Brief paper

Convergence of gradient observer for rotor position and magnet flux estimation of permanent magnet synchronous motors☆

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ABSTRACT

In Bernard and Praly (2017), we introduced a new sensorless rotor position observer for permanent magnet synchronous motors which does not require the knowledge of the magnet's flux : only electrical measurements and knowledge of the resistance and inductance are needed. In fact, this observer extends the gradient observer from Lee et al. (2010) with the estimation of the magnet's flux. In this paper, we prove its asymptotic stability provided the voltages/intensities (and some of their derivatives) are bounded, and the rotation speed remains away from zero. The proof relies on finding appropriate changes of coordinates allowing the construction of a weak Lyapunov function by backstepping, and the study of its invariant sets.

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1. Introduction

1.1. Context

To minimize the cost and increase the reliability of Permanent Magnet Synchronous Motors (PMSM), it is still important to make progress on estimating their state variables, in particular the rotor position and speed, with a minimum of sensors and fast algorithms. To this end, studies have been made for a long time on the so-called "sensorless" control which uses no mechanical variables measurement, only electrical ones. A review of the first used methods was given in Acarnley and Watson (2006), then a Luenberger observer was proposed in Poulain, Praly, and Ortega (2008). More recently, a very simple gradient observer, proposed in Lee et al. (2010) and analyzed in Ortega, Praly, Astolfi, Lee, and Nam (2011), has been shown to be extremely effective in practice as rotor position estimator. From the theoretical view point it is only conditionally convergent but it was shown in Malaizé, Praly, and Henwood (2012) how, via a very minor modification, it can be made globally convergent thanks to convexity properties.

These observers typically require the knowledge of the resistance, magnet flux and inductance. Unfortunately while the latter may be considered as known and constant (as long as there is no

magnetic saturation), the other two do vary significantly with the temperature and these variations should be taken into account in the observer. For example, for a given injected current, when the magnet's temperature increases, its magnetic flux decreases, and the produced torque becomes smaller. Therefore, an online estimation of the magnet's flux would enable to adapt the control law in real time and thus ensure a torque control which is robust to the machine's temperature, and also have an estimation of the rotor's temperature and magnet's magnetization degradation with time.

That is why efforts have been made to look for observers which do not rely on the knowledge of those parameters. The case where the magnet flux is unknown but resistance and inductance are known is addressed in Henwood, Malaizé, and Praly (2012) with the design of a Luenberger observer (see Henwood, 2014 for a much more detailed analysis), and in Bobtsov, Bazylev, Pyrkin, Aranovsky, and Ortega (2016), Bobtsov, Pyrkin, and Ortega (2015) and Bobtsov, Stankovic et al. (2015), with the design of an observer based on tools from parameter linear identification. In Bernard and Praly (2017), we proposed, for the same case, another observer which is a direct extension, with estimation of the magnet flux, of the gradient observer obtained in Lee et al. (2010). We claimed its convergence, and compared it to the other aforementioned observers in terms of sensitivity to errors in the parameters and to the presence of saliency. In particular we have shown that, when the currents in the rotating frame and the rotation speed are constant, an error in the values of the resistance and the inductance induces a bias on the estimated flux and rotor position that we have quantified. We have also reported on the performances achieved in open-loop via simulations using real data. In this paper, as

a complement of Bernard and Praly (2017), we concentrate our attention only on the proof, not provided in Bernard and Praly (2017), of convergence of the new observer in ideal conditions.

1.2. System model and problem statement

Using Joule's and Faraday's laws, a simple PMSM model expressed in a fixed $\alpha\beta$-frame reads

$$\dot{\Psi} = u - R\,i \tag{1}$$

where Ψ is the total flux generated by the windings and the permanent magnet, (u, i) are the voltage and intensity of the current in the fixed frame and R the stator winding resistance. The quantities u, i and Ψ are two dimensional vectors, and, for the case of a non-salient PMSM, the total flux may be expressed as

$$\Psi = Li + \Phi \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{2}$$

where L is the inductance, Φ the magnet's flux, and θ the electrical phase. This relation implies

$$|\Psi - Li|^2 - \Phi^2 = 0 \tag{3}$$

and the electrical phase θ is nothing but the argument of $\Psi - Li$. It follows that, in the case where L and i are known, θ can be recovered simply through an estimate of the total flux Ψ.

Therefore, our interest in this work is about observers of Ψ using measurements of u and i, knowledge of R and L but not of Φ. In fact, we go further and look for observers for the augmented system

$$\begin{cases} \dot{\Psi} = u - Ri \\ \dot{\Phi} = 0 \\ y = |\Psi - Li|^2 - \Phi^2 \end{cases} \tag{4}$$

with inputs (u, i), known parameters (R, L), state (Ψ, Φ) and output y which is known to be constantly zero according to (3).

Notations: The rotation matrix of angle θ is denoted $\mathcal{R}(\theta)$, i-e

$$\mathcal{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

2. Gradient observer

2.1. Main result

Originally, in Lee et al. (2010), the authors proposed the gradient observer

$$\dot{\hat{\Psi}} = u - Ri - 2q(\hat{\Psi} - Li)\left(\left|\hat{\Psi} - Li\right|^2 - \Phi^2\right) \tag{5}$$

for System (1), with q some strictly positive real number. This observer turned out to be quite efficient in practice but it was proved in Ortega et al. (2011) that it was only conditionally convergent. In particular it may admit several equilibrium points depending on the rotation speed ω. Later in Malaizé et al. (2012), it was shown that taking rather the following "convexified" gradient observer

$$\dot{\hat{\Psi}} = u - Ri - 2q(\hat{\Psi} - Li)\max\left(\left|\hat{\Psi} - Li\right|^2 - \Phi^2, 0\right) \tag{6}$$

enables to achieve global asymptotic stability.

But in Bernard and Praly (2017), we proposed rather to extend directly the gradient observer (5) with the estimation of Φ, namely

$$\begin{cases} \dot{\hat{\Psi}} = u - Ri - 2q(\hat{\Psi} - Li)\left(\left|\hat{\Psi} - Li\right|^2 - \hat{\Phi}^2\right) \\ \dot{\hat{\Phi}} = q\,\hat{\Phi}\left(\left|\hat{\Psi} - Li\right|^2 - \hat{\Phi}^2\right) \end{cases} \tag{7}$$

where q is an arbitrary strictly positive real number. We claimed that, without any convexification, this system is an asymptotically stable observer for System (4) as soon as the input signals (u, i) (and their derivatives) are bounded, and the rotor rotation speed is lower-bounded away from zero. More precisely:

Theorem 1. *Consider* (ψ, Φ) *in* $\mathbb{R}^2 \times (0, +\infty)$ *and inputs* $u, i : \mathbb{R} \to \mathbb{R}^2$ *such that there exist strictly positive numbers* $\overline{\omega}_1$, $\overline{\omega}_0$, *and* $\underline{\omega}_0$ *such that the solution* $(\Psi(\psi; t; u, i), \Phi)$ *of* (4) *verifies*

$$0 < \underline{\omega}_0 \leq \dot{\theta}(t) \leq \overline{\omega}_0 \quad , \quad \ddot{\theta}(t) \leq \overline{\omega}_1 \tag{8}$$

with

$$\theta(t) = \arg(\Psi(\psi; t; u, i) - Li(t)).$$

Then, this solution $(\Psi(\psi; t; u, i), \Phi)$ *of* (4) *is an asymptotically stable solution of* (7) *with basin of attraction containing the forward invariant set* $\Omega = \mathbb{R}^2 \times (0, +\infty)$.

The goal of this paper is to provide the proof of this result. The observer has a very simple expression and is cheap in terms of computing time. But as in Ortega et al. (2011), its convergence analysis has some tricky points. First, we do a change of coordinates to transform the problem of asymptotic stability of a solution into one of an equilibrium. A second transformation allows us to obtain a feedback form to which backstepping tools can be applied to obtain a (weak) Lyapunov function. This enables to establish stability, boundedness and convergence of some quantities. All these steps are very standard. But to conclude, we need a finer and ad hoc analysis of the ω-limit points.

Remark 1. According to (1) and (2), the assumption of boundedness in time of $\omega = \dot{\theta}$ and $\dot{\omega} = \ddot{\theta}$ is achieved as soon as the signals $(u, i, \dot{u}, \widehat{i}, \widehat{i})$ are bounded. The values of the bounds do not matter, as long as they exist. But they do have an effect on the behavior and in particular on the convergence speed and the magnitude of the solutions.

Remark 2. The fact that the rotation speed ω should stay away from zero is quite standard and related to the observability of the system. In Bobtsov et al. (2016), Bobtsov et al. (2015) and Bobtsov, Stankovic et al. (2015), this assumption appears through the persistent excitation condition, and in Henwood et al. (2012), it is a condition for the invertibility of the Luenberger transformation.

2.2. Change of coordinates

Consider any solution (Ψ, Φ) of (4) with Φ in $(0, \infty)$ and define

$$\theta(t) = \arg(\Psi(t) - Li(t)),$$

as in Theorem 1. Since we know that $y(t) = 0$ for all t, we have

$$\Psi(t) = Li(t) + \Phi \begin{pmatrix} \cos\theta(t) \\ \sin\theta(t) \end{pmatrix}. \tag{9}$$

To simplify the analysis, we transform the solution $(\Psi, \Phi) = \left(Li + \Phi \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \Phi\right)$ into an equilibrium. To that end, we consider the coordinates

$$\begin{pmatrix} X_d \\ X_q \end{pmatrix} = \mathcal{R}(-\theta)(\Psi - Li), \quad \begin{pmatrix} \hat{X}_d \\ \hat{X}_q \end{pmatrix} = \mathcal{R}(-\theta)(\hat{\Psi} - Li),$$

i.e the solution (Ψ, Φ) is transformed into the constant point $(\Phi, 0, \Phi)$. In those coordinates, the dynamics of the observer (7)

read

$$\begin{cases} \dot{\hat{X}}_d = \omega\hat{X}_q - 2q\hat{X}_d\left(\hat{X}_d^2 + \hat{X}_q^2 - \hat{\Phi}^2\right) \\ \dot{\hat{X}}_q = -\omega\hat{X}_d + \omega\Phi - 2q\hat{X}_q\left(\hat{X}_d^2 + \hat{X}_q^2 - \hat{\Phi}^2\right) \\ \dot{\hat{\Phi}} = q\,\hat{\Phi}\left(\hat{X}_d^2 + \hat{X}_q^2 - \hat{\Phi}^2\right) \end{cases} \qquad (10)$$

where $\omega(t) = \dot{\theta}(t)$ is considered an input satisfying (8).

When the model (1) is exact, the dynamics of the observer are fully described by (10). This shows very clearly that its behavior depend only on ω. Moreover, we see that these dynamics are invariant under the following transformation $(t, \omega, q, \hat{X}_d, \hat{X}_q, \hat{\Phi}) \mapsto (t/k, k\omega, kq, \hat{X}_d, \hat{X}_q, \hat{\Phi})$ where k is any strictly positive real number. We conclude that the time scale is entirely dictated by ω. Namely if the rotor "turns" k times faster, and we multiply q by k, then the estimates converge k times faster.

We finally conclude that Theorem 1 holds if we can prove the following lemma:

Lemma 2. *Consider a strictly positive real number Φ and a function $\omega : [0, +\infty) \to \mathbb{R}$ such that there exists $\underline{\omega}_0 > 0$, $\overline{\omega}_0 > 0$, and $\overline{\omega}_1 > 0$ such that for all t in $[0, +\infty)$*

$$\underline{\omega}_0 \leq \omega(t) \leq \overline{\omega}_0 \quad , \quad \dot{\omega}(t) \leq \overline{\omega}_1.$$

Then, $(\Phi, 0, \Phi)$ is a globally asymptotically stable equilibrium point of the dynamics (10) with basin of attraction containing the forward invariant set $\Omega = \mathbb{R}^2 \times (0, +\infty)$.

3. Proof of Lemma 2

3.1. Lyapunov function candidate

Our first step for the analysis is to look for a Lyapunov function. To help us find a possible candidate, we do another change of coordinates aiming at getting the dynamics in a triangular form, the so-called feedback form. Our motivation is that for this specific form, we have the backstepping methodology allowing us in particular to build Lyapunov functions. This task is easily achieved after noticing that we have, $\hat{\Phi}$ being non zero when the solution is in Ω,

$$\dot{\hat{X}}_d + 2\hat{X}_d\frac{\dot{\hat{\Phi}}}{\hat{\Phi}} = \omega\hat{X}_q \quad , \quad \dot{\hat{X}}_q + 2\hat{X}_q\frac{\dot{\hat{\Phi}}}{\hat{\Phi}} = -\omega\hat{X}_d + \omega\Phi$$

and therefore

$$\overline{\dot{\hat{X}}_d\hat{\Phi}^2} = \omega\hat{X}_q\hat{\Phi}^2 \quad , \quad \overline{\dot{\hat{X}}_q\hat{\Phi}^2} = -\omega\hat{X}_d\hat{\Phi}^2 + \omega\hat{\Phi}^2\Phi.$$

Formally, this leads us to the second set of coordinates $(\hat{x}_1, \hat{x}_2, \hat{x}_3) = (\hat{X}_d\hat{\Phi}^2, \hat{X}_q\hat{\Phi}^2, \hat{\Phi}^4)$. As desired, the dynamics take the following feedback form

$$\dot{\hat{x}}_1 = \omega\hat{x}_2$$
$$\dot{\hat{x}}_2 = -\omega\hat{x}_1 + \omega\Phi\sqrt{\hat{x}_3}$$
$$\dot{\hat{x}}_3 = -4q\left(\hat{x}_3^{\frac{3}{2}} - (\hat{x}_1^2 + \hat{x}_2^2)\right)$$

which we can write compactly as:

$$\dot{\hat{x}}_{12} = f_{12}(\hat{x}_{12}, \hat{x}_3)$$
$$\dot{\hat{x}}_3 = -4q\left(\hat{x}_3^{\frac{3}{2}} - (\hat{x}_1^2 + \hat{x}_2^2)\right)$$

with $\hat{x}_{12} = (\hat{x}_1, \hat{x}_2)$. Now, a necessary condition to have a Lyapunov function V such that

$$\frac{\partial V}{\partial \hat{x}_{12}}(\hat{x}_{12}, \hat{x}_3)f_{12}(\hat{x}_{12}, \hat{x}_3)$$
$$- 4q\frac{\partial V}{\partial \hat{x}_3}(\hat{x}_{12}, \hat{x}_3)\left(\hat{x}_3^{\frac{3}{2}} - (\hat{x}_1^2 + \hat{x}_2^2)\right) \leq 0$$

is to have (just pick $\hat{x}_3^{\frac{3}{2}} = \hat{x}_1^2 + \hat{x}_2^2$)

$$\frac{\partial V}{\partial \hat{x}_{12}}(\hat{x}_{12}, (\hat{x}_1^2 + \hat{x}_2^2)^{\frac{2}{3}})f_{12}(\hat{x}_{12}, (\hat{x}_1^2 + \hat{x}_2^2)^{\frac{2}{3}}) \leq 0.$$

This suggests to find first a Lyapunov function for the system

$$\dot{\hat{x}}_1 = \omega\hat{x}_2$$
$$\dot{\hat{x}}_2 = -\omega\hat{x}_1 + \omega\Phi(\hat{x}_1^2 + \hat{x}_2^2)^{\frac{1}{3}}.$$

The latter system admits periodic orbits which are level sets of

$$V_1(\hat{x}_1, \hat{x}_2) = \frac{3}{4}(\hat{x}_1^2 + \hat{x}_2^2)^{2/3} - \Phi\hat{x}_1 + \frac{\Phi^4}{4}$$

which is positive, 0 only at $(\hat{x}_1, \hat{x}_2) = (\Phi^3, 0)$ and proper in $(\hat{x}_1, \hat{x}_2)$.

Then, inspired by the backstepping methodology (see Praly, d'Andréa Novel, & Coron, 1991), we look for a Lyapunov function in the form

$$V(\hat{x}) = V_1(\hat{x}_1, \hat{x}_2) + V_2(\hat{x}_3, r)$$

with

$$V_2(\hat{x}_3, r) = \int_{r^{2/3}}^{\hat{x}_3} \varphi(s, r)ds \quad , \qquad r = \hat{x}_1^2 + \hat{x}_2^2$$

where φ is a C^1 function satisfying

$$\varphi(\hat{x}_3, r)\left(\hat{x}_3^{\frac{3}{2}} - r\right) > 0 \qquad \forall r \neq \hat{x}_3^{\frac{3}{2}}. \qquad (11)$$

Along the solutions, we obtain

$$\dot{V} = \frac{\partial V_1}{\partial \hat{x}_{12}}(\hat{x}_{12}, \hat{x}_3)f_{12}(\hat{x}_{12}, \hat{x}_3) - 4q\frac{\partial V_2}{\partial \hat{x}_3}(\hat{x}_3, r)\left(\hat{x}_3^{\frac{3}{2}} - r\right)$$
$$+ \frac{\partial V_2}{\partial r}(\hat{x}_3, r)2\Phi\omega\hat{x}_2\sqrt{\hat{x}_3}$$
$$= \Phi\omega\hat{x}_2\left(\frac{\sqrt{\hat{x}_3}}{r^{1/3}} - 1\right) + \left[\int_{r^{2/3}}^{\hat{x}_3}\frac{\partial\varphi}{\partial r}(s, r)ds\right]2\Phi\omega\hat{x}_2\sqrt{\hat{x}_3}$$
$$- 4q\varphi(\hat{x}_3, r)\left(\hat{x}_3^{\frac{3}{2}} - r\right).$$

In view of (11), $\dot{V}$ is non positive if we select the function φ satisfying (11) and

$$\left[\int_{r^{2/3}}^{\hat{x}_3}\frac{\partial\varphi}{\partial r}(s, r)ds\right]2\Phi\omega\hat{x}_2\sqrt{\hat{x}_3} = -\Phi\omega\hat{x}_2\left(\frac{\sqrt{\hat{x}_3}}{r^{1/3}} - 1\right)$$

and thus

$$\left[\int_{r^{2/3}}^{\hat{x}_3}\frac{\partial\varphi}{\partial r}(s, r)ds\right] = \frac{1}{2}\left(\frac{1}{\sqrt{\hat{x}_3}} - \frac{1}{r^{1/3}}\right).$$

It is necessary to have $\frac{\partial\varphi}{\partial r}(\hat{x}_3, r) = -\frac{1}{4}\frac{1}{\hat{x}_3^{3/2}}$ so that we take $\varphi(\hat{x}_3, r) = \frac{1}{4}\left[1 - \frac{r}{\hat{x}_3^{3/2}}\right]$. This gives us

$$V_2(\hat{x}_3, r) = \frac{1}{4}\left[\hat{x}_3 - r^{2/3} + 2\left(\frac{r}{\hat{x}_3^{1/2}} - r^{2/3}\right)\right]$$

and finally

$$V = V_1 + V_2 = \frac{1}{4}\hat{x}_3 + \frac{1}{2}\frac{r}{\sqrt{\hat{x}_3}} - \Phi x_1 + \frac{\Phi^4}{4}.$$

In the original coordinates, the expression of V becomes

$$V(\hat{X}_d, \hat{X}_q, \hat{\Phi}) = \frac{\hat{\Phi}^4}{4} + \frac{1}{2}\hat{\Phi}^2(\hat{X}_d^2 + \hat{X}_q^2) - \Phi\hat{\Phi}^2\hat{X}_d + \frac{\Phi^4}{4}.$$

3.2. Stability analysis

For any $(\hat{X}_q, \hat{\Phi})$, $\hat{X}_d \mapsto (\hat{X}_d^2 + \hat{X}_q^2 - 2\Phi\hat{X}_d)$ reaches its minimum for $\hat{X}_d = \Phi$. Thus,

$$V \geq \frac{1}{2}\hat{\Phi}^2\hat{X}_q^2 + \frac{1}{4}(\hat{\Phi}^2 - \Phi^2)^2 \geq 0. \tag{12}$$

Therefore, V is positive and vanishes only at the equilibrium of interest $(\Phi, 0, \Phi)$. Besides, it satisfies

$$\dot{V} = -q\,\hat{\Phi}^2(\hat{\Phi}^2 - (\hat{X}_d^2 + \hat{X}_q^2))^2 \leq 0. \tag{13}$$

We have thus found a weak Lyapunov function for System (10) associated to the equilibrium $(\Phi, 0, \Phi)$ which is thus stable. It remains to prove that it is attractive.

3.3. Boundedness of solutions

Consider a solution $(\hat{X}_q, \hat{X}_d, \hat{\Phi})$ of System (10) maximally defined on $[0, \bar{t})$ in Ω. Because of (13), V is bounded on $[0, \bar{t})$ when evaluated along the solution. However, the function V is not proper (the sets $\left\{(\hat{X}_d, \hat{X}_q, \hat{\Phi}), V(\hat{X}_d, \hat{X}_q, \hat{\Phi}) \leq c\right\}$ are compact only for $c \leq \frac{\Phi^4}{4}$), and thus we cannot directly infer the boundedness of the solution. Nevertheless, (12) says that $\hat{\Phi}$ is bounded on $[0, \bar{t})$, let us say by Φ_m. Besides,

$$\overbrace{\dot{\hat{X}_d^2 + \hat{X}_q^2}} = -4q(\hat{X}_d^2 + \hat{X}_q^2)(\hat{X}_d^2 + \hat{X}_q^2 - \hat{\Phi}^2) + 2\omega\Phi X_q$$
$$\leq -4q(\hat{X}_d^2 + \hat{X}_q^2)^2 + 4q\Phi_m^2(\hat{X}_d^2 + \hat{X}_q^2)$$
$$+ 2\bar{\omega}_0\Phi\sqrt{\hat{X}_d^2 + \hat{X}_q^2}.$$

The negative term dominates for large values of $(\hat{X}_d^2 + \hat{X}_q^2)$, which implies that $(\hat{X}_d, \hat{X}_q)$ is also bounded on $[0, \bar{t})$. Now assume that $\bar{t}$ is finite. Since the solution is bounded, it tends to the boundary of Ω when t tends to $\bar{t}$, i.e $\hat{\Phi}$ tends to 0 (in finite time). But this is impossible, because of uniqueness of solution, knowing that $\hat{\Phi} = 0$ is a solution. Therefore $\bar{t}$ is infinite and any solution is defined in Ω and bounded on $[0, +\infty)$.

It follows from these arguments that the equilibrium is stable and all the solutions are bounded whatever the bounds $0 < \underline{\omega}_0 \leq \bar{\omega}_0$ and $0 \leq \bar{\omega}_1$ are. Let us now show that $(\hat{X}_q, \hat{X}_d, \hat{\Phi})$ converges to $(\Phi, 0, \Phi)$.

3.4. Convergence analysis

Note that (10) is time-varying because of ω so that LaSalle invariance principle may not apply. But since V decreases and is lower-bounded, it converges. Besides, the solution and ω being bounded $\dot{V}$ is bounded. It follows according to Barbalat's lemma that

$$\lim_{t\to+\infty}\dot{V} = \lim_{t\to+\infty}\hat{\Phi}(\hat{\Phi}^2 - (\hat{X}_d^2 + \hat{X}_q^2)) = 0.$$

Using again Barbalat's lemma on $\ddot{V}$ ($\dot{V}$ converges and $V^{(3)}$ is bounded because $\dot{\omega}$ is bounded by assumption) gives $\lim_{t\to+\infty}\ddot{V} = 0$ and thus

$$\lim_{t\to+\infty}\omega\Phi\hat{\Phi}\hat{X}_q = 0,$$

which yields since ω is lower-bounded away from zero,

$$\lim_{t\to+\infty}\hat{\Phi}\hat{X}_q = 0.$$

Finally, applying again Barbalat's lemma to the derivative of this function, we end up with

$$\lim_{t\to+\infty}\omega\hat{\Phi}(\hat{X}_d - \Phi) = 0,$$

and again since ω is lower-bounded,

$$\lim_{t\to+\infty}\hat{\Phi}(\hat{X}_d - \Phi) = 0.$$

To sum up, we have established the following three limits

$$\lim_{t\to+\infty}\hat{\Phi}(\hat{\Phi}^2 - (\hat{X}_d^2 + \hat{X}_q^2)) = 0, \quad \lim_{t\to+\infty}\hat{\Phi}\hat{X}_q = 0$$

$$\lim_{t\to+\infty}\hat{\Phi}(\hat{X}_d - \Phi) = 0.$$

This is not enough to conclude since we could have $\liminf_{t\to+\infty}\hat{\Phi} = 0$. However, the following points give the result:

(1) The time function $(\hat{X}_d, \hat{X}_q, \hat{\Phi})$ is bounded and continuous. It follows that for any sequence (t_n) such that $\lim_{n\to\infty}t_n = +\infty$, the sequence $(\hat{X}_d(t_n), \hat{X}_q(t_n), \hat{\Phi}(t_n))$ admits at least one accumulation point.

(2) Let $P^* = (\hat{X}_d^*, \hat{X}_q^*, \hat{\Phi}^*)$ be such an accumulation point. Because of the limits we have established, it verifies:

$$\hat{\Phi}^*(\hat{\Phi}^{*2} - (\hat{X}_d^{*2} + \hat{X}_q^{*2})) = 0$$
$$\hat{\Phi}^*\hat{X}_q^* = 0$$
$$\hat{\Phi}^*(\hat{X}_d^* - \Phi) = 0.$$

Thus P^* is either of the type $(\hat{X}_d^*, \hat{X}_q^*, 0)$ with $(\hat{X}_d^*, \hat{X}_q^*)$ in $\mathbb{R}^2$ (type I) or equal to $P_0 = (\Phi, 0, \Phi)$.

(3) Since the solution $(\hat{X}_d, \hat{X}_q, \hat{\Phi})$ is bounded in time, the set of its accumulation points (ω-limit set) is connected (see Filippov, 1988, §12,4,Corollary for instance). It follows that they are either all of type I, or equal to P_0. If we manage to prove that the first option is not possible, then the only accumulation point will be P_0 and the convergence will be proved.

(4) So, assume that P_0 is not an accumulation point, i.e any accumulation point is of type I. The only possible accumulation value for $\hat{\Phi}$ is 0. Thus

$$\lim_{t\to+\infty}\hat{\Phi}(t) = 0.$$

To ease the notations let us denote the vector $\hat{X} = (\hat{X}_d, \hat{X}_q)$. Solving the differential equation ruling $\hat{\Phi}^2$, there exist a_0, b_0 strictly positive such that

$$\hat{\Phi}^2 = \frac{\phi(t)}{1 + b_0 + 2q\int_0^t\phi(s)ds}$$

with $\phi(t) = a_0b_0\exp\left(2q\int_0^t|\hat{X}(s)|^2ds\right)$. Since $\hat{\Phi}$ tends to 0, for any $\eta > 0$, there exists $\bar{t} > 0$ such that for all $t \geq \bar{t}$, we have $\hat{\Phi}^2(t) \leq \frac{\eta}{2}$. This means that for all $t \geq \bar{t}$,

$$\phi(t) \leq \underbrace{\frac{\eta}{2}\left(1 + b_0 + 2q\int_0^{\bar{t}}\phi(s)ds\right)}_{c_0} + \eta q\int_{\bar{t}}^t\phi(s)ds$$

and by Gronwall's lemma

$$\phi(t) \leq c_0\exp\left(\eta q(t - \bar{t})\right).$$

We conclude that for any $\eta > 0$, there exists $\bar{t} > 0$ such that for all $t \geq \bar{t}$

$$\int_0^t|\hat{X}(s)|^2ds \leq \frac{\eta}{2}(t - \bar{t}) + \frac{1}{2q}\log\left(\frac{c_0}{a_0b_0}\right). \tag{14}$$

But we are going to prove the existence of $t_0, \eta > 0$, and a sequence (t_k) such that $\lim_{k\to\infty}t_k = +\infty$ and

$$\int_{t_0}^{t_k}|\hat{X}(s)|^2ds \geq \eta(t_k - t_0), \tag{15}$$

which contradicts (14). Indeed, consider the dynamics of $\hat{X}$ with inputs ω and $\hat{\Phi}$ satisfying

$$0 < \underline{\omega}_0 \leq \omega(t) \leq \overline{\omega}_0 \quad , \quad 0 < \hat{\Phi}(t) \leq \Phi_m.$$

By choosing a such that

$$a(\overline{\omega}_0 + 2q(a^2 + \Phi_m^2)) \leq \frac{\underline{\omega}_0 \Phi}{2},$$

we have $\dot{\hat{X}}_q(t) \geq \frac{\underline{\omega}_0 \Phi}{2}$ when $|\hat{X}(t)| \leq a$ and the conditions of Lemma 3 given in Appendix are satisfied for

$$x = \hat{X} \quad , \quad \underline{b} = \frac{\underline{\omega}_0 \Phi}{2} \quad , \quad \overline{t}_+ = +\infty \quad , \quad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

If there exists $t_0 > 0$ such that for all $t \geq t_0$, $|\hat{X}(t)| \geq \frac{a}{2}$, then (15) is true for any sequence (t_k), and $\eta = \frac{a^2}{4}$. So assume rather that this is not the case, i.e for any t_2, there exists $t_3 \geq t_2$ such that $|\hat{X}(t)| \leq \frac{a}{2}$. In particular, there exists t_0 such that $|\hat{X}(t_0)| \leq \frac{a}{2}$. Applying successively Lemma 3, one can build sequences $(t_{k,1})$, $(t_{k,2})$, $(t_{k,3})$, each tending to $+\infty$ such that for all $k \geq 1$:

$$t_{k,1} < t_{k,2} < t_{k,3}$$
$$t_{0,3} = t_0$$
$$|\hat{X}(t)| \leq \frac{a}{2} \quad \forall t \in [t_{k-1,3}, t_{k,1}]$$
$$\frac{a}{2} \leq |\hat{X}(t)| \leq a \quad \forall t \in [t_{k,1}, t_{k,2}]$$
$$|\hat{X}(t_{k,2})| = a$$
$$|\hat{X}(t)| \geq \frac{a}{2} \quad \forall t \in [t_{k,2}, t_{k,3}]$$

and

$$\frac{3a}{\underline{b}} \geq t_{k,2} - t_{k-1,3} \quad , \quad t_{k,2} - t_{k,1} \geq \frac{a}{2\overline{b}}.$$

We denote

$$\overline{t}_{k,1} = t_{k,1} - t_{k-1,3} \quad , \quad \overline{t}_{k,2} = t_{k,2} - t_{k,1}$$

$$\overline{t}_{k,3} = t_{k,3} - t_{k,2}$$

and $\overline{t}_k = t_{k,3} - t_{k-1,3} = \overline{t}_{k,1} + \overline{t}_{k,2} + \overline{t}_{k,3}$ the duration of the cycle k. Then, the mean over a cycle

$$\frac{1}{\overline{t}_k} \int_{t_{k-1,3}}^{t_{k,3}} |\hat{X}(s)|^2 ds \geq \frac{a^2}{4} \frac{\overline{t}_{k,2} + \overline{t}_{k,3}}{\overline{t}_{k,1} + \overline{t}_{k,2} + \overline{t}_{k,3}}$$
$$\geq \frac{a^2}{4} \frac{\frac{a}{2\overline{b}} + \overline{t}_{k,3}}{\frac{3a}{\underline{b}} + \overline{t}_{k,3}} \geq \frac{a^2}{4} \min \left(\frac{\underline{b}}{6\overline{b}}, 1 \right)$$

is lower-bounded. Thus, (15) holds with $\eta = \frac{a^2}{4} \min \left(\frac{\underline{b}}{6\overline{b}}, 1 \right)$ and $t_k = t_{k,3}$.

Finally, with (14) and η given above, there exists $\overline{t}$ such that

$$\eta(t_k - t_0) \leq \frac{\eta}{2}(t_k - \overline{t}) + \frac{1}{2q} \log \left(\frac{c_0}{a_0 b_0} \right)$$

for all k greater than some k_0. This is impossible. Thus, P_0 is the only accumulation point.

4. Conclusion

We have proved asymptotic convergence of the gradient observer (7) when the intensities/voltages (and some of their derivatives) are properly bounded and the rotation speed stays away from zero. This observer enables to estimate the rotor position of a PMSM without knowing its magnet flux. Its efficiency was illustrated in Bernard and Praly (2017) as well as its robustness with respect to errors on the inductance/resistance and to the presence of saliency.

Appendix

Lemma 3. *Let a, $\underline{b}$ and $\overline{b}$ be three strictly positive real numbers, v be a unit vector in $\mathbb{R}^n$ and f be a continuous function such that[1]:*

$$v^\top f(x, t) \geq \underline{b} \quad , \quad \overline{b} \geq |f(x, t)| \quad \forall(x, t) \in \overline{B_a(0)} \times \mathbb{R}.$$

Let $x(t)$ be a solution of

$$\dot{x} = f(x, t)$$

defined on $(\overline{t}_-, \overline{t}_+)$ with values in $\mathbb{R}^n$. If there exists t_0 in $(\overline{t}_-, \overline{t}_+)$ such that $|x(t_0)| \leq \frac{a}{2}$, then there exist t_1 and t_2 both in $(\overline{t}_-, \overline{t}_+)$ such that

$$|x(t_1)| = \frac{a}{2} \quad , \quad |x(t_2)| = a$$

$$\frac{3a}{\underline{b}} \geq t_2 - t_0 \quad , \quad t_2 - t_1 \geq \frac{a}{2\overline{b}}$$

and $\frac{a}{2} \leq |x(t)| \leq a$ for all t in $[t_1, t_2]$.

Proof. Let $t_2 < \overline{t}_+$ be the maximum time such that $x(t)$ is in $B_a(0)$ for all t in $[t_0, t_2)$. We have

$$v^\top \dot{x}(t) \geq \underline{b} \quad , \qquad \forall t \in [t_0, t_2)$$

and

$$v^\top x(t_0) \geq -|x(t_0)| \geq -\frac{a}{2}.$$

This yields

$$|a| > |x(t)| \geq v^\top x(t) \geq -\frac{a}{2} + \underline{b}[t - t_0] \quad \forall t \in [t_0, t_2).$$

Thus t_2 is finite and by continuity,

$$|x(t_2)| = a \quad , \qquad \frac{3a}{2\underline{b}} \geq t_2 - t_0.$$

By continuity of solutions, there also exists t_1 in $[t_0, t_2)$, satisfying:

$$|x(t_1)| = \frac{a}{2} \quad , \qquad \frac{a}{\underline{b}} \geq t_1 - t_0.$$

But we also have

$$x(t_2) = x(t_1) + \int_{t_1}^{t_2} f(x(t), t) dt$$

so that

$$|a| = |x(t_2)| \leq \frac{a}{2} + \overline{b}[t_2 - t_1]$$

and therefore

$$t_2 - t_1 \geq \frac{a}{2\overline{b}}.$$

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[1] We denote $B_a(0)$ the open ball of $\mathbb{R}^n$ centered at the origin with radius a and $\overline{B_a(0)}$ its closure.

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