UMPC mercredi 19 avril 2017

M2 Mathématiques & Applications
UE (ANEDP, COCV): Analyse et contrôle de systèmes quantiques
Contrôle des connaissances, durée 2 heures.
Sujet donné par M. Mirrahimi et P. Rouchon

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Exercise 1

Consider the tensor product $\mathcal{H} = \mathcal{H}_3 \otimes \mathcal{H}_c$ where $\mathcal{H}_3 \sim \mathbb{C}^3$ admits $(|g\rangle, |e\rangle, |f\rangle)$ as Hilbert basis and $\mathcal{H}_c \sim L^2(\mathbb{R}, \mathbb{C}) \sim l^2(\mathbb{C})$ admits $(|n\rangle)_{n\geq 0}$ as Hilbert basis (Fock basis). Take the following Hamiltonian on $\mathcal{H}$ ($\omega_g, \omega_e, \omega_f, \omega_c, \chi$ real parameters)

$$\boldsymbol{H} = \big(\omega_g |g\rangle\langle g| + \omega_e |e\rangle\langle e| + \omega_f |f\rangle\langle f|\big) \otimes \boldsymbol{I}_c + \omega_c \, \boldsymbol{I}_3 \otimes \big(\boldsymbol{N} + \tfrac{\boldsymbol{I}_c}{2}\big) \\ + \chi\big(|g\rangle\langle f| + |f\rangle\langle g| + |e\rangle\langle f| + |f\rangle\langle e|\big) \otimes \big(\boldsymbol{N} + \tfrac{\boldsymbol{I}_c}{2}\big)$$

where $\boldsymbol{I}_3$ and $\boldsymbol{I}_c$ are identity operators on $\mathcal{H}_3$ and $\mathcal{H}_c$, $\boldsymbol{N} = \boldsymbol{a}^\dagger \boldsymbol{a}$ is the photon number operator on $\mathcal{H}_c$. We consider the Schrödinger equation $\frac{d}{dt}|\psi\rangle = -i\boldsymbol{H}|\psi\rangle$ where $|\psi\rangle \in \mathcal{H}$.

1. With $\boldsymbol{a} = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)$ and $|\psi\rangle \sim (\psi_g, \psi_e, \psi_f) \in L^2(\mathbb{R}, \mathbb{C}) \times L^2(\mathbb{R}, \mathbb{C}) \times L^2(\mathbb{R}, \mathbb{C})$ give the partial differential formulation of the Schrödinger equation.

2. With $|\psi\rangle = \sum_{n\geq 0} \psi_{g,n}|g\rangle \otimes |n\rangle + \psi_{e,n}|e\rangle \otimes |n\rangle + \psi_{f,n}|f\rangle \otimes |n\rangle$ give the infinite set of ordinary differential equations satisfied by $(\psi_{g,n}, \psi_{e,n}, \psi_{f,n})_{n\geq 0}$.

Exercise 2

Consider the 3-level system of Hilbert space $\mathcal{H} \sim \mathbb{C}^3$ with $(|g\rangle, |e\rangle, |f\rangle)$ as Hilbert basis with the following Hamiltonian

$$\boldsymbol{H}(t) = \omega_e|e\rangle\langle e| + \omega_f|f\rangle\langle f| + u(t)\Big(\mu_{ge}(|g\rangle\langle e| + |e\rangle\langle g|) + \mu_{ef}(|e\rangle\langle f| + |f\rangle\langle e|) + \mu_{fg}(|f\rangle\langle g| + |g\rangle\langle f|)\Big)$$

where $t \mapsto u(t) \in \mathbb{R}$ is the control input and $(\omega_e, \omega_f, \mu_{ge}, \mu_{ef}, \mu_{fg})$ are constant real parameters. Consider the Schrödinger equation $\frac{d}{dt}|\psi\rangle = -i\boldsymbol{H}(t)|\psi\rangle$ with $\omega_f > \omega_e > 0$ and $0 < |\mu_{ge}|, |\mu_{ef}|, |\mu_{fg}| \ll \min(\omega_e, \omega_f - \omega_e)$.

1. Take the passage to the interaction frame $|\psi\rangle \mapsto |\phi\rangle = e^{it\left(\omega_e|e\rangle\langle e| + \omega_f|f\rangle\langle f|\right)}|\psi\rangle$ and compute the interaction Hamiltonian $\boldsymbol{H}_{int}(t)$ governing the Schrödinger dynamics of $|\phi\rangle$: $\frac{d}{dt}|\phi\rangle = -i\boldsymbol{H}_{int}(t)|\phi\rangle$.

2. Assume that $u(t) = \bar{u}e^{-i\omega_f t} + \bar{u}^* e^{i\omega_f t}$ of constant amplitude $\bar{u} \in \mathbb{C}/\{0\}$ with $|\bar{u}| \leq 1$. Justify that one can approximate the time evolution of ϕ by $\frac{d}{dt}|\phi\rangle = -i\overline{\boldsymbol{H}}|\phi\rangle$ where $\overline{\boldsymbol{H}}$ is a constant Hamiltonian and provide its explicit expression.

3. We assume now that the state $|f\rangle$ is unstable and relaxes towards $|g\rangle$ or $|e\rangle$ with rates $\kappa_g, \kappa_e > 0$ much smaller that $\min(\omega_e, \omega_f - \omega_e)$. This open quantum quantum is described by the Lindbald master equation for the density operator ρ in the interaction frame:

$$\frac{d}{dt}\rho = -i\Big[\,\overline{\boldsymbol{H}}, \rho\,\Big] + \kappa_g \left(\boldsymbol{L}_g \rho \boldsymbol{L}_g^\dagger - \tfrac{1}{2}(\boldsymbol{L}_g^\dagger \boldsymbol{L}_g \rho + \rho \boldsymbol{L}_g^\dagger \boldsymbol{L}_g)\right) + \kappa_e \left(\boldsymbol{L}_e \rho \boldsymbol{L}_e^\dagger - \tfrac{1}{2}(\boldsymbol{L}_e^\dagger \boldsymbol{L}_e \rho + \rho \boldsymbol{L}_e^\dagger \boldsymbol{L}_e)\right)$$

with $\boldsymbol{L}_g = |g\rangle\langle f|$ and $\boldsymbol{L}_e = |e\rangle\langle f|$. Show that for any initial density operator $\rho_0 = \rho(0)$, the limit of $\rho(t)$ when t tends to $+\infty$ is the pure state $|e\rangle\langle e|$ (**Hint:** use the Lyapunov function $V(\rho) = 1 - \langle e|\rho|e\rangle$ and LaSalle's invariance principle).

Problem

We consider a quantum harmonic oscillator defined on the Hilbert space

$$\mathcal{H}_c = \left\{ \sum_{n=0}^{\infty} c_n |n\rangle \mid (c_n) \in l^2(\mathbb{C}) \right\}$$

where $|n\rangle$ corresponds to the Fock state with n photon(s). Driving it at its resonance, the Hamiltonian in the interaction frame is given by

$$\boldsymbol{H}_c = i(\bar{u}^* \boldsymbol{a} - \bar{u}\boldsymbol{a}^\dagger).$$

where $\bar{u} \in \mathbb{C}$ is a complex amplitude and $\boldsymbol{a}$ is the photon annihilator operator. As illustrated in the course, this Hamiltonian generates during $T \geq 0$ a unitary evolution $\boldsymbol{U}_T = \boldsymbol{D}_\alpha = e^{-iT\boldsymbol{H}_c} = e^{\alpha \boldsymbol{a}^\dagger - \alpha^* \boldsymbol{a}}$ with $\alpha = T\bar{u}$.

Through this problem, we will study the situation where this Hamiltonian evolution is accompanied by frequent measurements of a certain observable $\boldsymbol{O}_1 = |1\rangle\langle 1|$. Indeed, we will assume that this dynamics is performed in m steps of length T/m and labeled from $k = 0$ to $k = m-1$, together with a measurement after each step. In this aim, we consider the measurement operators $\boldsymbol{M}_g = \boldsymbol{I} - |1\rangle\langle 1|$, $\boldsymbol{M}_e = |1\rangle\langle 1|$. The dynamics of the system is modeled by the Markov chain of state $|\psi_k\rangle \in \boldsymbol{H}_c$ and measurement outcomes $y_k \in \{g, e\}$ at step k:

$$|\psi_{k+1/2}\rangle = \boldsymbol{D}_{\frac{\alpha}{m}} |\psi_k\rangle,$$

$$|\psi_{k+1}\rangle = \begin{cases} \dfrac{\boldsymbol{M}_g |\psi_{k+1/2}\rangle}{\sqrt{\left\langle \psi_{k+1/2} | \boldsymbol{M}_g^\dagger \boldsymbol{M}_g | \psi_{k+1/2} \right\rangle}} & \text{with } y_k = g, \text{ probability } \left\langle \psi_{k+1/2} | \boldsymbol{M}_g^\dagger \boldsymbol{M}_g | \psi_{k+1/2} \right\rangle; \\ \dfrac{\boldsymbol{M}_e |\psi_{k+1/2}\rangle}{\sqrt{\left\langle \psi_{k+1/2} | \boldsymbol{M}_e^\dagger \boldsymbol{M}_e | \psi_{k+1/2} \right\rangle}} & \text{with } y_k = e, \text{ probability } \left\langle \psi_{k+1/2} | \boldsymbol{M}_e^\dagger \boldsymbol{M}_e | \psi_{k+1/2} \right\rangle. \end{cases}$$

Furthermore, we assume the initial state to be given by $|\psi_0\rangle = |0\rangle$. Physically $|\psi_m\rangle$ corresponds then to the wave function at time T.

1. Show that the operators $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ represent an eligible Kraus map. Show that this measurement is quantum non-demolition for an observable $\boldsymbol{O}$ if and only if $\langle n|\boldsymbol{O}|1\rangle = 0$ for all $n \neq 1$.

2. Provide the state $|\psi_r^g\rangle$ of the system conditioned on r measurements giving as result $y_k = g$ for all $k = 0, \cdots, r-1$.

3. Show that the probability p_r^g of measuring $y_k = g$ for all $k = 0, \cdots, r-1$ is given by

$$p_r^g = \left\| \left(\boldsymbol{M}_g \boldsymbol{D}_{\frac{\alpha}{m}} \right)^r |0\rangle \right\|^2 .$$

4. Now, we aim at studying the limits $\lim_{m\to\infty} p_m^g$ and $\lim_{m\to\infty} |\psi_m^g\rangle$. Show that

$$\| \boldsymbol{M}_g \boldsymbol{D}_{\frac{\alpha}{m}} |0\rangle - |0\rangle \| = O\left(\frac{1}{m^2} \right).$$

5. Deduce that

$$\lim_{m\to\infty} p_m^g = 1 \quad \text{and} \quad \lim_{m\to\infty} |\psi_m^g\rangle = |0\rangle \text{ strongly in } \mathcal{H}_c.$$

   Hint: Use the fact that $\boldsymbol{D}_{\alpha/m}$ is a unitary and that $\boldsymbol{M}_g$ is a projection, and therefore they do not increase the norm of a state in $\mathcal{H}_c$.

6. Provide a simple and physical interpretation of the above limits.

7. Now we consider a different measurement process based on the observable $\boldsymbol{O}_2 = |2\rangle\langle 2|$. We consider the associated Kraus operators $\boldsymbol{M}_g = \boldsymbol{I} - |2\rangle\langle 2|$ and $\boldsymbol{M}_e = |2\rangle\langle 2|$. Also, for simplicity sakes, we assume α to be real.

   (a) Take $c_0, c_1 \in \mathbb{R}$ such that $|c_0|^2 + |c_1|^2 = 1$, and consider the wave functions

   $$|\psi\rangle = c_0|0\rangle + c_1|1\rangle \text{ and } |\tilde{\psi}\rangle = \frac{(c_0 - \alpha c_1/m)|0\rangle + (c_1 + \alpha c_0/m)|1\rangle}{\sqrt{1 + \alpha^2/m^2}}.$$

   Show that $\| \boldsymbol{M}_g \boldsymbol{D}_{\frac{\alpha}{m}} |\psi\rangle - |\tilde{\psi}\rangle \| = O\left(\frac{1}{m^2}\right)$ (**Hint:** Calculate $\boldsymbol{D}_{\alpha/m}|1\rangle$ by noting that $|1\rangle = \boldsymbol{a}^\dagger|0\rangle$ and using the commutation relations).

   (b) Deduce the limits $\lim_{m\to\infty} p_m^g$ and $\lim_{m\to\infty} |\psi_m^g\rangle$ (p_m^g and $|\psi_m^g\rangle$ are the probability to detect $y_k = g$ for $k = 0, \cdots, m-1$ and the corresponding quantum state at step m starting from $|\psi_0\rangle = |0\rangle$).

   (c) Provide a simple and physical interpretation of the above limits.

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Exercise 1

1. We have

$$
\begin{aligned}
i\frac{\partial \psi_g}{\partial t} &= \frac{\omega_g}{2}\psi_g + \frac{\omega_c}{2}(x^2 - \frac{\partial^2}{\partial x^2})\psi_g + \frac{\chi}{2}(x^2 - \frac{\partial^2}{\partial x^2})\psi_f \\
i\frac{\partial \psi_e}{\partial t} &= \frac{\omega_e}{2}\psi_e + \frac{\omega_c}{2}(x^2 - \frac{\partial^2}{\partial x^2})\psi_e + \frac{\chi}{2}(x^2 - \frac{\partial^2}{\partial x^2})\psi_f \\
i\frac{\partial \psi_f}{\partial t} &= \frac{\omega_f}{2}\psi_f + \frac{\omega_c}{2}(x^2 - \frac{\partial^2}{\partial x^2})\psi_f + \frac{\chi}{2}(x^2 - \frac{\partial^2}{\partial x^2})\psi_g + \frac{\chi}{2}(x^2 - \frac{\partial^2}{\partial x^2})\psi_e
\end{aligned}
$$

2. We have

$$
\begin{aligned}
i\frac{d}{dt}\psi_{g,n} &= ((n+1/2)\omega_c + \omega_g)\psi_{g,n} + \chi(n+1/2)\psi_{f,n} \\
i\frac{d}{dt}\psi_{e,n} &= ((n+1/2)\omega_c + \omega_e)\psi_{e,n} + \chi(n+1/2)\psi_{f,n} \\
i\frac{d}{dt}\psi_{f,n} &= ((n+1/2)\omega_c + \omega_f)\psi_{f,n} + \chi(n+1/2)\psi_{g,n} + \chi(n+1/2)\psi_{e,n}
\end{aligned}
$$

Exercise 2

1. We have

$$
\begin{aligned}
\boldsymbol{H}_{int}(t) = u(t)\mu_{ge}(e^{-i\omega_e t}|g\rangle\langle e| + e^{i\omega_e t}|e\rangle\langle g|) \\
+ u(t)\mu_{ef}(e^{-i(\omega_f-\omega_e)t}|e\rangle\langle f| + e^{i(\omega_f-\omega_e)t}|f\rangle\langle e|) \\
+ u(t)\mu_{fg}(e^{i\omega_f t}|f\rangle\langle g| + e^{-i\omega_f t}|g\rangle\langle f|).
\end{aligned}
$$

2. Since $|\mu_{ge}|, |\mu_{ef}|, |\mu_{fg}| \ll \min(\omega_e, \omega_f - \omega_e)$, we can use the rotating wave approximation and keep only the non-oscillating terms (secular terms) in $\boldsymbol{H}_{int}(t)$ where $u(t)$ is replaced by $\bar{u}e^{-i\omega_f t} + \bar{u}^* e^{i\omega_f t}$. This yields to $\overline{\boldsymbol{H}} = \mu_{fg}\Big(\bar{u}|f\rangle\langle g| + \bar{u}^*|g\rangle\langle f|\Big)|\phi\rangle$.

3. Since $\rho(t)$ remains non-negative and of trace one, $V(\rho)$ remains between 0 and 1. Moreover $V(\rho) = 0$ means that $\rho = |e\rangle\langle e|$. Since

$$
\frac{d}{dt}\rho = -i\mu_{fg}\Big[\bar{u}|f\rangle\langle g| + \bar{u}^*|g\rangle\langle f|, \rho\Big] + \langle f|\rho|f\rangle \ \Big(\kappa_g|g\rangle\langle g| + \kappa_e|e\rangle\langle e|\Big) - \tfrac{\kappa_g+\kappa_e}{2}\Big(|f\rangle\langle f|\rho + \rho|f\rangle\langle f|\Big)
$$

we have $\frac{d}{dt}V(\rho) = -\kappa_e \langle f|\rho|f\rangle \leq 0$. Thus V is a decreasing time function. Since the set of density operators is compact and $V \geq 0$, we can apply LaSalle's invariance principle: the trajectories converge towards the largest invariant set of density operators satisfying $\frac{d}{dt}V = 0$. When $\langle f|\rho|f\rangle = 0$ we have $\rho|f\rangle = 0$ and $\langle f|\rho = 0$ since ρ is a density operator (therefore non-negative). Then we have

$$\frac{d}{dt}\rho = -i\mu_{fg}\Big(\bar{u}|f\rangle\langle g|\rho - \bar{u}^*\rho|g\rangle\langle f|\Big)$$

and we get by differentiating $\rho|f\rangle = 0$ with respect to t: $\frac{d}{dt}\rho|f\rangle = 0$, i.e. $-\mu_{fg}\bar{u}^*\rho|g\rangle = 0$. This means that $|f\rangle$ and $|g\rangle$ are in the kernel of ρ. This implies that ρ is necessarily the projector on $|e\rangle$ since it must be of trace one and non-negative.

This Lindbald equation is the simplest dynamical model describing optical pumping, a simple and powerful idea due to Alfred Kastler (Physics Nobel Prize 1966) for preparing and stabilizing pure states.

Problem

1. It is easy to check that $\boldsymbol{M}_g^\dagger \boldsymbol{M}_g + \boldsymbol{M}_e^\dagger \boldsymbol{M}_e = \boldsymbol{I}$ and therefore they represent an eligible Kraus map. The measurement is non-demolition for an observable $\boldsymbol{O}$, if the Kraus operators $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ commute with $\boldsymbol{O}$. It is easy to check that this condition is equivalent to $\langle m|\boldsymbol{O}|1\rangle = 0$, $\forall m \neq 1$.

2. The state at the step r is given by

$$|\psi_r^g\rangle = \frac{\boldsymbol{M}_g \boldsymbol{D}_{\frac{\alpha}{m}}|\psi_{r-1}^g\rangle}{\|\boldsymbol{M}_g \boldsymbol{D}_{\frac{\alpha}{m}}|\psi_{r-1}^g\rangle\|}.$$

Therefore by induction, it is easy to see that

$$|\psi_r^g\rangle = \frac{\big(\boldsymbol{M}_g \boldsymbol{D}_{\frac{\alpha}{m}}\big)^r|0\rangle}{\Big\|\big(\boldsymbol{M}_g \boldsymbol{D}_{\frac{\alpha}{m}}\big)^r|0\rangle\Big\|}.$$

3. The probability for the first measurement to give $y_0 = g$ is clearly $\|\boldsymbol{M}_g \boldsymbol{D}_{\frac{\alpha}{m}}|0\rangle\|^2$. The probability to achieve r measurements giving all $y_k = g$ is given by

$$\begin{aligned} p_r^g &= \mathbb{P}(y_{r-1} = g, y_{r-2} = g, \cdots y_0 = g) \\ &= \mathbb{P}(y_{r-1} = g \mid y_{r-2} = g, \cdots y_0 = g)\mathbb{P}(y_{r-2} = g, y_{r-3} = g, \cdots y_0 = g). \end{aligned}$$

But

$$\mathbb{P}(y_{r-1} = g \mid y_{r-2} = g, \cdots y_0 = g) = \|\boldsymbol{M}_g \boldsymbol{D}_{\frac{\alpha}{m}}\psi_{r-1}^g\|^2 = \frac{\Big\|\big(\boldsymbol{M}_g \boldsymbol{D}_{\frac{\alpha}{m}}\big)^r|0\rangle\Big\|^2}{\Big\|\big(\boldsymbol{M}_g \boldsymbol{D}_{\frac{\alpha}{m}}\big)^{r-1}|0\rangle\Big\|^2}$$

and

$$\mathbb{P}(y_{r-2} = g, y_{r-3} = g, \cdots y_0 = g) = p_{r-1}^g.$$

The proof is clear by induction.

4. We have

$$\boldsymbol{M}_g\boldsymbol{D}_{\frac{\alpha}{m}}|0\rangle = (\boldsymbol{I} - |1\rangle\langle 1|)|\frac{\alpha}{m}\rangle = e^{-\frac{|\alpha|^2}{2m^2}}(\boldsymbol{I} - |1\rangle\langle 1|)\sum_{k=0}^{\infty}\frac{1}{\sqrt{k!}}\frac{\alpha^k}{m^k}|k\rangle$$
$$= e^{-\frac{|\alpha|^2}{2m^2}}|0\rangle + e^{-\frac{|\alpha|^2}{2m^2}}\sum_{k=2}^{\infty}\frac{1}{\sqrt{k!}}\frac{\alpha^k}{m^k}|k\rangle.$$

We note that $e^{-\frac{|\alpha|^2}{2m^2}} = 1 + O(1/m^2)$ and that

$$\|\sum_{k=2}^{\infty}\frac{1}{\sqrt{k!}}\frac{\alpha^k}{m^k}|k\rangle\| \leq \frac{|\alpha|^2}{m^2}\sum_{k=2}^{\infty}\frac{1}{\sqrt{k!}}\frac{|\alpha|^{k-2}}{m^{k-2}}.$$

Noting that the series is convergent, the result is clear.

5. One can write

$$\boldsymbol{M}_g\boldsymbol{D}_{\frac{\alpha}{m}}|0\rangle = |0\rangle + O(\frac{1}{m^2})|\chi_0\rangle,$$

where χ_0 is a normalized state in $\mathcal{H}_c$. Therefore

$$(\boldsymbol{M}_g\boldsymbol{D}_{\frac{\alpha}{m}})^2|0\rangle = (\boldsymbol{M}_g\boldsymbol{D}_{\frac{\alpha}{m}})(|0\rangle + O(\frac{1}{m^2})|\chi_0\rangle)$$
$$= |0\rangle + O(\frac{1}{m^2})|\chi_0\rangle + O(\frac{1}{m^2})(\boldsymbol{M}_g\boldsymbol{D}_{\frac{\alpha}{m}})|\chi_0\rangle.$$

We note that $\|(\boldsymbol{M}_g\boldsymbol{D}_{\frac{\alpha}{m}})|\chi_0\rangle\| \leq 1$, as $\boldsymbol{D}_{\frac{\alpha}{m}}$ is a unitary (therefore conserving the norm) and $\boldsymbol{M}_g$ is a projection (therefore reducing the norm). Thus $O(\frac{1}{m^2})(\boldsymbol{M}_g\boldsymbol{D}_{\frac{\alpha}{m}})|\chi_0\rangle$ can be written as $O(\frac{1}{m^2})|\chi_1\rangle$ for a normalized state $|\chi_1\rangle$ in $\mathcal{H}_c$. In the same manner

$$(\boldsymbol{M}_g\boldsymbol{D}_{\frac{\alpha}{m}})^m|0\rangle = |0\rangle + O(\frac{1}{m^2})(\sum_{k=0}^{m-1}|\chi_k\rangle),$$

where $|\chi_k\rangle$'s are normalized states in $\mathcal{H}_c$. Therefore

$$p_m^g = \|(\boldsymbol{M}_g\boldsymbol{D}_{\frac{\alpha}{m}})^m|0\rangle\|^2 = \||0\rangle + O(\frac{1}{m^2})(\sum_{k=0}^{m-1}|\chi_k\rangle)\|^2 \to 1 \text{ as } m \to \infty.$$

Furthermore

$$\||\psi_m^g\rangle - |0\rangle\| = \left\|\frac{(\boldsymbol{M}_g\boldsymbol{D}_{\frac{\alpha}{m}})^m|0\rangle}{\sqrt{p_m^g}} - |0\rangle\right\| \to 0 \text{ as } m \to \infty.$$

6. We have illustrated that, whenever we measure frequently the observable $\boldsymbol{O}_1$ during the unitary evolution, we freeze the state at time T ($T > 0$ being arbitrary) in $|0\rangle$ and remove the effect of the driving Hamiltonian. This is called the quantum Zeno effect.

7. (a) We have $\boldsymbol{M}_g\boldsymbol{D}_{\frac{\alpha}{m}}|\psi\rangle = c_0\boldsymbol{M}_g|\frac{\alpha}{m}\rangle + c_1\boldsymbol{M}_g\boldsymbol{D}_{\frac{\alpha}{m}}|1\rangle$. In order to calculate $\boldsymbol{D}_{\frac{\alpha}{m}}|1\rangle$, we note that

$$\boldsymbol{D}_{\frac{\alpha}{m}}|1\rangle = \boldsymbol{D}_{\frac{\alpha}{m}}\boldsymbol{a}^\dagger|0\rangle = \boldsymbol{D}_{\frac{\alpha}{m}}\boldsymbol{a}^\dagger\boldsymbol{D}_{-\frac{\alpha}{m}}\boldsymbol{D}_{\frac{\alpha}{m}}|0\rangle = (\boldsymbol{a}^\dagger - \frac{\alpha}{m})|\frac{\alpha}{m}\rangle.$$

As $\boldsymbol{M}_g = \boldsymbol{I} - |2\rangle\langle 2|$, we have

$$\boldsymbol{M}_g \boldsymbol{D}_{\frac{\alpha}{m}}|\psi\rangle = c_0(|0\rangle + \frac{1}{1!}\frac{\alpha}{m}|1\rangle) + c_1 \boldsymbol{M}_g(\boldsymbol{a}^\dagger - \frac{\alpha}{m})(|0\rangle + \frac{1}{1!}\frac{\alpha}{m}|1\rangle) + O(\frac{1}{m^2})|\chi_0\rangle,$$

where $|\chi_0\rangle$ is a normalized state in $\mathcal{H}_c$. Therefore

$$\begin{aligned}
\boldsymbol{M}_g \boldsymbol{D}_{\frac{\alpha}{m}}|\psi\rangle &= c_0(|0\rangle + \frac{\alpha}{m}|1\rangle) + c_1 \boldsymbol{M}_g(\boldsymbol{a}^\dagger - \frac{\alpha}{m})(|0\rangle + \frac{1}{1!}\frac{\alpha}{m}|1\rangle) + O(\frac{1}{m^2})|\chi_0\rangle \\
&= c_0(|0\rangle + \frac{\alpha}{m}|1\rangle) + c_1 \boldsymbol{M}_g(|1\rangle + \sqrt{2}\frac{\alpha}{m}|2\rangle - \frac{\alpha}{m}|0\rangle) + O(\frac{1}{m^2})|\chi_1\rangle \\
&= c_0(|0\rangle + \frac{\alpha}{m}|1\rangle) + c_1(|1\rangle - \frac{\alpha}{m}|0\rangle) + O(\frac{1}{m^2})|\chi_1\rangle,
\end{aligned}$$

where $|\chi_0\rangle$ is a normalized state in $\mathcal{H}_c$. This proves the relation

$$\|\boldsymbol{M}_g \boldsymbol{D}_{\frac{\alpha}{m}}|\psi\rangle - |\tilde{\psi}\rangle\| = O(1/m^2),$$

as $1/\sqrt{1+\alpha^2/m^2} = 1 + O(1/m^2)$.

(b) One has $|\tilde{\psi}\rangle = \boldsymbol{R}_\theta|\psi\rangle$, where

$$\boldsymbol{R}_\theta = \begin{pmatrix} \frac{1}{\sqrt{1+\alpha^2/m^2}} & -\frac{\alpha/m}{\sqrt{1+\alpha^2/m^2}} \\ \frac{\alpha/m}{\sqrt{1+\alpha^2/m^2}} & \frac{1}{\sqrt{1+\alpha^2/m^2}} \end{pmatrix}$$

is a rotation matrix with $\theta = \arctan(\alpha/m)$ in the space $\text{span}\{|0\rangle, |1\rangle\}$. Similarly to the question 5, we have $\lim_{m\to\infty} p_m^g = 1$. Also, we have

$$|\psi_m^g\rangle = \boldsymbol{R}^m_{\arctan(\alpha/m)}|0\rangle + O(1/m)|\chi\rangle = \boldsymbol{R}_{m\arctan(\alpha/m)}|0\rangle + O(1/m)|\chi\rangle,$$

where χ is a normalized state in $\mathcal{H}_c$. Now, note that $\boldsymbol{R}_{m\arctan(\alpha/m)}|0\rangle$ converges to $\boldsymbol{R}_\alpha|0\rangle$ for m tending to infinity.

(c) We have shown that the measuring frequently the observable $\boldsymbol{O}_2$, we confine the dynamics of the harmonic oscillator to the two-dimensional subspace spanned by $|0\rangle$ and $|1\rangle$. A unitary displacement of the cavity state is therefore replaced by a Rabi oscillation for this effective two-level system. This is called Quantum Zeno Dynamics.