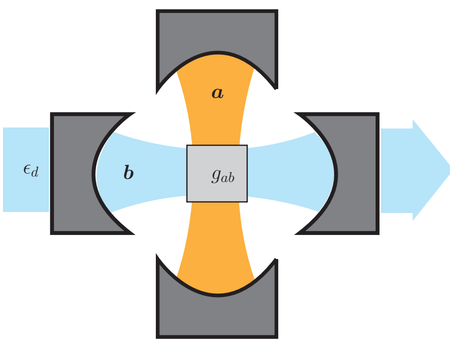
UMPC Wednesday 6 April 2016

M2 Mathématiques & Applications
UE (ANEDP, COCV): Analysis et control of quantum systems.
3-hour exam given by M. Mirrahimi and P. Rouchon

Lecture notes and other written documents are authorized. Access to internet and other networks is forbidden. The two problems are completely independent and can be treated in any order in French or English.

# Problem 1

We consider two quantum harmonic oscillators coupled through a nonlinear medium. We assume that one of the harmonic oscillators is driven off-resonance.




The Hamiltonian is given by

$$\frac{\boldsymbol{H}}{\hbar} = \omega_a\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right) + \omega_b\left(\boldsymbol{b}^\dagger\boldsymbol{b} + \frac{\boldsymbol{I}}{2}\right) + g_{ab}\Big((\boldsymbol{a}+\boldsymbol{a}^\dagger) + (\boldsymbol{b}+\boldsymbol{b}^\dagger)\Big)^4 + \left(\epsilon_d e^{-i\varpi t}\boldsymbol{b}^\dagger + \epsilon_d^* e^{i\varpi t}\boldsymbol{b}\right),$$

where $\omega_a$ and $\omega_b$ are the resonance frequencies of the harmonic oscillators associated to the annihilation operators $\boldsymbol{a}$ and $\boldsymbol{b}$. Also, $g_{ab} \ll \omega_a, \omega_b$ is the strength of the quartic coupling provided by the nonlinear medium. Finally $\epsilon_d$ represents the complex amplitude of the drive at frequency $\varpi$ applied to the harmonic oscillator $\boldsymbol{b}$.

1. Write the Schrödinger equation $i\frac{d}{dt}|\phi\rangle = \frac{\boldsymbol{H}}{\hbar}|\phi\rangle$ in the form of a partial differential equation for the complex-valued wave-function $\phi(x,y)$ depending on two real variables $x$ and $y$ (we will not use this PDE formulation in the sequel).

2. Express the Schrödinger equation $i\frac{d}{dt}|\phi\rangle = \frac{\boldsymbol{H}}{\hbar}|\phi\rangle$ in the rotating frame of the Hamiltonian $\boldsymbol{H}_0/\hbar = \varpi\boldsymbol{b}^\dagger\boldsymbol{b}$, i.e. with the new wave-function $|\widetilde{\phi}\rangle = e^{it\boldsymbol{H}_0/\hbar}|\phi\rangle$ instead of $|\phi\rangle$, i.e. compute $\widetilde{\boldsymbol{A}} = e^{it\boldsymbol{H}_0/\hbar}\left(\frac{\boldsymbol{H}-\boldsymbol{H}_0}{\hbar}\right)e^{-it\boldsymbol{H}_0/\hbar}$ where $i\frac{d}{dt}|\widetilde{\phi}\rangle = \widetilde{\boldsymbol{A}}(t)|\widetilde{\phi}\rangle$

3. Consider the displacement operator $\boldsymbol{D}_\beta = \exp(\beta\boldsymbol{b}^\dagger - \beta^*\boldsymbol{b})$ with $\beta = \epsilon_d/\Delta$, $\Delta = \omega_b - \varpi$. Show that the Schrödinger equation $i\frac{d}{dt}|\widetilde{\phi}\rangle = \widetilde{\boldsymbol{A}}(t)|\widetilde{\phi}\rangle$, after a change of variable $|\widetilde{\psi}\rangle = \boldsymbol{D}_\beta|\widetilde{\phi}\rangle$, and up to a change of global phase (i.e. up to $\omega\boldsymbol{I}$ with $\omega \in \mathbb{R}$) , can be written in the form $i\frac{d}{dt}|\widetilde{\psi}\rangle = \boldsymbol{A}(t)|\widetilde{\psi}\rangle$, where

$$\boldsymbol{A} = \omega_a\boldsymbol{a}^\dagger\boldsymbol{a} + \Delta\boldsymbol{b}^\dagger\boldsymbol{b} + g_{ab}\Big(\boldsymbol{a}+\boldsymbol{a}^\dagger + \boldsymbol{b}e^{-i\varpi t} + \boldsymbol{b}^\dagger e^{i\varpi t} - \beta e^{-i\varpi t} - \beta^* e^{i\varpi t}\Big)^4.$$

4. Once again, use the rotating frame of the Hamiltonian $\boldsymbol{A}_0 = \omega_a \boldsymbol{a}^\dagger \boldsymbol{a} + \Delta \boldsymbol{b}^\dagger \boldsymbol{b}$, write the above Schrödinger equation in the form $i\frac{d}{dt}|\psi\rangle = \boldsymbol{B}(t)|\psi\rangle$ with $|\psi\rangle = e^{it\boldsymbol{A}_0}|\widetilde{\psi}\rangle$ and provide the expression of $\boldsymbol{B}(t)$ versus $\boldsymbol{a}$ and $\boldsymbol{b}$.

5. We take $\overline{\omega} = 2\omega_a - \omega_b$ and we assume $|g_{ab}| \ll |n_a\omega_a - n_b\omega_b|$ for all $n_a, n_b = 0, 1, 2, 3, 4$ such that $n_a \neq n_b$. Show that the first-order averaging leads to an approximate dynamics of the form $i\frac{d}{dt}|\psi\rangle = \bar{\boldsymbol{B}}|\psi\rangle$, where, up to an irrelevant global phase,

$$\bar{\boldsymbol{B}} = \delta_a \boldsymbol{a}^\dagger \boldsymbol{a} + \delta_b \boldsymbol{b}^\dagger \boldsymbol{b} + \chi_{aa} \boldsymbol{a}^{\dagger\, 2} \boldsymbol{a}^2 + \chi_{bb} \boldsymbol{b}^{\dagger\, 2} \boldsymbol{b}^2 + \chi_{ab} \boldsymbol{a}^\dagger \boldsymbol{a} \boldsymbol{b}^\dagger \boldsymbol{b} + g_{2ph} \boldsymbol{a}^{\dagger\, 2} \boldsymbol{b} + g_{2ph}^* \boldsymbol{a}^2 \boldsymbol{b}^\dagger$$

Determine the parameters $\delta_a$, $\delta_b$, $\chi_{aa}$, $\chi_{bb}$, $\chi_{ab}$ and $g_{2ph}$ as a function of $g_{ab}$ and $\beta$. Indication: use $[\boldsymbol{a}, \boldsymbol{a}^\dagger] = 1$, $[\boldsymbol{b}, \boldsymbol{b}^\dagger] = 1$ and

$$\big(U + V + W\big)^4 = \sum_{n_U + n_V + n_W = 4} \left( \frac{4!}{n_U!\, n_V!\, n_W!} \right) U^{n_U} V^{n_V} W^{n_W}$$

where $U$, $V$ and $W$ are three operators that commute.

## Problem 2

Under the assumption of strong dissipation for the mode $\boldsymbol{b}$ in the previous problem, it is possible to eliminate the dynamics of the mode $\boldsymbol{b}$ to achieve an approximate Lindblad equation only for mode $\boldsymbol{a}$. This leads to a two-photon loss for the quantum harmonic oscillator $\boldsymbol{a}$ where the density operator $\rho(t)$ is governed by

$$\frac{d}{dt}\rho = \boldsymbol{L}\rho\boldsymbol{L}^\dagger - \tfrac{1}{2}(\boldsymbol{L}^\dagger\boldsymbol{L}\rho + \rho\boldsymbol{L}^\dagger\boldsymbol{L}) \triangleq \mathcal{L}(\rho), \quad \rho(0) = \rho_0$$

with $\boldsymbol{L} = \boldsymbol{a}^2$. We recall that for any integer $n \geq 1$, $\boldsymbol{a}|n\rangle = \sqrt{n}|n-1\rangle$ and $\boldsymbol{a}|0\rangle = 0$ where $\Big(|n\rangle\Big)_{n\in\mathbb{N}}$ is the Hilbert basis corresponding to photon-number states. We recall also that, for any scalar function $f$, $\boldsymbol{a} f(\boldsymbol{N}) = f(\boldsymbol{N}+1)\boldsymbol{a}$ where $\boldsymbol{N} = \boldsymbol{a}^\dagger \boldsymbol{a}$.

1. (a) Show that $\boldsymbol{L}^\dagger \boldsymbol{L} = \boldsymbol{N}(\boldsymbol{N} - 1)$.

   (b) Set $p_n = \langle n|\rho|n\rangle$ for $n \geq 0$. Show that

   $$\frac{d}{dt}p_n = (n+1)(n+2)p_{n+2} - n(n-1)p_n.$$

   (c) Deduce that the density operators $\bar{\rho}$ such that $\mathcal{L}(\bar{\rho}) = 0$ have their supports in $\mathrm{span}(|0\rangle, |1\rangle)$:

   $$\exists \bar{p}_0 \in [0,1], \exists \bar{c} \in \mathbb{C},\ \bar{\rho} = \bar{p}_0|0\rangle\langle 0| + (1 - \bar{p}_0)|1\rangle\langle 1| + \bar{c}|1\rangle\langle 0| + \bar{c}^*|0\rangle\langle 1|.$$

2. For any operator $J$ (not necessarily Hermitian) prove that $\frac{d}{dt}\big(\mathrm{Tr}\,(\rho J)\big) = \mathrm{Tr}\,(\rho\mathcal{L}^*(J))$ where $\mathcal{L}^*(J) = \boldsymbol{L}^\dagger J \boldsymbol{L} - \frac{1}{2}(\boldsymbol{L}^\dagger \boldsymbol{L} J + J \boldsymbol{L}^\dagger \boldsymbol{L})$ (the adjoint super-operator associated to $\mathcal{L}$ for the Frobenius scalar product between two Hermitian matrices).

3. (a) For any increasing scalar function $f$, prove that $\mathcal{L}^*(f(\boldsymbol{N})) \leq 0$.

   (b) Deduce that $V(\rho) = \mathrm{Tr}\,(N\rho)$ is a Lyapunov function.

(c) Prove that, formally, for any initial density operator $\rho_0$, $\lim_{t\mapsto+\infty}\rho(t)$ exists and corresponds to a steady state $\bar{\rho}$ characterized in question 1c.

(d) Show that $\bar{\rho}$ depends linearly on the initial condition $\rho_0$.

*Such dependence is denoted by $\bar{\rho} = \boldsymbol{K}(\rho_0)$. The remaining part of the problem consists in providing an explicit formulation of this map.*

4. An operator $J$ is said to be invariant if and only if $\mathcal{L}^*(J) = 0$. Show that, for any invariant operator $J$, $\mathrm{Tr}\,(\rho J)$ is a first integral of $\frac{d}{dt}\rho = \mathcal{L}(\rho)$.

5. Prove that $f(\boldsymbol{N})$ is an invariant operator if $f$ is 2-periodic. Show that $J_0 = \sum_{n\geq 0} |2n\rangle\langle 2n|$ is invariant and deduce that $\left\langle 0\middle|\boldsymbol{K}(\rho_0)\middle|0\right\rangle = \mathrm{Tr}\,(J_0\rho_0)$ and $\left\langle 1\middle|\boldsymbol{K}(\rho_0)\middle|1\right\rangle = 1 - \mathrm{Tr}\,(J_0\rho_0)$.

6. Prove that $f(\boldsymbol{N})\boldsymbol{a}$ is an invariant operator if $f(1) = 0$ and for all integer $n \geq 2$ we have $nf(n) = (n-1)f(n-2)$.

7. Consider a real function $f$ such that $f(0) = 1$ and, for all $n \geq 1$, $f(2n-1) = 0$ with $f(2n) = \prod_{k=1}^{n} \frac{2k-1}{2k}$.

(a) Show that the series $g_n = \sqrt{2n+1}f(2n)$ is strictly decreasing.

(b) Check that $J_1 = f(\boldsymbol{N})\boldsymbol{a}$ is a bounded and invariant operator.

(c) Prove that

$$\boldsymbol{K}(\rho_0) = \mathrm{Tr}\,(J_0\rho_0)\,|0\rangle\langle 0| + \Big(1 - \mathrm{Tr}\,(J_0\rho_0)\Big)|1\rangle\langle 1| + \mathrm{Tr}\,(\rho_0 J_1)\,|1\rangle\langle 0| + \mathrm{Tr}\,\left(\rho_0 J_1^\dagger\right)|0\rangle\langle 1|.$$

8. (a) Show that $\mathrm{Tr}\,(\rho_0 J_1) = \sum_{n\geq 0} g_n \left\langle 2n+1\middle|\rho_0\middle|2n\right\rangle = \left\langle 1\middle|\boldsymbol{K}(\rho_0)\middle|0\right\rangle$.

(b) Show that $\boldsymbol{K}(\rho_0)$ admits a Kraus formulation of the following form

$$\boldsymbol{K}(\rho_0) = \sum_{n\geq 0} A_n\rho_0 A_n^\dagger + B_n\rho_0 B_n^\dagger + C_n\rho_0 C_n^\dagger$$

where $A_n = a_n|0\rangle\langle 2n|$, $B_n = b_n|0\rangle\langle 2n+1|$ and $C_n = c_n\Big(|0\rangle\langle 2n| + |1\rangle\langle 2n+1|\Big)$ and express the values of the scalars $a_n$, $b_n$ and $c_n$ versus $g_n$. Check that $\sum_{n\geq 0} A_n^\dagger A_n + B_n^\dagger B_n + C_n^\dagger C_n = \boldsymbol{I}$.

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# Problem 1

1. Since $\boldsymbol{a}^\dagger\boldsymbol{a} + 1/2$ stands for the operator $-\frac{1}{2}\frac{\partial^2}{\partial x} + \frac{1}{2}x^2$ and $\boldsymbol{a}+\boldsymbol{a}^\dagger$ for $\sqrt{2}x$ (the same for $\boldsymbol{b}$ with $y$ instead of $x$), we get

$$i\frac{\partial \phi}{\partial t} = -\tfrac{\omega_a}{2}\frac{\partial^2\phi}{\partial x} - \tfrac{\omega_b}{2}\frac{\partial^2\phi}{\partial y} + \left(\tfrac{\omega_a}{2}x^2 + \tfrac{\omega_b}{2}y^2 + 4g_{ab}(x+y)^4\right)\phi + \sqrt{2}\Re(\epsilon_d e^{-i\varpi t})y\phi + i\sqrt{2}\Im(\epsilon_d e^{-i\varpi t})\frac{\partial\phi}{\partial y}.$$

2. In this frame $\boldsymbol{a}$ remains unchanged, i.e. $e^{it\boldsymbol{H}_0/\hbar}\boldsymbol{a}e^{-it\boldsymbol{H}_0/\hbar} = \boldsymbol{a}$ and $\boldsymbol{b}$ becomes $\boldsymbol{b}e^{-i\varpi t}$, i.e. $e^{it\boldsymbol{H}_0/\hbar}\boldsymbol{b}e^{-it\boldsymbol{H}_0/\hbar} = \boldsymbol{b}e^{-i\varpi t}$. Thus we have

$$\widetilde{A} = \omega_a\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \tfrac{\boldsymbol{I}}{2}\right) + (\omega_b - \varpi)\boldsymbol{b}^\dagger\boldsymbol{b} + \omega_b\tfrac{\boldsymbol{I}}{2} + g_{ab}\Big(\boldsymbol{a}+\boldsymbol{a}^\dagger + \boldsymbol{b}e^{-i\varpi t} + \boldsymbol{b}^\dagger e^{i\varpi t}\Big)^4 + \epsilon_d\boldsymbol{b}^\dagger + \epsilon_d^*\boldsymbol{b}.$$

3. In this frame $\boldsymbol{a}$ remains unchanged and $\boldsymbol{b}$ becomes $\boldsymbol{b}e^{-i\varpi t}$, i.e. $\boldsymbol{D}_\beta\boldsymbol{b}\boldsymbol{D}_{-\beta} = \boldsymbol{b} - \beta$. Thus we have

$$\begin{aligned}\boldsymbol{D}_\beta\widetilde{A}\boldsymbol{D}_{-\beta} &= \omega_a\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \tfrac{\boldsymbol{I}}{2}\right) + (\omega_b-\varpi)(\boldsymbol{b}^\dagger - \beta^*)(\boldsymbol{b}-\beta) + \omega_b\tfrac{\boldsymbol{I}}{2} \\ &\quad + g_{ab}\Big(\boldsymbol{a}+\boldsymbol{a}^\dagger + (\boldsymbol{b}-\beta)e^{-i\varpi t} + (\boldsymbol{b}^\dagger-\beta^*)e^{i\varpi t}\Big)^4 + \epsilon_d(\boldsymbol{b}^\dagger-\beta^*) + \epsilon_d^*(\boldsymbol{b}-\beta). \\ &= \omega_a\boldsymbol{a}^\dagger\boldsymbol{a} + (\omega_b-\varpi)\boldsymbol{b}^\dagger\boldsymbol{b} + g_{ab}\Big(\boldsymbol{a}+\boldsymbol{a}^\dagger + (\boldsymbol{b}-\beta)e^{-i\varpi t} + (\boldsymbol{b}^\dagger-\beta^*)e^{i\varpi t}\Big)^4 \\ &\quad + \left(\frac{\omega_a}{2} + (\omega_b-\varpi)|\beta|^2 + \frac{\omega_b}{2} - \epsilon_d\beta^* - \epsilon_d^*\beta\right)\boldsymbol{I}\end{aligned}$$

Thus, up-to the global phase term $\left(\frac{\omega_a}{2} + (\omega_b-\varpi)|\beta|^2 + \frac{\omega_b}{2} - \epsilon_d\beta^* - \epsilon_d^*\beta\right)\boldsymbol{I}$, we have $\boldsymbol{D}_\beta\widetilde{A}\boldsymbol{D}_{-\beta} = \boldsymbol{A}$.

4. In this frame $\boldsymbol{a}$ becomes $\boldsymbol{a}e^{-i\omega_a t}$, i.e. $e^{it\boldsymbol{A}_0/\hbar}\boldsymbol{a}e^{-it\boldsymbol{A}_0/\hbar} = \boldsymbol{a}e^{-i\omega_a t}$ and $\boldsymbol{b}$ becomes $\boldsymbol{b}e^{-i\Delta t}$, i.e. $e^{it\boldsymbol{A}_0/\hbar}\boldsymbol{b}e^{-it\boldsymbol{A}_0/\hbar} = \boldsymbol{b}e^{-i\Delta t}$. Thus

$$\boldsymbol{B}(t) = g_{ab}\Big(-\beta e^{-i\varpi t} - \beta^* e^{i\varpi t} + \boldsymbol{a}e^{-i\omega_a t} + \boldsymbol{a}^\dagger e^{i\omega_a t} + \boldsymbol{b}e^{-i\omega_b t} + \boldsymbol{b}^\dagger e^{i\omega_b t}\Big)^4.$$

5. Since the three operators $\boldsymbol{a}e^{-i\omega_a t} + \boldsymbol{a}^\dagger e^{i\omega_a t}$, $\boldsymbol{b}e^{-i\omega_b t} + \boldsymbol{b}^\dagger e^{i\omega_b t}$ and $-(\beta e^{-i\varpi t} + \beta^* e^{i\varpi t})\boldsymbol{I}$ commute, we have

$$\boldsymbol{B}(t) = \sum_{n+n_a+n_b=4} \frac{4!(-1)^n}{n_a!\,n_b!\,n!}\left(\beta e^{-i\varpi t} + \beta^* e^{i\varpi t}\right)^n \left(\boldsymbol{a}e^{-i\omega_a t} + \boldsymbol{a}^\dagger e^{i\omega_a t}\right)^{n_a}\left(\boldsymbol{b}e^{-i\omega_b t} + \boldsymbol{b}^\dagger e^{i\omega_b t}\right)^{n_b}.$$

Since $\overline{\omega} = 2\omega_a - \omega_b$ and $n_a\omega_a - n_b\omega_b \neq 0$, secular terms appear only in the monomials associated to the following values of $(n, n_a, n_b)$:

$$(n, n_a, n_b) \in \Big\{(4,0,0),\ (0,4,0),\ (0,0,4),\ (2,2,0),\ (2,0,2),\ (0,2,2),\ (1,2,1)\Big\}.$$

The secular term associated to $(n, n_a, n_b) = (4,0,0)$ is $6g_{ab}\beta^2\beta^{*2}$, i.e. an irrelevant global phase term.

The secular term associated to $(n, n_a, n_b) = (0,4,0)$ is given by

$$g_{ab}\Big(\boldsymbol{a}^\dagger(\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{a} + \boldsymbol{a}\boldsymbol{a}^\dagger\boldsymbol{a} + \boldsymbol{a}\boldsymbol{a}\boldsymbol{a}^\dagger) + \boldsymbol{a}(\boldsymbol{a}\boldsymbol{a}^\dagger\boldsymbol{a}^\dagger + \boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{a}^\dagger + \boldsymbol{a}^\dagger\boldsymbol{a}^\dagger\boldsymbol{a})\Big)$$

since $\boldsymbol{a}$ and $\boldsymbol{a}^\dagger$ do not commute. Using $\boldsymbol{a}\boldsymbol{a}^\dagger = 1 + \boldsymbol{a}^\dagger\boldsymbol{a}$, we have $\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{a} + \boldsymbol{a}\boldsymbol{a}^\dagger\boldsymbol{a} + \boldsymbol{a}\boldsymbol{a}\boldsymbol{a}^\dagger = 3(\boldsymbol{a}^\dagger\boldsymbol{a} + 1)\boldsymbol{a}$ and

$$\boldsymbol{a}^\dagger(\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{a} + \boldsymbol{a}\boldsymbol{a}^\dagger\boldsymbol{a} + \boldsymbol{a}\boldsymbol{a}\boldsymbol{a}^\dagger) = 3\boldsymbol{a}^{\dagger 2}\boldsymbol{a}^2 + 3\boldsymbol{a}^\dagger\boldsymbol{a}.$$

Similarly $\boldsymbol{a}\boldsymbol{a}^\dagger\boldsymbol{a}^\dagger + \boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{a}^\dagger + \boldsymbol{a}^\dagger\boldsymbol{a}^\dagger\boldsymbol{a} = 3\boldsymbol{a}^\dagger(\boldsymbol{a}^\dagger\boldsymbol{a} + 1)$ and thus

$$\boldsymbol{a}(\boldsymbol{a}\boldsymbol{a}^\dagger\boldsymbol{a}^\dagger + \boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{a}^\dagger + \boldsymbol{a}^\dagger\boldsymbol{a}^\dagger\boldsymbol{a}) = 3\boldsymbol{a}^{\dagger 2}\boldsymbol{a}^2 + 3\boldsymbol{a}^\dagger\boldsymbol{a} + 3$$

The secular term associated to $(n, n_a, n_b) = (0,4,0)$ reads, up-to an irrelevant global phase

$$6g_{ab}\left(\boldsymbol{a}^{\dagger 2}\boldsymbol{a}^2 + \boldsymbol{a}^\dagger\boldsymbol{a}\right).$$

The computation are the same for the secular term of $(n, n_a, n_b) = (0,0,4)$ (swap $\boldsymbol{a}$ and $\boldsymbol{b}$):

$$6g_{ab}\left(\boldsymbol{b}^{\dagger 2}\boldsymbol{b}^2 + \boldsymbol{b}^\dagger\boldsymbol{b}\right).$$

For $(n, n_a, n_b) = (2,2,0)$ and $(n, n_a, n_b) = (2,0,2)$ we get, up-to an irrelevant global phase,

$$24g_{ab}|\beta|^2\boldsymbol{a}^\dagger\boldsymbol{a} \quad \text{and} \quad 24g_{ab}|\beta|^2\boldsymbol{b}^\dagger\boldsymbol{b}$$

For $(n, n_a, n_b) = (0,2,2)$ we get, up-to an irrelevant global phase,

$$24g_{ab}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{b}^\dagger\boldsymbol{b} + 12g_{ab}\big(\boldsymbol{a}^\dagger\boldsymbol{a} + \boldsymbol{b}^\dagger\boldsymbol{b}\big)$$

For $(n, n_a, n_b) = (1,2,1)$ we have only two secular terms in

$$-12g_{ab}\big(\beta e^{-i(2\omega_a-\omega_b)t} + \beta^* e^{i(2\omega_a-\omega_b)t}\big)\left(\boldsymbol{a}e^{-i\omega_a t} + \boldsymbol{a}^\dagger e^{i\omega_a t}\right)^2\left(\boldsymbol{b}e^{-i\omega_b t} + \boldsymbol{b}^\dagger e^{i\omega_b t}\right)$$

that are $-12g_{ab}\beta\boldsymbol{a}^{\dagger 2}\boldsymbol{b}$ and its Hermitian conjugate.

Gathering these secular terms, we get

$$\bar{\boldsymbol{B}} = 6g_{ab}\Bigg((3 + 4|\beta|^2)\big(\boldsymbol{a}^\dagger\boldsymbol{a} + \boldsymbol{b}^\dagger\boldsymbol{b}\big) + \big(\boldsymbol{a}^{\dagger 2}\boldsymbol{a}^2 + \boldsymbol{b}^{\dagger 2}\boldsymbol{b}^2\big) + 4\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{b}^\dagger\boldsymbol{b} - 2\big(\beta\boldsymbol{a}^{\dagger 2}\boldsymbol{b} + \beta^*\boldsymbol{a}^2\boldsymbol{b}^\dagger\big)\Bigg).$$

Thus

$$\delta_a = \delta_b = 6g_{ab}(3 + 4|\beta|^2),\ \chi_{aa} = \chi_{bb} = 6g_{ab},\ \chi_{ab} = 24g_{ab},\ g_{2ph} = -12g_{ab}\beta.$$

## Problem 2

1. (a) We have

$$\boldsymbol{L}^\dagger \boldsymbol{L} = \boldsymbol{a}^{\dagger\,2}\boldsymbol{a}^2 = \boldsymbol{a}^\dagger(\boldsymbol{a}^\dagger\boldsymbol{a})\boldsymbol{a} = \boldsymbol{a}^\dagger\boldsymbol{N}\boldsymbol{a} = \boldsymbol{a}^\dagger\boldsymbol{a}(\boldsymbol{N}-1) = \boldsymbol{N}(\boldsymbol{N}-1).$$

(b) From $\frac{d}{dt}p_n = \langle n\,|\frac{d}{dt}\rho|\,n\rangle$ and $\boldsymbol{a}^{\dagger\,2}|n\rangle = \sqrt{(n+1)(n+2)}|n+2\rangle$ we get

$$\begin{aligned}\frac{d}{dt}p_n &= \left\langle n\left|\boldsymbol{a}^2\rho\boldsymbol{a}^{\dagger\,2}\right|n\right\rangle - \frac{1}{2}\langle n\,|\boldsymbol{N}(\boldsymbol{N}-1)\rho + \rho\boldsymbol{N}(\boldsymbol{N}-1)|\,n\rangle \\ &= (n+1)(n+2)\langle n+2\,|\rho|\,n+2\rangle - n(n-1)\langle n\,|\rho|\,n\rangle \\ &= (n+1)(n+2)p_{n+2} - n(n-1)p_n.\end{aligned}$$

(c) Set $\bar{p}_n = \langle n\,|\bar{\rho}|\,n\rangle$. Then $(n+1)(n+2)\bar{p}_{n+2} = n(n-1)\bar{p}_n$ for all $n \geq 0$. Thus for all $n \geq 2$, $\bar{p}_n = 0$. Since $\bar{\rho} \geq 0$, this means that for all $n, m$ with $n \geq 2$ or $m \geq 2$, we have $\langle n\,|\bar{\rho}|\,m\rangle = 0$. The support of $\bar{\rho}$ is in span$(|0\rangle, |1\rangle)$ and thus reads $\bar{\rho} = \bar{p}_0|0\rangle\langle 0| + (1-\bar{p}_0)|1\rangle\langle 1| + \bar{c}|1\rangle\langle 0| + \bar{c}^*|0\rangle\langle 1|$ since $\mathrm{Tr}\,(\bar{\rho}) = 1$ and $\bar{\rho} \geq 0$. Moreover $\bar{p}_0(1-\bar{p}_0) \geq |\bar{c}|^2$.

2. We have

$$\begin{aligned}\frac{d}{dt}\big(\,\mathrm{Tr}\,(\rho J)\,\big) &= \mathrm{Tr}\left(J\left(\boldsymbol{L}\rho\boldsymbol{L}^\dagger - \tfrac{1}{2}(\boldsymbol{L}^\dagger\boldsymbol{L}\rho + \rho\boldsymbol{L}^\dagger\boldsymbol{L})\right)\right) \\ &= \mathrm{Tr}\left(\boldsymbol{L}^\dagger J\boldsymbol{L}\rho - \tfrac{1}{2}(J\boldsymbol{L}^\dagger\boldsymbol{L}\rho + J\rho\boldsymbol{L}^\dagger\boldsymbol{L}J)\right) = \mathrm{Tr}\,(\mathcal{L}^*(J)\rho)\,.\end{aligned}$$

3. (a) We have

$$\begin{aligned}\mathcal{L}^*(f(\boldsymbol{N})) &= \boldsymbol{a}^{\dagger\,2}f(\boldsymbol{N})\boldsymbol{a}^2 - \tfrac{1}{2}(\boldsymbol{N}(\boldsymbol{N}-1)f(\boldsymbol{N}) + f(\boldsymbol{N})\boldsymbol{N}(\boldsymbol{N}-1)) \\ &= \boldsymbol{a}^{\dagger\,2}\boldsymbol{a}^2 f(\boldsymbol{N}-2) - \boldsymbol{N}(\boldsymbol{N}-1)f(\boldsymbol{N}) = \boldsymbol{N}(\boldsymbol{N}-1)(f(\boldsymbol{N}-2) - f(\boldsymbol{N})).\end{aligned}$$

Since $\boldsymbol{N}(\boldsymbol{N}-1) \geq 0$ and $f(\boldsymbol{N}-2) \leq f(\boldsymbol{N})$ we have $\mathcal{L}^*(f(\boldsymbol{N})) \leq 0$.

(b) We have $\frac{d}{dt}V(\rho) = \mathrm{Tr}\,(\mathcal{L}^*(\boldsymbol{N})\rho) = -2\,\mathrm{Tr}\,(\boldsymbol{N}(\boldsymbol{N}-1)\rho)$ since $\mathcal{L}^*(\boldsymbol{N}) = -2\boldsymbol{N}(\boldsymbol{N}-1)$. Thus $\frac{d}{dt}V \leq 0$ and $V$ is a Lyapunov function in the sense that its time-derivative is non-positive.

(c) Assume that $\frac{d}{dt}V = 0$. Then $\mathrm{Tr}\,(\boldsymbol{N}(\boldsymbol{N}-1)\rho) = 0$. Since $\boldsymbol{N}(\boldsymbol{N}-1)$ and $\rho$ are non-negative Hermitian operators,$\mathrm{Tr}\,(\boldsymbol{N}(\boldsymbol{N}-1)\rho) = 0$ implies that $\boldsymbol{N}(\boldsymbol{N}-1)\rho = 0$: the range of $\rho$ is included in the kernel of $\boldsymbol{N}(\boldsymbol{N}-1)$, i.e. in span$(|0\rangle, |1\rangle)$. According to question 1c, this means that $\rho$ is a steady state, i.e., $\mathcal{L}(\rho) = 0$.

(d) Since for each $t \geq 0$, $\rho(t)$ depends linearly on its initial condition $\rho_0$, its limits for $t$ tending to infinity depends also linearly on $\rho_0$.

4. This results from the fact that $\frac{d}{dt}\,\mathrm{Tr}\,(J\rho) = \mathrm{Tr}\,(J\mathcal{L}(\rho)) = \mathrm{Tr}\,(\mathcal{L}^*(J)\rho) = 0$.

5. With question 3a, we have $\mathcal{L}^*(f(\boldsymbol{N})) = 0$ when $f(\boldsymbol{N}-2) = f(\boldsymbol{N})$. The operator $J_0 = (1 + (-1)^{\boldsymbol{N}})/2$ is defined via a function $f$ that is 2-periodic. For any $t \geq 0$, $\mathrm{Tr}\,(J_0\rho(t)) = \mathrm{Tr}\,(J_0\rho_0)$. Since $\lim_{t\mapsto+\infty}\rho(t) = \boldsymbol{K}(\rho_0)$, we get the result since $\mathrm{Tr}\,(J_0\boldsymbol{K}(\rho_0)) = \left\langle 0\middle|\boldsymbol{K}(\rho_0)\middle|0\right\rangle$ and $1 = \mathrm{Tr}\,(\boldsymbol{K}(\rho_0)) = \left\langle 0\middle|\boldsymbol{K}(\rho_0)\middle|0\right\rangle + \left\langle 1\middle|\boldsymbol{K}(\rho_0)\middle|1\right\rangle$.

6. We have

$$\mathcal{L}^*(f(\boldsymbol{N})\boldsymbol{a}) = \boldsymbol{a}^{\dagger\,2} f(\boldsymbol{N})\boldsymbol{a}^3 - \tfrac{1}{2}(\boldsymbol{N}(\boldsymbol{N}-1)f(\boldsymbol{N})\boldsymbol{a} + f(\boldsymbol{N})\boldsymbol{a}\boldsymbol{N}(\boldsymbol{N}-1))$$
$$= \Big(\boldsymbol{a}^{\dagger\,2}\boldsymbol{a}^2 f(\boldsymbol{N}-2) - \tfrac{1}{2}\boldsymbol{N}(\boldsymbol{N}-1)f(\boldsymbol{N}) - \tfrac{1}{2}f(\boldsymbol{N})\boldsymbol{N}(\boldsymbol{N}+1)\Big)\boldsymbol{a}$$
$$= \boldsymbol{N}\Big((\boldsymbol{N}-1)f(\boldsymbol{N}-2) - \boldsymbol{N}f(\boldsymbol{N})\Big)\boldsymbol{a}.$$

This means that $\mathcal{L}^*(f(\boldsymbol{N})\boldsymbol{a})|0\rangle = 0$, $\mathcal{L}^*(f(\boldsymbol{N})\boldsymbol{a})|1\rangle = 0$ and

$$\forall n \geq 2, \quad \mathcal{L}^*(f(\boldsymbol{N})\boldsymbol{a})|n\rangle = (n-1)\sqrt{n}\big((n-2)f(n-3) - (n-1)f(n-1)\big)|n-1\rangle.$$

Thus when $f(1) = 0$ and $nf(n) = (n-1)f(n-2)$ for all $n \geq 2$, we have $\mathcal{L}^*(f(\boldsymbol{N})\boldsymbol{a})|n\rangle = 0$ for all $n \geq 0$.

7. (a) For $n \geq 1$, we have $g_{n+1}/g_n = \frac{\sqrt{(2n+1)(2n+3)}}{2n+2} < 1$ (geometric mean smaller than arithmetic mean). Thus $g$ is strictly decreasing and $g_n < g_1 = \sqrt{3/4} < 1 = g_0$.

   (b) For $n \geq 0$, $J_1|2n\rangle = 0$ and $J_1|2n+1\rangle = g_n|2n\rangle$. Since $g_n$ is bounded, $J_1$ is bounded. Since for any $n \geq 0$ the norm of $J_1|n\rangle$ is less than 1 and $J_1|1\rangle = |0\rangle$, this implies that
   $$\sup_{\langle\psi|\psi\rangle=1} \left\langle \psi|J_1^\dagger J_1|\psi \right\rangle = 1.$$

   (c) From question 1c, we have
   $$\boldsymbol{K}(\rho_0) = \bar{p}_0|0\rangle\langle 0| + (1-\bar{p}_0)|1\rangle\langle 1| + \bar{c}|1\rangle\langle 0| + \bar{c}^*|0\rangle\langle 1|.$$
   By definition of $J_0$ and $J_1$, we have $\bar{p}_0 = \operatorname{Tr}(J_0\boldsymbol{K}(\rho_0))$ and $\bar{c} = \operatorname{Tr}(J_1\boldsymbol{K}(\rho_0))$. Since $J_0$ and $J_1$ are invariant operators, we have $\operatorname{Tr}(J_s\boldsymbol{K}(\rho_0)) = \operatorname{Tr}(J_s\rho_0)$ for $s = 0, 1$.

8. (a) This results directly from the fact that for all $n$, $J_1|2n\rangle = 0$ and $J_1|2n+1\rangle = g_n|2n\rangle$.

   (b) From $\operatorname{Tr}(J_0\rho_0) = \sum_n \langle 2n|\rho_0|2n\rangle$ and $\operatorname{Tr}(J_1\rho_0) = \sum_n g_n \left\langle 2n+1\middle|\rho_0\middle|2n\right\rangle$ we have
   $$\boldsymbol{K}(\rho_0) = \sum_{n\geq 0} \langle 2n|\rho_0|2n\rangle\,|0\rangle\langle 0| + \langle 2n+1|\rho_0|2n+1\rangle\,|1\rangle\langle 1|$$
   $$+ \sum_{n\geq 0} g_n \left\langle 2n+1\middle|\rho_0\middle|2n\right\rangle |1\rangle\langle 0| + g_n \left\langle 2n\middle|\rho_0\middle|2n+1\right\rangle |0\rangle\langle 1|.$$
   where we have used $1 - \operatorname{Tr}(J_0\rho_0) = \operatorname{Tr}(\rho_0) - \sum_n \langle 2n|\rho_0|2n\rangle = \sum_n \langle 2n+1|\rho_0|2n+1\rangle$. For each $n$ we have
   $$\left\langle 2n+1\middle|\rho_0\middle|2n\right\rangle |1\rangle\langle 0| + \left\langle 2n\middle|\rho_0\middle|2n+1\right\rangle |0\rangle\langle 1|$$
   $$= \Big(|0\rangle\langle 2n| + |1\rangle\langle 2n+1|\Big)\rho_0\Big(|2n\rangle\langle 0| + |2n+1\rangle\langle 0|\Big)$$
   $$- \langle 2n|\rho_0|2n\rangle\,|0\rangle\langle 0| - \langle 2n+1|\rho_0|2n+1\rangle\,|1\rangle\langle 1|.$$
   With identity $\langle n'|\rho_0|n'\rangle\,|0\rangle\langle 0| = |0\rangle\langle n'|\rho_0|n'\rangle\langle 0|$, we get
   $$\boldsymbol{K}(\rho_0) = \sum_{n\geq 0}(1-g_n)|0\rangle\langle 2n|\rho_0|2n\rangle\langle 0| + (1-g_n)|1\rangle\langle 2n+1|\rho_0|2n+1\rangle\langle 1|$$
   $$+ \sum_{n\geq 0} g_n\Big(|0\rangle\langle 2n| + |1\rangle\langle 2n+1|\Big)\rho_0\Big(|2n\rangle\langle 0| + |2n+1\rangle\langle 0|\Big).$$

Since $0 < g_n \leq 1$, we have $a_n = b_n = \sqrt{1 - g_n}$ and $c_n = \sqrt{g_n}$.
From $A_n^\dagger A_n = (1 - g_n)|2n\rangle\langle 2n|$, $B_n^\dagger B_n = (1 - g_n)|2n+1\rangle\langle 2n+1|$ and $C_n^\dagger C_n = g_n|2n\rangle\langle 2n| + g_n|2n+1\rangle\langle 2n+1|$ we get

$$\sum_{n\geq 0} A_n^\dagger A_n + B_n^\dagger B_n + C_n^\dagger C_n = \sum_{n\geq 0} |2n\rangle\langle 2n| + |2n+1\rangle\langle 2n+1| = \boldsymbol{I}.$$