Mathematical methods for modeling and control of open quantum systems[1]

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[1] Lecture-notes, slides and Matlab simulation scripts available at: http://cas.ensmp.fr/~rouchon/LIASFMA/index.html

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Outline

1. Single-frequency averaging and Kapitza's pendulum
2. Averaging and open-loop control of Schrödinger systems
   - Almost periodic open-loop control of Schrödinger systems
   - Lemmas underlying first and second order approximations
   - Rotating Wave Approximation (RWA) recipes
3. Resonant control of a qubit
4. Averaging and control of spin/spring system
   - The spin/spring model
     - Resonant interaction
     - Dispersive interaction
   - Law-Eberly control of a single trapped ion

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Time-periodic non-linear systems

We consider a non-linear ODE of the form:

$$\frac{d}{dt}x = \epsilon f(x,t), \qquad x \in \mathbb{R}^n, \qquad 0 < \epsilon \ll 1,$$

where f is T-periodic in t and depends smoothly on x.

We will see how its solution is well-approximated by the solution of the time-independent system, the averaged system:

$$\frac{d}{dt}z = \epsilon \bar{f}(z)$$

where $\bar{f}(z) = \frac{1}{T}\int_0^T f(z,t)dt$.

Consider $\frac{d}{dt}x = \epsilon f(x, t)$ with $x \in U \subset \mathbb{R}^n$, $0 \leq \epsilon \ll 1$, and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ smooth and period $T > 0$ in t. Also assume U to be bounded.

- If z is the solution of $\frac{d}{dt}z = \epsilon \bar{f}(z)$ with the initial condition z_0, and assuming $|x_0 - z_0| = \mathcal{O}(\epsilon)$, we have $|x(t) - z(t)| = \mathcal{O}(\epsilon)$ on a time-scale $t \sim 1/\epsilon$.
- If $\bar{z}$ is a hyperbolic fixed point of the averaged system then there exists $\epsilon_0 > 0$ such that, for all $0 < \epsilon \leq \epsilon_0$, the main system possesses a unique hyperbolic periodic orbit $\gamma_\epsilon(t) = \bar{z} + \mathcal{O}(\epsilon)$ of the same stability type as $\bar{z}$.

J. Guckenheimer and P. Holmes, Nonlinear oscillations, Dynamical systems and Bifurcation of Vector Fields, Springer, 1983.

Fixed suspension point:

$$\frac{d^2}{dt^2}\theta = \frac{g}{l}\sin\theta$$

g: free fall acceleration, l: pendulum's length, θ: angle to the vertical; $\theta = \pi$ stable and $\theta = 0$ unstable equilibrium.

Suspension point in vertical oscillation:

Dynamics of the suspension point: $z = \frac{v}{\Omega}\cos(\Omega t)$ ($a = v/\Omega > 0$ amplitude and Ω frequency).

Pendulum's dynamics: replace acceleration g by $g + \ddot{z} = g - v\Omega\cos(\Omega t)$,

$$\frac{d}{dt}\theta = \omega, \quad \frac{d}{dt}\omega = \frac{g - v\Omega\cos(\Omega t)}{l}\sin\theta.$$

Replacing the velocity ω by the momentum $p_\theta = \omega + \frac{v\sin(\Omega t)}{l}\sin\theta$:

$$\frac{d}{dt}\theta = p_\theta - \frac{v\sin(\Omega t)}{l}\sin\theta,$$
$$\frac{d}{dt}p_\theta = \left(\frac{g}{l} - \frac{v^2\sin^2(\Omega t)}{l^2}\cos\theta\right)\sin\theta + \frac{v\sin(\Omega t)}{l}p_\theta\cos\theta.$$

For large enough Ω, we can average these time-periodic dynamics over $[t - \pi/\Omega, t + \pi/\Omega]$:

$$\frac{d}{dt}\theta = p_\theta, \quad \frac{d}{dt}p_\theta = \left(\frac{g}{l} - \frac{v^2}{2l^2}\cos\theta\right)\sin\theta.$$

Around $\theta = 0$ the approximation of small angles gives $\frac{d^2}{dt^2}\theta = \frac{g - v^2/2l}{l}\theta$.
If $v^2/2l > g$ then the system becomes stable around $\theta = 0$.

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Bilinear Schrödinger equation

Un-measured quantum system $\rightarrow$ **Bilinear Schrödinger equation**

$$i\frac{d}{dt}|\psi\rangle = (\boldsymbol{H}_0 + u(t)\boldsymbol{H}_1)|\psi\rangle,$$

- $|\psi\rangle \in \mathcal{H}$ the system's wavefunction with $\left\| |\psi\rangle \right\|_{\mathcal{H}} = 1$;
- the free Hamiltonian, $\boldsymbol{H}_0$, is a Hermitian operator defined on $\mathcal{H}$;
- the control Hamiltonian, $\boldsymbol{H}_1$, is a Hermitian operator defined on $\mathcal{H}$;
- the control $u(t) : \mathbb{R}^+ \mapsto \mathbb{R}$ is a scalar control.

Formal computations $\dim(\mathcal{H})$ arbitrary. Mathematical proofs $\dim(\mathcal{H})$ finite

Two key examples:

- Qubit: $\boldsymbol{H}_0 + u(t)\boldsymbol{H}_1 = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \frac{u(t)}{2}\boldsymbol{\sigma_x}$.
- Quantum harmonic oscillator:
  $\boldsymbol{H}_0 + u(t)\boldsymbol{H}_1 = \omega_c(\boldsymbol{a}^\dagger \boldsymbol{a} + \frac{1}{2}) + u(t)(\boldsymbol{a} + \boldsymbol{a}^\dagger)$.

Almost periodic control

We consider the controls of the form

$$u(t) = \epsilon \left(\sum_{j=1}^{r} \boldsymbol{u}_j e^{i\omega_j t} + \boldsymbol{u}_j^* e^{-i\omega_j t} \right)$$

- $\epsilon > 0$ is a small parameter;
- $\epsilon \boldsymbol{u}_j$ is the constant complex amplitude associated to the pulsation $\omega_j \geq 0$;
- r stands for the number of independent frequencies ($\omega_j \neq \omega_k$ for $j \neq k$).

We are interested in approximations, for ϵ tending to 0^+, of trajectories $t \mapsto |\psi_\epsilon\rangle_t$ of

$$\frac{d}{dt}|\psi_\epsilon\rangle = \left(\boldsymbol{A}_0 + \epsilon \left(\sum_{j=1}^{r} \boldsymbol{u}_j e^{i\omega_j t} + \boldsymbol{u}_j^* e^{-i\omega_j t} \right) \boldsymbol{A}_1 \right) |\psi_\epsilon\rangle$$

where $\boldsymbol{A}_0 = -i\boldsymbol{H}_0$ and $\boldsymbol{A}_1 = -i\boldsymbol{H}_1$ are skew-Hermitian.

Rotating frame

Consider the following change of variables

$$|\psi_\epsilon\rangle_t = e^{\boldsymbol{A}_0 t}|\phi_\epsilon\rangle_t.$$

The resulting system is said to be in the "interaction frame"

$$\frac{d}{dt}|\phi_\epsilon\rangle = \epsilon \boldsymbol{B}(t)|\phi_\epsilon\rangle$$

where $\boldsymbol{B}(t)$ is a skew-Hermitian operator whose time-dependence is almost periodic:

$$B(t) = \sum_{j=1}^{r} \boldsymbol{u}_j e^{i\omega_j t} e^{-\boldsymbol{A}_0 t} \boldsymbol{A}_1 e^{\boldsymbol{A}_0 t} + \boldsymbol{u}_j^* e^{-i\omega_j t} e^{-\boldsymbol{A}_0 t} \boldsymbol{A}_1 e^{\boldsymbol{A}_0 t}.$$

Main idea

We can write

$$\boldsymbol{B}(t) = \bar{\boldsymbol{B}} + \frac{d}{dt}\widetilde{\boldsymbol{B}}(t),$$

where $\bar{\boldsymbol{B}}$ is a constant skew-Hermitian matrix and $\widetilde{\boldsymbol{B}}(t)$ is a bounded almost periodic skew-Hermitian matrix.

Multi-frequency averaging: first order

Consider the two systems

$$\frac{d}{dt}|\phi_\epsilon\rangle = \epsilon\left(\bar{\boldsymbol{B}} + \frac{d}{dt}\widetilde{\boldsymbol{B}}(t)\right)|\phi_\epsilon\rangle,$$

and

$$\frac{d}{dt}|\phi_\epsilon^{1^{\mathrm{st}}}\rangle = \epsilon\bar{\boldsymbol{B}}|\phi_\epsilon^{1^{\mathrm{st}}}\rangle,$$

initialized at the same state $|\phi_\epsilon^{1^{\mathrm{st}}}\rangle_0 = |\phi_\epsilon\rangle_0$.

Theorem: first order approximation (Rotating Wave Approximation)

Consider the functions $|\phi_\epsilon\rangle$ and $|\phi_\epsilon^{1^{\mathrm{st}}}\rangle$ initialized at the same state and following the above dynamics. Then, there exist $M > 0$ and $\eta > 0$ such that for all $\epsilon \in]0, \eta[$ we have

$$\max_{t\in\left[0,\frac{1}{\epsilon}\right]} \left\| |\phi_\epsilon\rangle_t - |\phi_\epsilon^{1^{\mathrm{st}}}\rangle_t \right\| \le M\epsilon$$

Multi-frequency averaging: first order

Proof's idea

Almost periodic change of variables:

$$|\chi_\epsilon\rangle = (1 - \epsilon\widetilde{\boldsymbol{B}}(t))|\phi_\epsilon\rangle$$

well-defined for $\epsilon > 0$ sufficiently small.
The dynamics can be written as

$$\frac{d}{dt}|\chi_\epsilon\rangle = (\epsilon\bar{\boldsymbol{B}} + \epsilon^2\boldsymbol{F}(\epsilon, t))|\chi_\epsilon\rangle$$

where $\boldsymbol{F}(\epsilon, t)$ is uniformly bounded in time.

Multi-frequency averaging: second order

More precisely, the dynamics of $|\chi_\epsilon\rangle$ is given by

$$\frac{d}{dt}|\chi_\epsilon\rangle = \left(\epsilon\bar{\boldsymbol{B}} + \epsilon^2[\bar{\boldsymbol{B}}, \widetilde{\boldsymbol{B}}(t)] - \epsilon^2\widetilde{\boldsymbol{B}}(t)\frac{d}{dt}\widetilde{\boldsymbol{B}}(t) + \epsilon^3\boldsymbol{E}(\epsilon, t)\right)|\chi_\epsilon\rangle$$

- $\boldsymbol{E}(\epsilon, t)$ is still almost periodic but its entries are no more linear combinations of time-exponentials;
- $\widetilde{\boldsymbol{B}}(t)\frac{d}{dt}\widetilde{\boldsymbol{B}}(t)$ is an almost periodic operator whose entries are linear combinations of oscillating time-exponentials.

We can write

$$\widetilde{\boldsymbol{B}}(t) = \frac{d}{dt}\widetilde{\boldsymbol{C}}(t) \qquad \text{and} \qquad \widetilde{\boldsymbol{B}}(t)\frac{d}{dt}\widetilde{\boldsymbol{B}}(t) = \bar{\boldsymbol{D}} + \frac{d}{dt}\widetilde{\boldsymbol{D}}(t)$$

where $\widetilde{\boldsymbol{C}}(t)$ and $\widetilde{\boldsymbol{D}}(t)$ are almost periodic. We have

$$\frac{d}{dt}|\chi_\epsilon\rangle = \left(\epsilon\bar{\boldsymbol{B}} - \epsilon^2\bar{\boldsymbol{D}} + \epsilon^2\frac{d}{dt}\left([\bar{\boldsymbol{B}}, \widetilde{\boldsymbol{C}}(t)] - \widetilde{\boldsymbol{D}}(t)\right) + \epsilon^3\boldsymbol{E}(\epsilon, t)\right)|\chi_\epsilon\rangle$$

where the skew-Hermitian operators $\bar{\boldsymbol{B}}$ and $\bar{\boldsymbol{D}}$ are constants and the other ones $\widetilde{\boldsymbol{C}}$, $\widetilde{\boldsymbol{D}}$, and $\boldsymbol{E}$ are almost periodic.

Multi-frequency averaging: second order

Consider the two systems

$$\frac{d}{dt}|\phi_\epsilon\rangle = \epsilon\left(\bar{\boldsymbol{B}} + \frac{d}{dt}\widetilde{\boldsymbol{B}}(t)\right)|\phi_\epsilon\rangle,$$

and

$$\frac{d}{dt}|\phi_\epsilon^{2^{\text{nd}}}\rangle = (\epsilon\bar{\boldsymbol{B}} - \epsilon^2\bar{\boldsymbol{D}})|\phi_\epsilon^{2^{\text{nd}}}\rangle,$$

both initialized at $|\phi_\epsilon\rangle_0$.

Theorem: second order approximation

Consider $|\phi_\epsilon\rangle_t$ and $|\phi_\epsilon^{2^{\text{nd}}}\rangle_t$ solutions of the above dynamics. Then, there exist $M > 0$ and $\eta > 0$ such that for all $\epsilon \in]0, \eta]$ we have

$$\max_{t\in\left[0,\frac{1}{\epsilon}\right]} \left\| |\phi_\epsilon\rangle_t - (I + \epsilon\widetilde{\boldsymbol{B}}(t))|\phi_\epsilon^{2^{\text{nd}}}\rangle_t \right\| \leq M\boldsymbol{\epsilon^2}$$

$$\max_{t\in\left[0,\frac{1}{\epsilon^2}\right]} \left\| |\phi_\epsilon\rangle_t - |\phi_\epsilon^{2^{\text{nd}}}\rangle_t \right\| \leq M\boldsymbol{\epsilon}$$

Multi-frequency averaging: second order

Proof's idea

Another almost periodic change of variables

$$|\xi_\epsilon\rangle = \left(\boldsymbol{I} - \epsilon^2 \left([\bar{\boldsymbol{B}}, \widetilde{\boldsymbol{C}}(t)] - \widetilde{\boldsymbol{D}}(t)\right)\right) |\chi_\epsilon\rangle.$$

The dynamics can be written as

$$\frac{d}{dt}|\xi_\epsilon\rangle = \left(\epsilon\bar{\boldsymbol{B}} - \epsilon^2\bar{\boldsymbol{D}} + \epsilon^3\boldsymbol{F}(\epsilon, t)\right)|\xi_\epsilon\rangle$$

where $\epsilon\bar{\boldsymbol{B}} - \epsilon^2\bar{\boldsymbol{D}}$ is skew Hermitian and $\boldsymbol{F}$ is almost periodic and therefore uniformly bounded in time.

The Rotating Wave Approximation (RWA) recipes

Schrödinger dynamics $i\frac{d}{dt}|\psi\rangle = \boldsymbol{H}(t)|\psi\rangle$, with

$$\boldsymbol{H}(t) = \boldsymbol{H}_0 + \sum_{k=1}^{m} u_k(t)\boldsymbol{H}_k, \qquad u_k(t) = \sum_{j=1}^{r} \boldsymbol{u}_{k,j}e^{i\omega_j t} + \boldsymbol{u}^*_{k,j}e^{-i\omega_j t}.$$

The Hamiltonian in interaction frame

$$\boldsymbol{H}_{\text{int}}(t) = \sum_{k,j} \left(\boldsymbol{u}_{k,j}e^{i\omega_j t} + \boldsymbol{u}^*_{k,j}e^{-i\omega_j t}\right) e^{i\boldsymbol{H}_0 t}\boldsymbol{H}_k e^{-i\boldsymbol{H}_0 t}$$

We define the first order Hamiltonian

$$\boldsymbol{H}_{\text{rwa}}^{1^{\text{st}}} = \overline{\boldsymbol{H}_{\text{int}}} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \boldsymbol{H}_{\text{int}}(t)dt,$$

and the second order Hamiltonian

$$\boldsymbol{H}_{\text{rwa}}^{2^{\text{nd}}} = \boldsymbol{H}_{\text{rwa}}^{1^{\text{st}}} - i\overline{\left(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}}\right)\left(\int_t (\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}})\right)}$$

Choose the amplitudes $\boldsymbol{u}_{k,j}$ and the frequencies ω_j such that the propagators of $\boldsymbol{H}_{\text{rwa}}^{1^{\text{st}}}$ or $\boldsymbol{H}_{\text{rwa}}^{2^{\text{nd}}}$ admit simple explicit forms that are used to find $t \mapsto u(t)$ steering $|\psi\rangle$ from one location to another one.

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RWA and resonant control

In

$$i\frac{d}{dt}|\psi\rangle = \left(\frac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \frac{u(t)}{2}\boldsymbol{\sigma_x}\right)|\psi\rangle,$$

take a resonant control $u(t) = \boldsymbol{u}e^{i\omega_{eg}t} + \boldsymbol{u}^* e^{-i\omega_{eg}t}$ with $\boldsymbol{u}$ slowly varying complex amplitude $\left|\frac{d}{dt}\boldsymbol{u}\right| \ll \omega_{eg}|\boldsymbol{u}|$. Set $H_0 = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z}$ and $\epsilon H_1 = \frac{u}{2}\boldsymbol{\sigma_x}$ and consider $|\psi\rangle = e^{-\frac{i\omega_{eg}t}{2}\boldsymbol{\sigma_z}}|\phi\rangle$ to eliminate the drift H_0 and to get the Hamiltonian in the interaction frame:

$$i\frac{d}{dt}|\phi\rangle = \frac{u(t)}{2}e^{\frac{i\omega_{eg}t}{2}\boldsymbol{\sigma_z}}\boldsymbol{\sigma_x}e^{-\frac{i\omega_{eg}t}{2}\boldsymbol{\sigma_z}}|\phi\rangle = \boldsymbol{H}_{\text{int}}|\phi\rangle$$

with $\boldsymbol{H}_{\text{int}} = \frac{u(t)}{2}e^{i\omega_{eg}t}\overbrace{\frac{\boldsymbol{\sigma_x} + i\boldsymbol{\sigma_y}}{2}}^{\boldsymbol{\sigma_+}=|e\rangle\langle g|} + \frac{u(t)}{2}e^{-i\omega_{eg}t}\overbrace{\frac{\boldsymbol{\sigma_x} - i\boldsymbol{\sigma_y}}{2}}^{\boldsymbol{\sigma_-}=|g\rangle\langle e|}$

The RWA consists in neglecting the oscillating terms at frequency $2\omega_{eg}$ when $|\boldsymbol{u}| \ll \omega_{eg}$:

$$H_{int} = \left(\frac{\boldsymbol{u}e^{2i\omega_{eg}t} + \boldsymbol{u}^*}{2}\right)\boldsymbol{\sigma_+} + \left(\frac{\boldsymbol{u} + \boldsymbol{u}^* e^{-2i\omega_{eg}t}}{2}\right)\boldsymbol{\sigma_-}.$$

Thus

$$\overline{H_{int}} = \frac{\boldsymbol{u}^*\boldsymbol{\sigma_+} + \boldsymbol{u}\boldsymbol{\sigma_-}}{2}.$$

The decomposition of $\boldsymbol{H}_{\text{int}}$,

$$\boldsymbol{H}_{\text{int}} = \underbrace{\frac{u^*}{2}\boldsymbol{\sigma}_+ + \frac{u}{2}\boldsymbol{\sigma}_-}_{\overline{\boldsymbol{H}_{\text{int}}}} + \underbrace{\frac{u e^{2i\omega_{\text{eg}}t}}{2}\boldsymbol{\sigma}_+ + \frac{u^* e^{-2i\omega_{\text{eg}}t}}{2}\boldsymbol{\sigma}_-}_{\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}}},$$

provides the first order approximation (RWA) $\boldsymbol{H}_{\text{rwa}}^{1^{\text{st}}} = \overline{\boldsymbol{H}_{\text{int}}} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \boldsymbol{H}_{\text{int}}(t)dt$, and also the second order approximation $\boldsymbol{H}_{\text{rwa}}^{2^{\text{nd}}} = \boldsymbol{H}_{\text{rwa}}^{1^{\text{st}}} - i\overline{\left(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}}\right)\left(\int_t (\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}})\right)}$. Since $\int_t \boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}} = \frac{u e^{2i\omega_{\text{eg}}t}}{4i\omega_{\text{eg}}}\boldsymbol{\sigma}_+ - \frac{u^* e^{-2i\omega_{\text{eg}}t}}{4i\omega_{\text{eg}}}\boldsymbol{\sigma}_-$, we have

$$\overline{\left(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}}\right)\left(\int_t (\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}})\right)} = -\frac{|u|^2}{8i\omega_{\text{eg}}}\boldsymbol{\sigma}_z$$

(use ${\boldsymbol{\sigma}_+}^2 = {\boldsymbol{\sigma}_-}^2 = 0$ and $\boldsymbol{\sigma}_z = \boldsymbol{\sigma}_+\boldsymbol{\sigma}_- - \boldsymbol{\sigma}_-\boldsymbol{\sigma}_+$).
The second order approximation reads:

$$\boldsymbol{H}_{\text{rwa}}^{2^{\text{nd}}} = \boldsymbol{H}_{\text{rwa}}^{1^{\text{st}}} + \left(\frac{|u|^2}{8\omega_{\text{eg}}}\right)\boldsymbol{\sigma}_z = \frac{u^*}{2}\boldsymbol{\sigma}_+ + \frac{u}{2}\boldsymbol{\sigma}_- + \left(\frac{|u|^2}{8\omega_{\text{eg}}}\right)\boldsymbol{\sigma}_z.$$

The 2nd order correction $\frac{|u|^2}{4\omega_{\text{eg}}}(\boldsymbol{\sigma}_z/2)$ is called the Bloch-Siegert shift.

Exercise: controllability of the 2-level systems and Rabi oscillation

Take the first order approximation

$$(\Sigma) \quad i\frac{d}{dt}|\phi\rangle = \frac{(\boldsymbol{u}^*\boldsymbol{\sigma_+} + \boldsymbol{u}\boldsymbol{\sigma_-})}{2}|\phi\rangle = \frac{(\boldsymbol{u}^*|e\rangle\langle g| + \boldsymbol{u}|g\rangle\langle e|)}{2}|\phi\rangle$$

with control $\boldsymbol{u} \in \mathbb{C}$.

1. Take constant control $\boldsymbol{u}(t) = \Omega_r e^{i\theta}$ for $t \in [0, T]$, $T > 0$. Show that $i\frac{d}{dt}|\phi\rangle = \frac{\Omega_r(\cos\theta\boldsymbol{\sigma_x} + \sin\theta\boldsymbol{\sigma_y})}{2}|\phi\rangle$.
2. Set $\Theta_r = \frac{\Omega_r}{2}T$. Show that the solution at T of the propagator $\boldsymbol{U}_t \in SU(2)$, $i\frac{d}{dt}\boldsymbol{U} = \frac{\Omega_r(\cos\theta\boldsymbol{\sigma_x} + \sin\theta\boldsymbol{\sigma_y})}{2}\boldsymbol{U}$, $\boldsymbol{U}_0 = \boldsymbol{I}$ is given by
$$\boldsymbol{U}_T = \cos\Theta_r\boldsymbol{I} - i\sin\Theta_r\left(\cos\theta\boldsymbol{\sigma_x} + \sin\theta\boldsymbol{\sigma_y}\right),$$
3. Take a wave function $|\bar{\phi}\rangle$. Show that exist Ω_r and θ such that $U_T|g\rangle = e^{i\alpha}|\bar{\phi}\rangle$, where α is some global phase.
4. Prove that for any given two wave functions $|\phi_a\rangle$ and $|\phi_b\rangle$ exists a piece-wise constant control $[0, 2T] \ni t \mapsto \boldsymbol{u}(t) \in \mathbb{C}$ such that the solution of (Σ) with $|\phi\rangle_0 = |\phi_a\rangle$ satisfies $|\phi\rangle_{2T} = e^{i\beta}|\phi_b\rangle$ for some global phase β.

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The spin/spring model

The Schrödinger system

$$i\frac{d}{dt}|\psi\rangle = \left(\frac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\mathbf{I}}{2}\right) + i\frac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a})\right)|\psi\rangle$$

corresponds to two coupled scalar PDE's:

$$i\frac{\partial\psi_e}{\partial t} = +\frac{\omega_{eg}}{2}\psi_e + \frac{\omega_c}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_e - i\frac{\Omega}{\sqrt{2}}\frac{\partial}{\partial x}\psi_g$$

$$i\frac{\partial\psi_g}{\partial t} = -\frac{\omega_{eg}}{2}\psi_g + \frac{\omega_c}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_g - i\frac{\Omega}{\sqrt{2}}\frac{\partial}{\partial x}\psi_e$$

since $\boldsymbol{a} = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)$ and $|\psi\rangle$ corresponds to $(\psi_e(x,t), \psi_g(x,t))$ where $\psi_e(.,t), \psi_g(.,t) \in L^2(\mathbb{R},\mathbb{C})$ and $\|\psi_e\|^2 + \|\psi_g\|^2 = 1$.

In

$$\boldsymbol{H} = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\mathbf{I}}{2}\right) + i\frac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a})$$

take $\omega_{eg} = \omega_c + \Delta$ with $|\Omega|, |\Delta| \ll \omega_c$. Then $\boldsymbol{H} = \boldsymbol{H}_0 + \epsilon\boldsymbol{H}_1$ where ϵ is a small parameter and

$$\boldsymbol{H}_0 = \frac{\omega}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\mathbf{I}}{2}\right)$$

$$\epsilon\boldsymbol{H}_1 = \frac{\Delta}{2}\boldsymbol{\sigma_z} + i\frac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a}).$$

$\boldsymbol{H}_{\text{int}}$ is obtained by setting $|\psi\rangle = e^{-i\omega_c t\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\mathbf{I}}{2}\right)} e^{\frac{-i\omega_c t}{2}\boldsymbol{\sigma_z}}|\phi\rangle$ in $i\frac{d}{dt}|\psi\rangle = \boldsymbol{H}|\psi\rangle$ to get $i\frac{d}{dt}|\phi\rangle = \boldsymbol{H}_{\text{int}}|\phi\rangle$ with

$$\boldsymbol{H}_{\text{int}} = \frac{\Delta}{2}\boldsymbol{\sigma_z} + i\frac{\Omega}{2}\left(e^{-i\omega_c t}\boldsymbol{\sigma_-} + e^{i\omega_c t}\boldsymbol{\sigma_+}\right)\left(e^{i\omega_c t}\boldsymbol{a}^\dagger - e^{-i\omega_c t}\boldsymbol{a}\right)$$

where we used

$$e^{\frac{i\theta}{2}\boldsymbol{\sigma_z}}\,\boldsymbol{\sigma_x} e^{-\frac{i\theta}{2}\boldsymbol{\sigma_z}} = e^{-i\theta}\boldsymbol{\sigma_-} + e^{i\theta}\boldsymbol{\sigma_+}, \quad e^{i\theta\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\mathbf{I}}{2}\right)}\,\boldsymbol{a}\,e^{-i\theta\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\mathbf{I}}{2}\right)} = e^{-i\theta}\boldsymbol{a}$$

Resonant case: first order (Jaynes-Cummings Hamiltonian)

The secular terms in $\boldsymbol{H}_{\text{int}}$ are given by (RWA, first order approximation)

$$\boldsymbol{H}^{1^{\text{st}}}_{\text{rwa}} = \tfrac{\Delta}{2}\boldsymbol{\sigma_z} + i\tfrac{\Omega}{2}\left(\boldsymbol{\sigma_- a}^\dagger - \boldsymbol{\sigma_+ a}\right).$$

Since quantum state $|\phi\rangle = e^{+i\omega_c t\left(\boldsymbol{a}^\dagger \boldsymbol{a} + \frac{\mathbf{I}}{2}\right)} e^{\frac{+i\omega_c t}{2}\boldsymbol{\sigma_z}}|\psi\rangle$ obeys approximatively to $i\frac{d}{dt}|\phi\rangle = \boldsymbol{H}^{1^{\text{st}}}_{\text{rwa}}|\phi\rangle$, the original quantum state $|\psi\rangle$ is governed by

$$i\frac{d}{dt}|\psi\rangle = \left(\tfrac{\omega_c+\Delta}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\mathbf{I}}{2}\right) + i\tfrac{\Omega}{2}\left(\boldsymbol{\sigma_- a}^\dagger - \boldsymbol{\sigma_+ a}\right)\right)|\psi\rangle$$

The Jaynes-Cummings Hamiltonian ($\omega_{\text{eg}} = \omega_c + \Delta$ with $|\Delta|, |\Omega| \ll \omega_c$) reads:

$$\boldsymbol{H}_{JC} = \tfrac{\omega_c+\Delta}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\mathbf{I}}{2}\right) + i\tfrac{\Omega}{2}\left(\boldsymbol{\sigma_- a}^\dagger - \boldsymbol{\sigma_+ a}\right)$$

The corresponding PDE is (case $\Delta = 0$) :

$$i\frac{\partial\psi_e}{\partial t} = \frac{\omega_c+\Delta}{2}\psi_e + \frac{\omega_c}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_e - i\tfrac{\Omega}{2\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right)\psi_g$$

$$i\frac{\partial\psi_g}{\partial t} = -\frac{\omega_c+\Delta}{2}\psi_g + \frac{\omega_c}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_g + i\tfrac{\Omega}{2\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)\psi_e$$

Dispersive case: passage to the interaction frame

For $\omega_c \gg |\Delta| \gg |\Omega|$, the dominant term in

$$\boldsymbol{H}_{\text{rwa}}^{1^{\text{st}}} = \frac{\Delta}{2}\boldsymbol{\sigma_z} + i\frac{\Omega}{2}\left(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}\right)$$

is an isolated qubit. To make the interaction dominant, we go to the interaction frame with $(\omega_{\text{eg}} = \omega_c + \Delta)$

$$\boldsymbol{H}_0 = \frac{\omega_{\text{eg}}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right), \quad \epsilon\boldsymbol{H}_1 = i\frac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a}).$$

By setting $|\psi\rangle = e^{-i\omega_c t\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right)} e^{\frac{-i\omega_{\text{eg}}t}{2}\boldsymbol{\sigma_z}}|\phi\rangle$ we get $i\frac{d}{dt}|\phi\rangle = \boldsymbol{H}_{\text{int}}|\phi\rangle$ with

$$\begin{aligned}\boldsymbol{H}_{\text{int}} &= i\frac{\Omega}{2}\left(e^{-i\omega_{\text{eg}}t}\boldsymbol{\sigma_-} + e^{i\omega_{\text{eg}}t}\boldsymbol{\sigma_+}\right)\left(e^{i\omega_c t}\boldsymbol{a}^\dagger - e^{-i\omega_c t}\boldsymbol{a}\right)\\ &= i\frac{\Omega}{2}\left(e^{-i\Delta t}\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - e^{i\Delta t}\boldsymbol{\sigma_+}\boldsymbol{a} + e^{i(2\omega_c+\Delta)t}\boldsymbol{\sigma_+}\boldsymbol{a}^\dagger - e^{-i(2\omega_c+\Delta)t}\boldsymbol{\sigma_-}\boldsymbol{a}\right)\end{aligned}$$

Thus $\boldsymbol{H}_{\text{rwa}}^{1^{\text{st}}} = \overline{\boldsymbol{H}_{\text{int}}} = 0$: no secular term. We have to compute $\boldsymbol{H}_{\text{rwa}}^{2^{\text{nd}}} = \overline{\boldsymbol{H}_{\text{int}}} - i\overline{\left(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}}\right)\left(\int_t(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}})\right)}$ where $\int_t(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}})$ corresponds to

$$\frac{-\Omega}{2}\left(\frac{e^{-i\Delta t}}{\Delta}\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger + \frac{e^{i\Delta t}}{\Delta}\boldsymbol{\sigma_+}\boldsymbol{a} - \frac{e^{i(2\omega_c+\Delta)t}}{2\omega_c+\Delta}\boldsymbol{\sigma_+}\boldsymbol{a}^\dagger - \frac{e^{-i(2\omega_c+\Delta)t}}{2\omega_c+\Delta}\boldsymbol{\sigma_-}\boldsymbol{a}\right)$$

The secular terms in $\boldsymbol{H}_{\text{rwa}}^{2^{\text{nd}}}$ are

$$\frac{-\Omega^2}{4\Delta}\left(\boldsymbol{\sigma_-\sigma_+a^\dagger a}-\boldsymbol{\sigma_+\sigma_-aa^\dagger}\right)+\frac{-\Omega^2}{4(\omega_c+\omega_{\text{eg}})}\left(\boldsymbol{\sigma_-\sigma_+aa^\dagger}-\boldsymbol{\sigma_+\sigma_-a^\dagger a}\right)$$

Since $|\Omega| \ll |\Delta| \ll \omega_{\text{eg}}, \omega_c$, we have $\frac{\Omega^2}{4(\omega_c+\omega_{\text{eg}})} \ll \frac{\Omega^2}{4\Delta}$

$$\boldsymbol{H}_{\text{rwa}}^{2^{\text{nd}}} \approx \frac{\Omega^2}{4\Delta}\left(\boldsymbol{\sigma_z}\left(\boldsymbol{N}+\tfrac{\boldsymbol{I}}{2}\right)+\tfrac{\boldsymbol{I}}{2}\right).$$

Since quantum state $|\phi\rangle = e^{+i\omega_c t\left(\boldsymbol{N}+\frac{\boldsymbol{I}}{2}\right)} e^{\frac{+i\omega_{\text{eg}}t}{2}\boldsymbol{\sigma_z}}|\psi\rangle$ obeys approximatively to $i\frac{d}{dt}|\phi\rangle = \boldsymbol{H}_{\text{rwa}}^{2^{\text{nd}}}|\phi\rangle$, the original quantum state $|\psi\rangle$ is governed by $i\frac{d}{dt}|\psi\rangle = \left(\boldsymbol{H}_{disp}+\frac{\Omega^2}{8\Delta}\right)|\psi\rangle$ with

$$\boldsymbol{H}_{disp} = \frac{\omega_{\text{eg}}}{2}\boldsymbol{\sigma_z}+\omega_c\left(\boldsymbol{N}+\tfrac{\boldsymbol{I}}{2}\right)-\frac{\chi}{2}\boldsymbol{\sigma_z}\left(\boldsymbol{N}+\tfrac{\boldsymbol{I}}{2}\right) \quad \text{and } \chi = \frac{-\Omega^2}{2\Delta}$$

The corresponding PDE is :

$$i\frac{\partial\psi_e}{\partial t} = +\frac{\omega_{\text{eg}}}{2}\psi_e+\frac{1}{2}\left(\omega_c-\frac{\chi}{2}\right)\left(x^2-\frac{\partial^2}{\partial x^2}\right)\psi_e$$
$$i\frac{\partial\psi_g}{\partial t} = -\frac{\omega_{\text{eg}}}{2}\psi_g+\frac{1}{2}\left(\omega_c+\frac{\chi}{2}\right)\left(x^2-\frac{\partial^2}{\partial x^2}\right)\psi_g$$

Exercise: resonant spin-spring system with controls

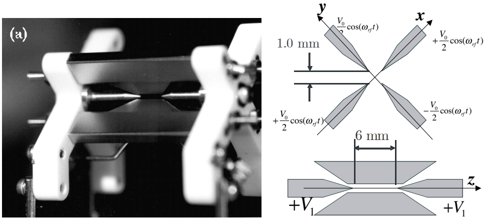
Consider the resonant spin-spring model with $\Omega \ll |\omega|$:

$$\boldsymbol{H} = \frac{\omega}{2}\boldsymbol{\sigma_z} + \omega\left(\boldsymbol{a}^\dagger \boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right) + i\frac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a}) + u(\boldsymbol{a} + \boldsymbol{a}^\dagger)$$

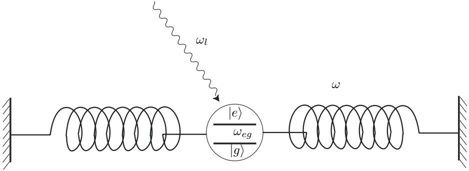
with a real control input $u(t) \in \mathbb{R}$:

1. Show that with the resonant control $u(t) = \boldsymbol{u}e^{-i\omega t} + \boldsymbol{u}^* e^{i\omega t}$ with complex amplitude $\boldsymbol{u}$ such that $|\boldsymbol{u}| \ll \omega$, the first order RWA approximation yields the following dynamics in the interaction frame:
$$i\frac{d}{dt}|\psi\rangle = \left(i\frac{\Omega}{2}(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}) + \boldsymbol{u}\boldsymbol{a}^\dagger + \boldsymbol{u}^*\boldsymbol{a}\right)|\psi\rangle$$
2. Set $\mathbf{v} \in \mathbb{C}$ solution of $\frac{d}{dt}\mathbf{v} = -i\boldsymbol{u}$ and consider the following change of frame $|\phi\rangle = D_{-\mathbf{v}}|\psi\rangle$ with the displacement operator $D_{-\mathbf{v}} = e^{-\mathbf{v}\boldsymbol{a}^\dagger + \mathbf{v}^*\boldsymbol{a}}$. Show that, up to a global phase change, we have, with $\tilde{\boldsymbol{u}} = i\frac{\Omega}{2}\mathbf{v}$,
$$i\frac{d}{dt}|\phi\rangle = \left(\frac{i\Omega}{2}(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}) + (\tilde{\boldsymbol{u}}\boldsymbol{\sigma_+} + \tilde{\boldsymbol{u}}^*\boldsymbol{\sigma_-})\right)|\phi\rangle$$
3. Take the orthonormal basis $\{|g,n\rangle, |e,n\rangle\}$ with $n \in \mathbb{N}$ being the photon number and where for instance $|g,n\rangle$ stands for the tensor product $|g\rangle \otimes |n\rangle$. Set $|\phi\rangle = \sum_n \phi_{g,n}|g,n\rangle + \phi_{e,n}|e,n\rangle$ with $\phi_{g,n}, \phi_{e,n} \in \mathbb{C}$ depending on t and $\sum_n |\phi_{g,n}|^2 + |\phi_{e,n}|^2 = 1$. Show that, for $n \geq 0$
$$i\frac{d}{dt}\phi_{g,n+1} = i\frac{\Omega}{2}\sqrt{n+1}\phi_{e,n} + \tilde{\boldsymbol{u}}^*\phi_{e,n+1}, \quad i\frac{d}{dt}\phi_{e,n} = -i\frac{\Omega}{2}\sqrt{n+1}\phi_{g,n+1} + \tilde{\boldsymbol{u}}\phi_{g,n}$$
and $i\frac{d}{dt}\phi_{g,0} = \tilde{\boldsymbol{u}}^*\phi_{e,0}$.
4. Assume that $|\phi\rangle_0 = |g,0\rangle$. Construct an open-loop control $[0,T] \ni t \mapsto \tilde{\boldsymbol{u}}(t)$ such that $|\phi\rangle_T \approx |g,1\rangle$ (hint: use an impulse for $t \in [0,\epsilon]$ followed by 0 on $[\epsilon, T]$ with $\epsilon \ll T$ and well chosen T).
5. Generalize the above open-loop control when the goal state $|\phi\rangle_T$ is $|g,n\rangle$ with any arbitrary photon number n.

A single trapped ion



1D ion trap, picture borrowed from S. Haroche course at CDF.



A classical cartoon of spin-spring system.

A single trapped ion

A composite system:

internal degree of freedom+vibration inside the 1D trap

Hilbert space:

$$\mathbb{C}^2 \otimes L^2(\mathbb{R}, \mathbb{C})$$

Hamiltonian:

$$\boldsymbol{H} = \omega_m \left(\boldsymbol{a}^\dagger \boldsymbol{a} + \frac{\mathbf{I}}{2} \right) + \frac{\omega_{\text{eg}}}{2} \boldsymbol{\sigma_z} + \left(u_l e^{i(\omega_l t - \eta_l (\boldsymbol{a} + \boldsymbol{a}^\dagger))} + u_l^* e^{-i(\omega_l t - \eta_l (\boldsymbol{a} + \boldsymbol{a}^\dagger))} \right) \boldsymbol{\sigma_x}$$

Parameters:
ω_m: harmonic oscillator of the trap,
ω_{eg}: optical transition of the internal state,
ω_l: lasers frequency,
$\eta_l = \omega_l / c$: Lamb-Dicke parameter.
Scales:

$$|\omega_l - \omega_{\text{eg}}| \ll \omega_{\text{eg}}, \quad \omega_m \ll \omega_{\text{eg}}, \quad |u_l| \ll \omega_{\text{eg}}, \quad \left| \frac{d}{dt} u_l \right| \ll \omega_{\text{eg}} |u_l|.$$

PDE formulation

The Schrödinger equation $i\frac{d}{dt}|\psi\rangle = \boldsymbol{H}|\psi\rangle$, with

$$\boldsymbol{H} = \omega_m \left(\boldsymbol{a}^\dagger \boldsymbol{a} + \frac{\mathbf{I}}{2} \right) + \frac{\omega_{eg}}{2} \boldsymbol{\sigma_z} + \left(u_l e^{i(\omega_l t - \eta_l (\boldsymbol{a} + \boldsymbol{a}^\dagger))} + u_l^* e^{-i(\omega_l t - \eta_l (\boldsymbol{a} + \boldsymbol{a}^\dagger))} \right) \boldsymbol{\sigma_x}$$

can be written in the form

$$i\frac{\partial \psi_g}{\partial t} = \frac{\omega_m}{2} \left(x^2 - \frac{\partial^2}{\partial x^2} \right) \psi_g - \frac{\omega_{eg}}{2} \psi_g + \left(u_l e^{i(\omega_l t - \sqrt{2}\eta_l x)} + u_l^* e^{-i(\omega_l t - \sqrt{2}\eta_l x)} \right) \psi_e,$$

$$i\frac{\partial \psi_e}{\partial t} = \frac{\omega_m}{2} \left(x^2 - \frac{\partial^2}{\partial x^2} \right) \psi_e + \frac{\omega_{eg}}{2} \psi_e + \left(u_l e^{i(\omega_l t - \sqrt{2}\eta_l x)} + u_l^* e^{-i(\omega_l t - \sqrt{2}\eta_l x)} \right) \psi_g.$$

- This system is approximately controllable in $(L^2(\mathbb{R}, \mathbb{C}))^2$:
  S. Ervedoza and J.-P. Puel, Annales de l'IHP (c), 26(6): 2111-2136, 2009.

Law-Eberly method

Main idea

Control is superposition of 3 mono-chromatic plane waves with:

1. frequency ω_{eg} (ion transition frequency) and amplitude u;
2. frequency $\omega_{\text{eg}} - \omega_m$ (red shift by a vibration quantum) and amplitude u_r;
3. frequency $\omega_{\text{eg}} + \omega_m$ (blue shift by a vibration quantum) and amplitude u_b;

Control Hamiltonian:

$$
\begin{aligned}
\boldsymbol{H} =& \omega_m \left(\boldsymbol{a}^\dagger \boldsymbol{a} + \frac{\mathbf{I}}{2} \right) + \frac{\omega_{\text{eg}}}{2} \boldsymbol{\sigma_z} + \left(u e^{i(\omega_{\text{eg}} t - \eta(\boldsymbol{a}+\boldsymbol{a}^\dagger))} + u^* e^{-i(\omega_{\text{eg}} t - \eta(\boldsymbol{a}+\boldsymbol{a}^\dagger))} \right) \boldsymbol{\sigma_x} \\
&+ \left(u_b e^{i((\omega_{\text{eg}}+\omega_m)t - \eta_b(\boldsymbol{a}+\boldsymbol{a}^\dagger))} + u_b^* e^{-i((\omega_{\text{eg}}+\omega_m)t - \eta_b(\boldsymbol{a}+\boldsymbol{a}^\dagger))} \right) \boldsymbol{\sigma_x} \\
&+ \left(u_r e^{i((\omega_{\text{eg}}-\omega_m)t - \eta_r(\boldsymbol{a}+\boldsymbol{a}^\dagger))} + u_r^* e^{-i((\omega_{\text{eg}}-\omega_m)t - \eta_r(\boldsymbol{a}+\boldsymbol{a}^\dagger))} \right) \boldsymbol{\sigma_x}.
\end{aligned}
$$

Lamb-Dicke parameters:

$$\eta = \eta_{eg} \approx \eta_r \approx \eta_b \ll 1.$$

Law-Eberly method: rotating frame

Rotating frame: $|\psi\rangle = e^{-i\omega_m t\left(\boldsymbol{a}^\dagger \boldsymbol{a} + \frac{\mathbf{I}}{2}\right)} e^{\frac{-i\omega_{\text{eg}} t}{2}\boldsymbol{\sigma_z}} |\phi\rangle$

$$\boldsymbol{H}_{\text{int}} = e^{i\omega_m t\left(\boldsymbol{a}^\dagger \boldsymbol{a}\right)} \left(u e^{i\omega_{\text{eg}} t} e^{-i\eta(\boldsymbol{a}+\boldsymbol{a}^\dagger)} + u^* e^{-i\omega_{\text{eg}} t} e^{i\eta(\boldsymbol{a}+\boldsymbol{a}^\dagger)} \right)$$
$$e^{-i\omega_m t\left(\boldsymbol{a}^\dagger \boldsymbol{a}\right)} \left(e^{i\omega_{\text{eg}} t} |e\rangle\langle g| + e^{-i\omega_{\text{eg}} t} |g\rangle\langle e| \right)$$
$$+ e^{i\omega_m t\left(\boldsymbol{a}^\dagger \boldsymbol{a}\right)} \left(u_b e^{i(\omega_{\text{eg}}+\omega_m) t} e^{-i\eta_b(\boldsymbol{a}+\boldsymbol{a}^\dagger)} + u_b^* e^{-i(\omega_{\text{eg}}+\omega_m) t} e^{i\eta_b(\boldsymbol{a}+\boldsymbol{a}^\dagger)} \right)$$
$$e^{-i\omega_m t\left(\boldsymbol{a}^\dagger \boldsymbol{a}\right)} \left(e^{i\omega_{\text{eg}} t} |e\rangle\langle g| + e^{-i\omega_{\text{eg}} t} |g\rangle\langle e| \right)$$
$$+ e^{i\omega_m t\left(\boldsymbol{a}^\dagger \boldsymbol{a}\right)} \left(u_r e^{i(\omega_{\text{eg}}-\omega_m) t} e^{-i\eta_r(\boldsymbol{a}+\boldsymbol{a}^\dagger)} + u_r^* e^{-i(\omega_{\text{eg}}-\omega_m) t} e^{i\eta_r(\boldsymbol{a}+\boldsymbol{a}^\dagger)} \right)$$
$$e^{-i\omega_m t\left(\boldsymbol{a}^\dagger \boldsymbol{a}\right)} \left(e^{i\omega_{\text{eg}} t} |e\rangle\langle g| + e^{-i\omega_{\text{eg}} t} |g\rangle\langle e| \right)$$

Law-Eberly method: RWA

Commutation of exponentials in $(\boldsymbol{a} + \boldsymbol{a}^\dagger)$ and $(\boldsymbol{a}^\dagger \boldsymbol{a})$ is non-trivial.

- Approximation $e^{i\epsilon(\boldsymbol{a}+\boldsymbol{a}^\dagger)} \approx 1 + i\epsilon(\boldsymbol{a} + \boldsymbol{a}^\dagger)$ for $\epsilon = \pm\eta, \eta_b, \eta_r$

Then averaging: neglecting highly oscillating terms of frequencies $2\omega_{\rm eg}$, $2\omega_{\rm eg} \pm \omega_m$, $2(\omega_{\rm eg} \pm \omega_m)$ and $\pm\omega_m$, as

$$|u|, |u_b|, |u_r| \ll \omega_m, \quad \left|\frac{d}{dt}u\right| \ll \omega_m|u|, \quad \left|\frac{d}{dt}u_b\right| \ll \omega_m|u_b|, \quad \left|\frac{d}{dt}u_r\right| \ll \omega_m|u_r|.$$

First order approximation:

$$\boldsymbol{H}_{\rm rwa} = u|g\rangle\langle e| + u^*|e\rangle\langle g| + \bar{u}_b \boldsymbol{a}|g\rangle\langle e| + \bar{u}_b^* \boldsymbol{a}^\dagger|e\rangle\langle g| + \bar{u}_r \boldsymbol{a}^\dagger|g\rangle\langle e| + \bar{u}_r^* \boldsymbol{a}|e\rangle\langle g|$$

where

$$\bar{u}_b = -i\eta_b u_b \quad \text{and} \quad \bar{u}_r = -i\eta_r u_r$$

PDE form

$$i\frac{\partial \phi_g}{\partial t} = \left(u + \frac{\bar{u}_b}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right) + \frac{\bar{u}_r}{\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right)\right)\phi_e$$

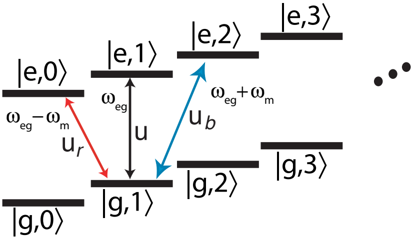
$$i\frac{\partial \phi_e}{\partial t} = \left(u^* + \frac{\bar{u}_b^*}{\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right) + \frac{\bar{u}_r^*}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)\right)\phi_g$$

Hilbert basis: $\{|g,n\rangle, |e,n\rangle\}_{n=0}^{\infty}$

Dynamics:

$$i\frac{d}{dt}\phi_{g,n} = u\phi_{e,n} + \bar{u}_r\sqrt{n}\phi_{e,n-1} + \bar{u}_b\sqrt{n+1}\phi_{e,n+1}$$
$$i\frac{d}{dt}\phi_{e,n} = u^*\phi_{g,n} + \bar{u}_r^*\sqrt{n+1}\phi_{g,n+1} + \bar{u}_b^*\sqrt{n}\phi_{g,n-1}$$

Physical interpretation:



Law-Eberly method: spectral controllability

Truncation to *n*-phonon space:

$$\mathcal{H}_n = \text{span}\,\{|g,0\rangle, |e,0\rangle, \ldots, |g,n\rangle, |e,n\rangle\}$$

We consider $|\phi\rangle_0, |\phi\rangle_T \in \mathcal{H}_n$ and we look for u, $\bar{u}_b$ and $\bar{u}_r$, s.t.

$$\text{for } |\phi\rangle(t=0) = |\phi\rangle_0 \text{ we have } |\phi\rangle(t=T) = |\phi\rangle_T.$$

- If u^1, $\bar{u}_b^1$ and $\bar{u}_r^1$ bring $|\phi\rangle_0$ to $|g,0\rangle$ at time $T/2$,
- and u^2, $\bar{u}_b^2$ and $\bar{u}_r^2$ bring $|\phi\rangle_T$ to $|g,0\rangle$ at time $T/2$,

then

$$u = u^1, \qquad u_b = u_b^1, \qquad u_r = u_r^1 \quad \text{for } t \in [0, T/2],$$
$$u = -u^2, \qquad u_b = -u_b^2, \qquad u_r = -u_r^2 \quad \text{for } t \in [T/2, T],$$

bring $|\phi\rangle_0$ to $|\phi\rangle_T$ at time T.

Take $|\phi_0\rangle \in \mathcal{H}_n$ and $\overline{T} > 0$:

- For $t \in [0, \frac{\overline{T}}{2}]$, $\bar{u}_r(t) = \bar{u}_b(t) = 0$, and

$$\bar{u}(t) = \frac{2i}{\overline{T}} \arctan \left| \frac{\phi_{e,n}(0)}{\phi_{g,n}(0)} \right| e^{i \arg(\phi_{g,n}(0)\phi^*_{e,n}(0))}$$

  implies $\phi_{e,n}(\overline{T}/2) = 0$;

- For $t \in [\frac{\overline{T}}{2}, \overline{T}]$, $\bar{u}_b(t) = \bar{u}(t) = 0$, and

$$\bar{u}_r(t) = \frac{2i}{\overline{T}\sqrt{n}} \arctan \left| \frac{\phi_{g,n}(\frac{\overline{T}}{2})}{\phi_{e,n-1}(\frac{\overline{T}}{2})} \right| e^{i \arg\left(\phi_{g,n}(\frac{\overline{T}}{2})\phi^*_{e,n-1}(\frac{\overline{T}}{2})\right)}$$

  implies that $\phi_{e,n}(\overline{T}) = 0$ and that $\phi_{g,n}(\overline{T}) = 0$.

The two pulses $\bar{u}$ and $\bar{u}_r$ lead to some $|\phi\rangle(\overline{T}) \in \mathcal{H}_{n-1}$.

Law-Eberly method

Repeating n times, we have

$$|\phi\rangle(n\overline{T}) \in \mathcal{H}_0 = \text{span}\{|g,0\rangle, \langle e,0|\}.$$

- for $t \in [n\overline{T}, (n+\frac{1}{2})\overline{T}]$, the control

$$\bar{u}_r(t) = \bar{u}_b(t) = 0,$$
$$\bar{u}(t) = \frac{2i}{\overline{T}} \arctan \left| \frac{\phi_{e,0}(n\overline{T})}{\phi_{g,0}(n\overline{T})} \right| e^{i \arg\left(\phi_{g,0}(n\overline{T})\phi^*_{e,0}(n\overline{T})\right)}$$

implies $|\phi\rangle_{(n+\frac{1}{2})\overline{T}} = e^{i\theta}|g,0\rangle$.