lindbladian dynamics on $\mathcal{H}_A$
- AB-interaction Hamiltonian $\boldsymbol{H}_{\text{int}} = \sum_{k=1}^{m} \boldsymbol{A}_k \otimes \boldsymbol{B}_k$, with $\boldsymbol{A}_k$ and $\boldsymbol{B}_k$ Hermitian operators on $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively.

When $\epsilon = 0$, for all initial state $\boldsymbol{\rho}_0$ on $\mathcal{H}_A \otimes \mathcal{H}_B$, the solution of $\frac{d}{dt}\boldsymbol{\rho} = \mathcal{L}_B(\boldsymbol{\rho})$ converges towards the separable steady-state $\text{Tr}_B(\boldsymbol{\rho}_0) \otimes \overline{\rho}_B$.
For $0 < \epsilon \ll 1$, the attractive steady-state manifold

$$\bar{\Sigma} = \Big\{\boldsymbol{\rho}_A \otimes \overline{\rho}_B \mid \rho_A \text{ density operator on } \mathcal{H}_A\Big\}$$

becomes Σ_ϵ, an attractive invariant manifold where the evolution is slow.
Σ_ϵ can be parameterized via density operators $\boldsymbol{\xi}$ on $\mathcal{H}_A$ with a slow evolution.
Approximation of such parametrization and slow evolution can be done via asymptotic expansion in ϵ. Is-it always possible to preserve positivity of $\boldsymbol{\rho}$?
Always OK for second order expansion.



Lindbladian slow dynamics on a density operator ξ on $\mathcal{H}_A$,

$$\frac{d}{dt}\xi = \epsilon\mathcal{F}_1(\xi) + \epsilon^2\mathcal{F}_2(\xi) + \dots$$

with **a Kraus map** giving density operator ρ on $\mathcal{H}_A \otimes \mathcal{H}_B$ from ξ:

$$\rho = \mathcal{K}(\xi) = \mathcal{K}_0(\xi) + \epsilon\mathcal{K}_1(\xi) + \epsilon^2\mathcal{K}_2(\xi) + \dots$$

[14] Azouit, R. / Chittaro, F. / Sarlette, A. / PR: Towards generic adiabatic elimination for bipartite open quantum systems 2017, Quantum Science and Technology , Vol. 2, p. 044011

An iterative procedure based on center manifold approximation

Plug

$$\boldsymbol{\rho} = \mathcal{K}(\boldsymbol{\xi}) = \boldsymbol{\xi} \otimes \overline{\rho}_B + \epsilon \mathcal{K}_1(\boldsymbol{\xi}) + \ldots, \quad \text{and} \quad \frac{d}{dt}\boldsymbol{\xi} = \mathcal{F}(\boldsymbol{\xi}) = \epsilon \mathcal{F}_1(\boldsymbol{\xi}) + \epsilon^2 \mathcal{F}_2(\boldsymbol{\xi}) + \ldots$$

into invariance condition

$$\mathcal{L}_B(\mathcal{K}(\boldsymbol{\xi})) - \epsilon i \left[\boldsymbol{H}_{\text{int}}, \mathcal{K}(\boldsymbol{\xi})\right] + \epsilon \mathcal{L}_A(\mathcal{K}(\boldsymbol{\xi})) = \frac{d}{dt}\boldsymbol{\rho} = \mathcal{K}(\mathcal{F}(\boldsymbol{\xi}))$$

and identify terms of same orders:

order 1: $\mathcal{L}_B(\mathcal{K}_1(\boldsymbol{\xi})) - i\left[\boldsymbol{H}_{\text{int}}, \mathcal{K}_0(\boldsymbol{\xi})\right] + \mathcal{L}_A(\mathcal{K}_0(\boldsymbol{\xi})) = \mathcal{K}_0(\mathcal{F}_1(\boldsymbol{\xi}))$

order 2: $\mathcal{L}_B(\mathcal{K}_2(\boldsymbol{\xi})) - i\left[\boldsymbol{H}_{\text{int}}, \mathcal{K}_1(\boldsymbol{\xi})\right] + \mathcal{L}_A(\mathcal{K}_1(\boldsymbol{\xi})) = \mathcal{K}_0(\mathcal{F}_2(\boldsymbol{\xi})) + \mathcal{K}_1(\mathcal{F}_1(\boldsymbol{\xi}))$

$\ldots$

At each order

1. take the trace versus B to get the correction to $\mathcal{F}$
2. compute the correction to $\mathcal{K}$ via $-\mathcal{L}_B^{-1}$, a super operator for zero-trace operators $\boldsymbol{W}$ on $\mathcal{H}_A$

$$-\mathcal{L}_B^{-1}(\boldsymbol{W}) = \int_0^{+\infty} e^{t\mathcal{L}_B}(\boldsymbol{W})dt$$

that coincides with the restriction to zero-trace operators of a completely positive (CP) map.

Second order approximation when B is a low-Q mode

For $\mathcal{L}_B(\boldsymbol{\rho}) = \kappa_b\Big(\boldsymbol{b}\rho\boldsymbol{b}^\dagger - \frac{1}{2}(\boldsymbol{b}^\dagger\boldsymbol{b}\rho + \rho\boldsymbol{b}^\dagger\boldsymbol{b})\Big)$ one gets using $\overline{\rho}_B = |0_b\rangle\langle 0_b|$,

$$\frac{d}{dt}\xi = -i\epsilon\left[\sum_k \beta_k \boldsymbol{A}_k\ ,\ \xi\right] + \epsilon\mathcal{L}_A(\xi) + \frac{4\epsilon^2}{\kappa_b}\left(\sum_k \boldsymbol{L}_k\xi\boldsymbol{L}_k^\dagger - \frac{1}{2}\left(\boldsymbol{L}_k^\dagger\boldsymbol{L}_k\xi + \xi\boldsymbol{L}_k^\dagger\boldsymbol{L}_k\right)\right) + O(\epsilon^3)$$

with

$$\boldsymbol{\rho} = e^{i\epsilon\boldsymbol{W}_1}\left(\xi\otimes|0_b\rangle\langle 0_b|\right)e^{-i\epsilon\boldsymbol{W}_1} + O(\epsilon^2)$$

and where

- $\beta_k = \langle 0_b|\boldsymbol{B}_k|0_b\rangle$,
- $\boldsymbol{L}_k = \sum_{k'=1}^m \Lambda_{k,k'}\boldsymbol{A}_{k'}$ based on Cholesky factorization $\Lambda^\dagger\Lambda = G$ of the following Gram matrix

$$G_{kk'} == \sum_{n_b=1}^{+\infty}\left(\frac{1}{\sqrt{n_b}}\langle n_b|\boldsymbol{B}_k|0_b\rangle\right)^*\left(\frac{1}{\sqrt{n_b}}\langle n_b|\boldsymbol{B}_{k'}|0_b\rangle\right).$$

- $\boldsymbol{W}_1 = \frac{2}{\kappa_b}\sum_{k=1}^m \boldsymbol{A}_k\otimes\left((\boldsymbol{b}^\dagger\boldsymbol{b})^{-1}\boldsymbol{B}_k + \boldsymbol{B}_k(\boldsymbol{b}^\dagger\boldsymbol{b})^{-1}\right)$ using $(\boldsymbol{b}^\dagger\boldsymbol{b})^{-1} = \sum_{n_b>1}\frac{1}{n_b}|n_b\rangle\langle n_b|$.

Outline

Super-conducting circuit stabilizing a cat-qubit [15]

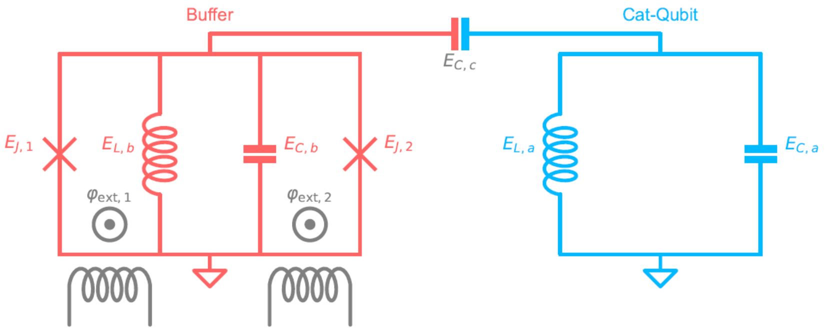


Figure S3. Equivalent circuit diagram. The cat-qubit (blue), a linear resonator, is capacitively coupled to the buffer (red). One recovers the circuit of Fig. 2 by replacing the buffer inductance with a 5-junction array and by setting $\varphi_\Sigma = (\varphi_{\text{ext},1} + \varphi_{\text{ext},2})/2$ and $\varphi_\Delta = (\varphi_{\text{ext},1} - \varphi_{\text{ext},2})/2$. Not shown here: the buffer is capacitively coupled to a transmission line, the cat-qubit resonator is coupled to a transmon qubit

[15] R. Lescanne, ..., M. Mirrahimi, M. and Z. Leghtas: Exponential suppression of bit-flips in a qubit encoded in an oscillator. 2020, Nat. Phys. , Vol. 16, p. 509-513. See also the patent underlying the startup Alice&Bob.

Quantum analysis of the circuit stabilizing a cat-qubit (1)

Quantum Hamiltonian: two commuting annihilation operators $\boldsymbol{a} = (q_a + \frac{\partial}{\partial q_a})/\sqrt{2}$ and $\boldsymbol{b} = (q_b + \frac{\partial}{\partial q_b})/\sqrt{2}$ with $[\boldsymbol{a}, \boldsymbol{a}^\dagger] = \boldsymbol{I}$, $[\boldsymbol{b}, \boldsymbol{b}^\dagger] = \boldsymbol{I}$

$$\boldsymbol{H}_1(t) = \omega_a \boldsymbol{a}^\dagger \boldsymbol{a} + \omega_b \boldsymbol{b}^\dagger \boldsymbol{b} + 2g \cos\left(\phi_a(\boldsymbol{a} + \boldsymbol{a}^\dagger) + \phi_b(\boldsymbol{b} + \boldsymbol{b}^\dagger) + (2\omega_a - \omega_b)t\boldsymbol{I}\right)$$

Change of frame for $\frac{d}{dt}\boldsymbol{\rho}_1 = -i[\boldsymbol{H}_1(t), \boldsymbol{\rho}_1]$: new density operator

$$\boldsymbol{\rho}_2 = \exp\left(i\omega_a t\boldsymbol{a}^\dagger \boldsymbol{a} + i\omega_b t\boldsymbol{b}^\dagger \boldsymbol{b}\right) \boldsymbol{\rho}_1 \exp\left(-i\omega_a t\boldsymbol{a}^\dagger \boldsymbol{a} - i\omega_b t\boldsymbol{b}^\dagger \boldsymbol{b}\right)$$

is governed by $\frac{d}{dt}\boldsymbol{\rho}_2 = -i[\boldsymbol{H}_2(t), \boldsymbol{\rho}_2]$ with

$$\boldsymbol{H}_2(t) = g e^{i(2\omega_a - \omega_b)t} \exp\left(i\phi_a(e^{-i\omega_a t}\boldsymbol{a} + e^{i\omega_a t}\boldsymbol{a}^\dagger) + i\phi_b(e^{-i\omega_b t}\boldsymbol{b} + e^{i\omega_b t}\boldsymbol{b}^\dagger)\right) + h.c.$$

Expansion up-to order 3 versus $\phi_a, \phi_b \ll 1$:

$$\boldsymbol{H}_2(t) \approx g e^{i(2\omega_a - \omega_b)t}\left(\boldsymbol{I} + i\phi_a(e^{-i\omega_a t}\boldsymbol{a} + e^{i\omega_a t}\boldsymbol{a}^\dagger) - \frac{\phi_a^2}{2}(e^{-i\omega_a t}\boldsymbol{a} + e^{i\omega_a t}\boldsymbol{a}^\dagger)^2 - \frac{i\phi_a^3}{6}(e^{-i\omega_a t}\boldsymbol{a} + e^{i\omega_a t}\boldsymbol{a}^\dagger)^3\right) \dots$$
$$\left(\boldsymbol{I} + i\phi_b(e^{-i\omega_b t}\boldsymbol{b} + e^{i\omega_b t}\boldsymbol{b}^\dagger) - \frac{\phi_b^2}{2}(e^{-i\omega_b t}\boldsymbol{b} + e^{i\omega_b t}\boldsymbol{b}^\dagger)^2 - \frac{i\phi_b^3}{6}(e^{-i\omega_b t}\boldsymbol{b} + e^{i\omega_b t}\boldsymbol{b}^\dagger)^3\right) + h.c.$$

$$\boldsymbol{H}_2(t) \approx g e^{i(2\omega_a - \omega_b)t} \ldots$$
$$\Big(\boldsymbol{I} + i\phi_a(e^{-i\omega_a t}\boldsymbol{a} + e^{i\omega_a t}\boldsymbol{a}^\dagger) - \tfrac{\phi_a^2}{2}(e^{-i\omega_a t}\boldsymbol{a} + e^{i\omega_a t}\boldsymbol{a}^\dagger)^2 - \tfrac{i\phi_a^3}{6}(e^{-i\omega_a t}\boldsymbol{a} + e^{i\omega_a t}\boldsymbol{a}^\dagger)^3\Big)\ldots$$
$$\Big(\boldsymbol{I} + i\phi_b(e^{-i\omega_b t}\boldsymbol{b} + e^{i\omega_b t}\boldsymbol{b}^\dagger) - \tfrac{\phi_b^2}{2}(e^{-i\omega_b t}\boldsymbol{b} + e^{i\omega_b t}\boldsymbol{b}^\dagger)^2 - \tfrac{i\phi_b^3}{6}(e^{-i\omega_b t}\boldsymbol{b} + e^{i\omega_b t}\boldsymbol{b}^\dagger)^3\Big)$$
$$+ h.c.$$

When ω_a/ω_b irrational **only two secular terms** (i.e. non-oscillatory): $-ig_2\boldsymbol{a}^2\boldsymbol{b}^\dagger$ and its Hermitian conjugate $ig_2(\boldsymbol{a}^\dagger)^2\boldsymbol{b}$ where $g_2 = g\phi_a^2\phi_b/2$ (order exceeding 3 in $\phi_a, \phi_b \ll 1$ are neglected).

Justify the following approximate time-invariant Hamiltonian for $\boldsymbol{H}_2$ (**rotating wave approximation**): :

$$\boldsymbol{H}_2(t) \approx -ig_2\boldsymbol{a}^2\boldsymbol{b}^\dagger + ig_2(\boldsymbol{a}^\dagger)^2\boldsymbol{b}.$$

Finer approximations via high-order **averaging** techniques.

Analysis of the circuit stabilizing a cat-qubit (1)

Cat-qubit stored in oscillator $\boldsymbol{a}$, **controller based on a damped oscillator $\boldsymbol{b}$** stabilizing against one decoherence channel (bit-fip):

$$\frac{d}{dt}\rho = -[g_2\boldsymbol{a}^2\boldsymbol{b}^\dagger - g_2(\boldsymbol{a}^\dagger)^2\boldsymbol{b}\,,\ \rho] + [u\boldsymbol{b}^\dagger - u^*\boldsymbol{b}\,,\ \rho] + \kappa_b\Big(\boldsymbol{b}\rho\boldsymbol{b}^\dagger - (\boldsymbol{b}^\dagger\boldsymbol{b}\rho + \rho\boldsymbol{b}^\dagger\boldsymbol{b})/2\Big)$$
$$= -\Big[g_2(\boldsymbol{a}^2 - \alpha^2)\boldsymbol{b}^\dagger - g_2((\boldsymbol{a}^\dagger)^2 - (\alpha)^2)\boldsymbol{b}\,,\ \rho\Big] + \kappa_b\Big(\boldsymbol{b}\rho\boldsymbol{b}^\dagger - (\boldsymbol{b}^\dagger\boldsymbol{b}\rho + \rho\boldsymbol{b}^\dagger\boldsymbol{b})/2\Big)$$

with $\alpha \in \mathbb{C}$ such that $\alpha^2 = u/g_2$, the drive amplitude $u \in \mathbb{C}$ applied to mode $\boldsymbol{b}$ and $1/\kappa_b > 0$ the short life-time of photon in mode $\boldsymbol{b}$.

Any density operator $\bar{\rho} = \bar{\rho}_a \otimes |0\rangle\langle 0|_b$ **is a steady-state as soon as the support of $\bar{\rho}_a$ belongs to the two dimensional vector space spanned by the coherent states $|\alpha\rangle$ and $|\text{-}\alpha\rangle$** (range$(\bar{\rho}_a) \subset$ span$\{|\alpha\rangle, |\text{-}\alpha\rangle\}$) (Schrödinger phase-cat).

Analysis of the circuit stabilizing a cat-qubit (2)

Cat-qubit stored in oscillator ***a***, **controller based low-Q mode *b***:

$$\frac{d}{dt}\boldsymbol{\rho} == -\Big[g_2(\boldsymbol{a}^2-\alpha^2)\boldsymbol{b}^\dagger - g_2((\boldsymbol{a}^\dagger)^2-(\alpha)^2)\boldsymbol{b}\,,\ \boldsymbol{\rho}\Big] + \kappa_b\Big(\boldsymbol{b}\boldsymbol{\rho}\boldsymbol{b}^\dagger - (\boldsymbol{b}^\dagger\boldsymbol{b}\boldsymbol{\rho}+\boldsymbol{\rho}\boldsymbol{b}^\dagger\boldsymbol{b})/2\Big)$$

with $\alpha \in \mathbb{C}$ and $\kappa_b \gg g_2$.

- Any density operators $\bar{\rho} = \bar{\rho}_a \otimes |0\rangle\langle 0|_b$ **is a steady-state as soon as the support of $\bar{\rho}_a$ belongs to the two dimensional vector space spanned by the quasi-classical wave functions $|\alpha\rangle$ and $|\text{-}\alpha\rangle$** (range$(\bar{\rho}_a) \subset$ span$\{|\alpha\rangle, |\text{-}\alpha\rangle\}$) (Schrödinger cat-qubit).
- Usually $\kappa_b \gg |g_2|$, mode ***b*** relaxes rapidly to vaccuum $|0\rangle\langle 0|_b$, can be eliminated adiabatically (**singular perturbations**, second order corrections) to provides the slow evolution of mode ***a***:

$$\frac{d}{dt}\boldsymbol{\rho}_a = \frac{4|g_2|^2}{\kappa_b}\Big((\boldsymbol{a}^2-\alpha^2)\boldsymbol{\rho}_a(\boldsymbol{a}^2-\alpha^2)^\dagger - \tfrac{1}{2}((\boldsymbol{a}^2-\alpha^2)^\dagger(\boldsymbol{a}^2-\alpha^2)\boldsymbol{\rho}_a + \boldsymbol{\rho}_a(\boldsymbol{a}^2-\alpha^2)^\dagger(\boldsymbol{a}^2-\alpha^2))\Big).$$

  Exponential convergence toward the code space span$\{|\alpha\rangle, |\text{-}\alpha\rangle\}$ based on the following exponential Lyapunov function[16]

$$V(\rho_a) = \text{Tr}\left((\boldsymbol{a}^2-\alpha^2)^\dagger(\boldsymbol{a}^2-\alpha^2)\rho_a\right), \qquad \frac{d}{dt}V \leq -\frac{8|g_2|^2}{\kappa_b}V.$$

- Photon-number parity $\text{Tr}\left(e^{i\pi\boldsymbol{a}^\dagger\boldsymbol{a}}\rho\right)$ is invariant since $[\boldsymbol{a}^2, e^{i\pi\boldsymbol{a}^\dagger\boldsymbol{a}}] \equiv 0$.

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[16] For a mathematical proof of convergence analysis in an adapted Banach space, see : R. Azouit, A. Sarlette, PR: Well-posedness and convergence of the Lindblad master equation for a quantum harmonic oscillator with multi-photon drive and damping. 2016, ESAIM: COCV , Vol. 22, No. 4, p. 1353-1369.

Numerical simulation and exponentially protection against bit-flips

Take $|\alpha| \gg 1$ (with $|\alpha| > 3$ one has $\langle \alpha | \text{-}\alpha \rangle \leq e^{-18}$) and the following logical state

$$|0\rangle_L \approx |\alpha\rangle, \qquad |1\rangle_L \approx |\text{-}\alpha\rangle$$

Even and odd cats read

$$|+\rangle_L = \tfrac{1}{\sqrt{2}}(|0\rangle_L + |1\rangle_L) \quad \text{and} \quad |-\rangle_L = \tfrac{1}{\sqrt{2}}(|0\rangle_L - |1\rangle_L).$$

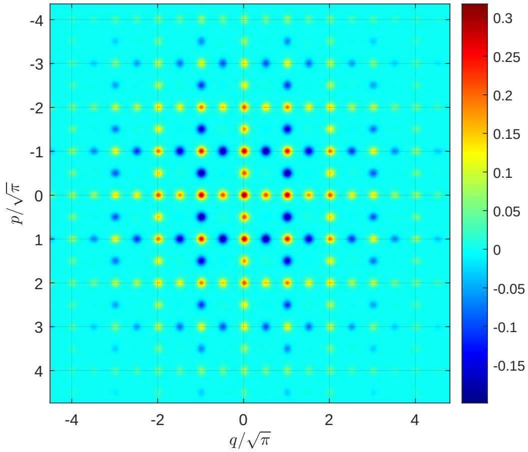
Dynamic governed by the following Lindblad master equation

$$\frac{d}{dt}\boldsymbol{\rho} = \mathbb{D}_{\boldsymbol{L}_0}(\rho) + \kappa_1 \mathbb{D}_{\boldsymbol{L}1}(\boldsymbol{\rho})$$

with $\mathbb{D}_{\boldsymbol{L}}(\rho) \triangleq \boldsymbol{L}\rho\boldsymbol{L}^\dagger - \frac{1}{2}\left(\boldsymbol{L}^\dagger\boldsymbol{L}\rho + \rho\boldsymbol{L}^\dagger\boldsymbol{L}\right)$, two-photon pumping $\boldsymbol{L}_0 = \boldsymbol{a}^2 - \alpha^2$ and the main error channel $\boldsymbol{L}_1 = \boldsymbol{a}$ corresponding to photon losses.
Matlab script `CatQubit.m`:

- $\alpha^2 = 25/2$, $k_1 = 1/10$.
- truncation to $n_{\max} \approx \alpha^2 + 15\alpha$ of the Fock basis
- discretization time $dt = 10^{-3}/\alpha^2$
- numerical integration between $t = 0$ to $t = 10/\alpha^2$ starting from vacuum, $|+\rangle_L$ and $|0\rangle_L$.

Wigner function of a GKP grid-state



Magic logical-qubit state endoded in a GKP grid-state with finite energy:

$$\psi(q) \propto e^{-\epsilon q^2} \left(\sum_k \cos(\tfrac{\pi}{8}) e^{-\frac{(q-2k\sqrt{\pi})^2}{\epsilon}} + \sin(\tfrac{\pi}{8}) e^{-\frac{(q-(2k+1)\sqrt{\pi})^2}{\epsilon}} \right) \text{ with } 0 < \epsilon \ll 1.$$

Outline

Conclusion of these lectures

Topics partially covered:

- Models of open quantum systems based on density operators, Kraus maps and Stochastic Master Equation (SME).
- Positivity preserving numerical schemes for simulation with classical computers.
- Two key quantum systems: qubit (two-level system) and harmonic oscillator (cavity mode).
- Two approximation methods, averaging (RWA) and singular perturbations (adiabatic elimination), for open-loop control and closed-loop stabilization with a quantum controller.
- Convergence analysis based on Lyapunov techniques and super-martingales.

Absent topics;

- Open-loop control: adiabatic control, optimal control, ensemble control and parametric robustness
- Stabilisation with a classical controller: measurement based feedback, quantum error correction.
- State and parameter estimation: quantum filtering and tomography.
- ...