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Robust design of nonlinear internal models without adaptation ☆

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ABSTRACT

We propose a solution to the problem of semiglobal output regulation for nonlinear minimum-phase systems driven by uncertain exosystems that does not rely upon conventional adaptation schemes to estimate the frequency of the exogenous signals. Rather, the proposed approach relies upon regression-like arguments used to derive a nonlinear internal model able to offset the presence of an *unknown* number of harmonic exogenous inputs of uncertain amplitude, phase and frequency. The design methodology guarantees asymptotic regulation if the dimension of the regulator exceeds a lower bound determined by the actual number of harmonic components of the exogenous input. If this is not the case, a bounded steady-state regulation error is ensured whose amplitude, though, can be arbitrarily decreased by acting on a design parameter of the regulator.

1. Introduction

In this paper, we propose a new approach to the problem of semiglobal output regulation for minimum-phase nonlinear systems in the presence of exogenous inputs consisting of a superposition of uncertain harmonic oscillations. The problem of asymptotically tracking/rejecting uncertain exogenous inputs has received increased attention in recent years. A detailed and thorough review of relevant earlier works, in the context of adaptive control theory as well as of output regulation theory, can be found in the recent paper (Marino & Tomei, 2011). Interest in the problem in question was particularly boosted by the appearance of the paper (Serrani, Isidori, & Marconi, 2001), in which the problem was addressed, for nonlinear systems under suitable assumptions, by means of a controller that includes an internal model of the exogenous input (as in the classical theory of output regulation) whose frequencies are adaptively tuned so as to obtain perfect tracking in steady-state. Since then, a number of further contributions have appeared, for linear as well as for nonlinear systems. Most of these contributions use a variety of tools and ideas typical of the adaptive control literature, to the purpose of adaptively tuning the parameters of an internal model of the otherwise uncertain exosystem. It is perhaps for this reason that this area of research has been sometimes referred to, with an abuse of terminology, as adaptive output regulation. In particular, the theory of adaptive observers was efficiently used in Marino and Tomei (2003), for linear systems, and subsequently in Ding (2003) and in Delli Priscoli, Marconi, and Isidori (2006), for nonlinear systems, to solve the problem in question, under appropriate assumptions, in a global and/or semi-global setting. A case of "large-scale systems" was dealt with in Ye and Huang (2003). In the recent works (Marino & Tomei, 2008, 2011) the problem of adaptive output regulation for uncertain minimum-phase linear systems was addressed in the relevant case in which the number of harmonics characterizing the exosystem is not exactly known. An adaptive error feedback control algorithm was proposed that guarantees *exponential* convergence of the error when the regulator exactly models all components of the exogenous input that are actually excited, while *asymptotic* convergence is guaranteed if the exosystem is overmodeled by the regulator. When the adaptive internal model undermodels the actual exosystem, the regulation error is reduced to a residual bound that decreases as the exosystem modeling error decreases. Applications of the theory of adaptive output regulation can be found in Isidori, Marconi, and Serrani (2003), to the problem of controlling the autonomous landing of a helicopter on a rolling ship, in Bonivento, Isidori, Marconi, and Paoli (2004), to the problem of rejecting spurious oscillations in a faulty induction motor, and in Serrani (2006), to the purpose of compensating for the effect of measurement noises. Implicit in the results of these

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papers (see e.g. in Serrani, 2006, Marino & Tomei, 2011, Marino & Tomei, 2008 and also in Marconi & Praly, 2008) is the remark that, to the purpose of achieving asymptotic decay of the error, the condition that all frequencies of the exosystem be excited is not necessary, in contrast to what would be requested if the purpose of the adaptive internal model were that of estimating the unknown frequencies of the actual exosystem.

In this paper we suggest an alternative design methodology, which does not rely upon the technique of adaptively tuning the frequencies of an internal model, but rather relies upon certain results of nonlinear high-gain observers, which have already been effectively used in the context of nonlinear output regulation (see e.g. in Byrnes & Isidori, 2004). We develop the theory in a general framework comprising the case of over- and under-dimensioned internal models. In the first scenario, capturing the case in which the number of actual exogenous harmonics is possibly over-estimated, we show that asymptotic regulation is achieved. On the other hand, in case the number of actual exogenous harmonics is under-estimated, the proposed controller guarantees a bounded steady-state regulation error whose amplitude, though, can be arbitrarily decreased by acting on a design parameter of the regulator. The latter feature is interesting "per se" and, to the best knowledge of the authors, never addressed in the related literature about nonlinear output regulation. When specialized to the class of linear systems, the results we obtain can be seen as an alternative to those recently presented in Marino and Tomei (2011), whose design methods rely on well-established "explicit" adaptive techniques. Our design strategy, though, is substantially different. In fact, we are not seeking adaptation of the frequencies of an internal model of an uncertain exosystem, but rather seek direct synthesis of a device able to generate all inputs needed to force a zero steady state error. As emphasized later in the paper, one of the advantages of our approach is that convergence of the error to zero is *exponential* even when the internal model is over-dimensioned.

The work is organized as follows. In Sections 2 and 3 the problem is formulated and a set of results that are instrumental to the design of the output regulator are presented. The design of internal models for uncertain oscillators is then presented in Section 4 while Sections 5 and 6 present a simulative example and conclusive remarks. All the proofs of relevant results are deferred to Appendix.

2. Problem setting

Consistently with most of the literature on output regulation, in this work we consider smooth systems modeled by equations of the form

$$\begin{aligned} \dot{\mathbf{w}} &= s(\mathbf{w}) \\ \dot{z} &= f(\mathbf{w}, z, e_1) \\ \dot{e}_i &= e_{i+1} \quad i = 1, \ldots, r-1 \\ \dot{e}_r &= q(\mathbf{w}, z, e) + b(\mathbf{w}, z, e)u \end{aligned} \tag{1}$$

with state $(z, e_1, \ldots, e_r) \in \mathbb{R}^n \times \mathbb{R}^r$, control input $u \in \mathbb{R}$, regulated output $e_1 \in \mathbb{R}$, in which $\mathbf{w} \in \mathbf{W}$ is a vector of exogenous inputs, modeling references/disturbances to be asymptotically tracked/rejected. The set $\mathbf{W}$ is a *fixed* compact set, invariant under the dynamics of $\dot{\mathbf{w}} = s(\mathbf{w})$. The real-valued function $b(\mathbf{w}, z, e)$, usually referred to as "high-frequency gain", is assumed to be bounded away from zero, namely there exists a $\bar{b}$ (which is assumed positive without loss of generality) such that

$$b(\mathbf{w}, z, e) \geq \bar{b} > 0 \quad \text{for all } (\mathbf{w}, z, e) \in \mathbf{W} \times \mathbb{R}^n \times \mathbb{R}^r.$$

In this setting we address the following problem (*semiglobal output regulation*): given any (arbitrary) pair of sets $Z \subset \mathbb{R}^n$ and $E \subset \mathbb{R}^r$, find an integer d, a compact set $\Xi \subset \mathbb{R}^d$ and an error-feedback controller of the form

$$\dot{\xi} = \alpha(\xi, e_1) \qquad u = \beta(\xi, e_1) \tag{2}$$

yielding a closed-loop system in which all trajectories originating from $\mathbf{W} \times Z \times E \times \Xi$ are bounded and satisfy $\lim_{t\to\infty} e_1(t) = 0$.

We approach the problem under assumptions that are quite common. First, we assume the existence of a continuously differentiable function $\pi : \mathbf{W} \to \mathbb{R}^n$ solution of the *regulator equations*

$$\frac{\partial \pi(\mathbf{w})}{\partial \mathbf{w}} s(\mathbf{w}) = f(\mathbf{w}, \pi(\mathbf{w}), 0), \tag{3}$$

and we let $u^\star(\mathbf{w})$ denote the function

$$u^\star(\mathbf{w}) := -\frac{q(\mathbf{w}, \pi(\mathbf{w}), 0)}{b(\mathbf{w}, \pi(\mathbf{w}), 0)},$$

which, as it is well known, provides the (unique) control that renders the set $\{(\mathbf{w}, z, e) : \mathbf{w} \in \mathbf{W}, z = \pi(\mathbf{w}), e = 0\}$ invariant for (1).

Furthermore, we require that system (1), viewed as a system with input u and output e_1, is "*minimum-phase*", i.e. the dynamics of

$$\begin{aligned} \dot{\mathbf{w}} &= s(\mathbf{w}) \\ \dot{z} &= f(\mathbf{w}, z, 0) \end{aligned} \tag{4}$$

satisfy:

Assumption 1. The set

$$\text{graph}(\pi) = \{(\mathbf{w}, z) \in \mathbf{W} \times \mathbb{R}^n : z = \pi(\mathbf{w})\}$$

is locally asymptotically stable for the system (4) with a domain of attraction of the form $\mathbf{W} \times \mathcal{D}$ where $\mathcal{D}$ is an open set satisfying $\mathcal{D} \supset Z$. □

In what follows, we address the problem of semiglobal output regulation in the simpler case in which $r = 1$. The reason why this can be done without loss of generality follows from classical results about output feedback stabilization which, for the sake of completeness, are briefly summarized here.

For system (1) consider the change of variables

$$\begin{aligned} e_i &\mapsto \zeta_i := k^{-(i-1)} e_i, \quad i = 1, \ldots, r-1, \\ e_r &\mapsto \theta := e_r + k^{r-1} a_0 e_1 + k^{r-2} a_1 e_2 + \cdots + k a_{r-2} e_{r-1}, \end{aligned}$$

in which $k > 1$ is a design parameter and the a_i, $i = 0, \ldots, r-2$, are such that all roots of the polynomial $\lambda^{r-1} + a_{r-2}\lambda^{r-1} + \cdots + a_1\lambda + a_0 = 0$ have a negative real part. This change of variables transforms system (1) into a system of the form

$$\begin{aligned} \dot{\mathbf{w}} &= s(\mathbf{w}) \\ \dot{z} &= f(\mathbf{w}, z, \zeta_1) \\ \dot{\zeta} &= k A_H \zeta + B\theta \\ \dot{\theta} &= \tilde{q}(\mathbf{w}, z, \zeta, \theta, k) + \tilde{b}(\mathbf{w}, z, \zeta, \theta, k)u \end{aligned} \tag{5}$$

in which $\zeta = (\zeta_1, \ldots, \zeta_{r-1})$, A_H is a Hurwitz matrix, and $\tilde{q}, \tilde{b}$ are smooth functions, with $\tilde{b}(\mathbf{w}, z, \zeta, \theta, k) \geq \bar{b}$ for all $(\mathbf{w}, z, \zeta, \theta) \in \mathbf{W} \times \mathbb{R}^n \times \mathbb{R}^{r-1} \times \mathbb{R}$ and for all $k > 0$. Note that, by definition, $\zeta_1 = e_1$. Let $\tilde{E} \in \mathbb{R}^{r-1}$ be a compact set such that $e \in E \Rightarrow \zeta \in \tilde{E}$ and note that, if $k > 1$, the set $\tilde{E}$ can be taken to be independent of k. System (5), regarded as a system with input u and output θ, has relative degree one and zero dynamics

$$\begin{aligned} \dot{\mathbf{w}} &= s(\mathbf{w}) \\ \dot{z} &= f(\mathbf{w}, z, \zeta_1) \\ \dot{\zeta} &= k A_H \zeta. \end{aligned} \tag{6}$$

For this system, under Assumption 1, classical results (see Byrnes & Isidori, 1991) can be invoked to show the existence of a $k^\star > 1$ such that for all $k \geq k^\star$ the set graph$(\pi) \times \{0\}$ is locally asymptotically stable (locally exponentially if graph(π) is such for system (4)) for (6), with a domain of attraction of the form $\mathbf{W} \times \tilde{\mathcal{D}}$, with $\tilde{\mathcal{D}} \supset Z \times \tilde{E}$. Pick, and fix, a value of $k > k^\star$. If the relative degree-r system (1), with input u and output e_1, satisfies Assumption 1, also the relative degree-1 system (5), with input u and output θ, satisfies a similar assumption, the set graph(π) being replaced by the set graph$(\pi) \times \{0\}$.

Suppose now that a controller

$$\dot{\tilde{\xi}} = \tilde{\alpha}(\tilde{\xi}, \theta) \qquad u = \tilde{\beta}(\tilde{\xi}, \theta) \tag{7}$$

solves the problem of output regulation for system (5). This controller is driven by the "dummy" regulated output θ and not by the actual regulated output ζ_1. However, by construction, θ is a fixed linear combination of the components $\zeta_1, \ldots, \zeta_r$ of the "partial state" e of (1). By definition, e_i coincides with the $(i-1)$-th derivative, with respect to time, of the actual regulated output e_1. As it is well known, to the purpose of securing asymptotic convergence to the desired target set, $e_1, \ldots, e_r$ can be replaced by appropriate "estimates" $\hat{e}_1, \ldots, \hat{e}_r$ provided by a "high-gain observer" driven only by e_1. Using these estimates to replace the expression of θ in (7) yields a controller able to solve the problem for the original plant (1). Details, and precautions meant to avoid finite escape times, can be found in Esfandiari and Khalil (1992), Teel and Praly (1995), Isidori (1999), and Marconi, Praly, and Isidori (2010) and need not to be repeated here.

On the basis of these arguments, we can in what follows restrict the discussion to the case of systems having relative degree $r = 1$ which, for notational convenience, are rewritten as

$$\begin{aligned} \dot{\mathbf{w}} &= s(\mathbf{w}) \\ \dot{z} &= f(\mathbf{w}, z, e) \quad z \in \mathbb{R}^n \\ \dot{e} &= q(\mathbf{w}, z, e) + b(\mathbf{w}, z, e)u \quad e \in \mathbb{R}. \end{aligned} \tag{8}$$

3. Preliminaries

For the class of systems (8) it is known that the problem in question can be solved if one is able to find an integer d, a continuous function $F : \mathbb{R}^d \to \mathbb{R}^d$, a continuous function $\gamma : \mathbb{R}^d \to \mathbb{R}$, a column vector $G \in \mathbb{R}^{d\times 1}$, and a continuously differentiable function $\tau : \mathbf{W} \to \mathbb{R}^d$ satisfying

$$\begin{aligned} \frac{\partial \tau}{\partial \mathbf{w}} s(\mathbf{w}) &= F(\tau(\mathbf{w})) + G\gamma(\tau(\mathbf{w})) \quad \forall \mathbf{w} \in \mathbf{W} \\ u^\star(\mathbf{w}) &= \gamma(\tau(\mathbf{w})) \end{aligned} \tag{9}$$

and such that

$$\text{graph}(\tau) = \{(\mathbf{w}, \xi) \in \mathbf{W} \times \mathbb{R}^d : \xi = \tau(\mathbf{w})\}$$

is locally asymptotically stable for the system

$$\begin{aligned} \dot{\mathbf{w}} &= s(\mathbf{w}) \\ \dot{\xi} &= F(\xi) + Gu^\star(\mathbf{w}) \end{aligned} \tag{10}$$

with a domain of attraction of the form $\mathbf{W} \times \mathcal{D}'$ with $\mathcal{D}' \subset \mathbb{R}^d$ an open set. As a matter of fact the following result holds (see Marconi, Praly, & Isidori, 2007).

Theorem 1. *Let the minimum-phase Assumption 1 hold. Let $(F(\cdot), G, \gamma(\cdot))$ be chosen to satisfy (9) for some map $\tau(\cdot)$ and so that graph(τ) is locally asymptotically stable for (10) with domain of attraction $\mathbf{W} \times \mathcal{D}'$. Then there exists a continuous function $\kappa : \mathbb{R} \to \mathbb{R}$ such that the controller*

$$\begin{aligned} \dot{\xi} &= F(\xi) + G(v + \gamma(\xi)) \\ u &= \gamma(\xi) + v \\ v &= -\kappa(e) \end{aligned} \tag{11}$$

solves the problem of nonlinear output regulation with $\Xi \subset \mathcal{D}'$.

Remark 1. Under the additional assumptions that the sets graph (π) and graph(τ) are also locally *exponentially* stable for (4) and (10), respectively, and that the function $\gamma(\cdot)$ is locally Lipschitz, the result in the previous proposition holds with $v = -\kappa e$ with κ a sufficiently large number.

According to the previous result, the problem of output regulation, for the considered class of systems, reduces to the problem of designing a triplet $(F(\cdot), G, \gamma(\cdot))$ with the required properties. A triplet fulfilling the properties in question is usually said to have the *internal model property*. A number of methodologies for the design of the triplets with the internal model property have been proposed so far. In this respect the following result (proved in Marconi et al., 2007) is conceptually relevant as it shows that a triplet with the required properties can always be designed.

Proposition 1. *Let $d \geq 2s+2$. There is a positive $\bar{\ell} \in \mathbb{R}$ such that, for almost all choices (see Marconi et al., 2007 for details) of a controllable pair $(F, G) \in \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times 1}$, with F a Hurwitz matrix whose eigenvalues have a real part which is less than $-\bar{\ell}$, then there exists a* continuous *function $\gamma : \mathbb{R}^d \to \mathbb{R}$ such that the triplet $(F\xi, G, \gamma(\xi))$ has the internal model property.*

Although conceptually relevant, the previous result is weak from a practical viewpoint because the actual construction of the function $\gamma(\cdot)$ is not simple (see Marconi & Praly, 2008 for approximate expressions of practical interest) and because the function in question is only guaranteed to be continuous.

More constructive design methodologies and smoothness in the controller can be obtained at the price of restricting the class of possible functions $u^\star(\mathbf{w})$ entering in the design of the regulator. Specifically, it is known (see Byrnes & Isidori, 2004) that the design of a triplet having the internal model property can be effectively carried out in the case there exist an integer d and a locally Lipschitz map $\phi : \mathbb{R}^d \to \mathbb{R}$ such that

$$L_s^d u^\star(\mathbf{w}) = \phi(u^\star(\mathbf{w}), L_s u^\star(\mathbf{w}), \ldots, L_s^{d-1} u^\star(\mathbf{w})) \tag{12}$$

$\forall\, \mathbf{w} \in \mathbf{W}$. In fact, set

$$\tau(\mathbf{w}) = \begin{pmatrix} \tau_0(\mathbf{w}) \\ \vdots \\ \tau_{d-1}(\mathbf{w}) \end{pmatrix} := \begin{pmatrix} u^\star(\mathbf{w}) \\ \vdots \\ L_{s(\mathbf{w})}^{d-1} u^\star(\mathbf{w}) \end{pmatrix} \tag{13}$$

and let $\phi_c : \mathbb{R}^d \to \mathbb{R}$ be any locally Lipschitz bounded function that agrees with ϕ on $\tau(\mathbf{W})$. Then, it turns out that the choice

$$F(\xi) = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{d-1} \\ \phi_c(\xi_0, \ldots, \xi_{d-1}) \end{pmatrix} - G\xi_0 \tag{14}$$

with $\xi = (\xi_0, \ldots, \xi_{d-1})$ makes (9) fulfilled with $\gamma(\xi) = \xi_0$ and for any vector G. Furthermore, the theory of high-gain observers (see Esfandiari & Khalil, 1992, Gauthier, Hammouri, & Othman, 1992, Teel & Praly, 1995) can be successfully used in this context to show that if the vector G is chosen as

$$G = \begin{pmatrix} g\lambda_0 & g^2\lambda_1 & \cdots & g^d\lambda_{d-1} \end{pmatrix}^T \tag{15}$$

where $(\lambda_0, \lambda_1, \ldots, \lambda_{d-1})$ are coefficients of a Hurwitz polynomial and $g > 0$ is a high-gain parameter, then the set graph(τ) is locally exponentially stable for the system (10) with a domain of attraction which can be made arbitrarily large by increasing g.

We summarize the general (constructive) result in the forthcoming Theorem, whose proof is provided in Appendix A. The result in question is an extension of a result originally proved in Byrnes and Isidori (2004) to the more general case in which relation (12) is satisfied modulo a residual bias (see (16)), introduced to handle the general theory (internal models with "under-estimated" dimension) presented in Section 3.

Theorem 2. *Let Assumption 1 be fulfilled. Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz function and $\nu : \mathbf{W} \to \mathbb{R}$ a continuous function such that*

$$L_s^d u^\star(\mathbf{w}) = \phi(u^\star(\mathbf{w}), L_s u^\star(\mathbf{w}), \dots, L_s^{d-1}u^\star(\mathbf{w})) + \nu(\mathbf{w}) \qquad (16)$$

for all $\mathbf{w} \in \mathbf{W}$. Then, there exist a $g^\star > 0$, a $c > 0$, and a continuous function $\kappa : \mathbb{R} \to \mathbb{R}$ such that for all $g \geq g^\star$ the trajectories of the system (8) in closed-loop with the regulator (11), (14), (15) and $\gamma(\xi) = \xi_0$ originating from $\mathbf{W} \times Z \times E \times \Xi$ are bounded and such that

$$\lim_{t\to\infty} \sup |e(t)| \leq \frac{c}{g^{d+1}} \max_{\mathbf{w}\in\mathbf{W}} |\nu(\mathbf{w})|. \qquad (17)$$

Furthermore, if $\nu = 0$ and graph(π) is also locally exponentially stable for system (4), the error converges to zero exponentially.

It is worth noting that, if (12) holds, the previous result provides an effective way to design asymptotic regulators (with exponential convergence of the error if Assumption 1 is strengthened by asking exponential stability of the set graph(π)). On the other hand, in case (16) is satisfied with a non-zero function $\nu(\mathbf{w})$, relation (17) shows the presence of a persistent steady-state regulation error whose amplitude, though, can be arbitrarily decreased by increasing the high-gain parameter g.

We conclude this section by emphasizing a further robustness property of the internal model-based regulators, resulting from the previous design, to possible dynamical uncertainties characterizing the exosystem. Specifically, we consider the case in which the exogenous input $\mathbf{w}$ is generated by an exosystem of the form

$$\dot{\mathbf{w}} = s(\mathbf{w}) + \varepsilon\delta(\mathbf{w}) \qquad (18)$$

where $\varepsilon\delta(\mathbf{w})$ is a continuous function modeling a dynamical mismatch between a "nominal" exosystem $\dot{\mathbf{w}} = s(\mathbf{w})$ and the actual exosystem (18). In this setting we are interested to investigate the asymptotic properties of the system in closed loop with a regulator designed on the basis of the nominal exosystem, when the actual exogenous input is instead generated by (18). Consistently with the basic approach, we assume the existence of a compact set $\mathbf{W}_\varepsilon$ that is invariant for (18) and such that the dynamics of

$$\begin{aligned} \dot{\mathbf{w}} &= s(\mathbf{w}) + \varepsilon\delta(\mathbf{w}) \\ \dot{z} &= f(\mathbf{w}, z, 0). \end{aligned} \qquad (19)$$

satisfy the following assumption.

Assumption 1-bis. There exists a map $\pi_\varepsilon : \mathbf{W}_\varepsilon \to \mathbb{R}^n$ such that the set

$$\text{graph}(\pi_\varepsilon) = \{(\mathbf{w}, z) \in \mathbf{W}_\varepsilon \times \mathbb{R}^n : z = \pi_\varepsilon(\mathbf{w})\}$$

is locally asymptotically stable for the system (19) with a domain of attraction of the form $\mathbf{W}_\varepsilon \times \mathcal{D}$ where $\mathcal{D}$ is an open set satisfying $\mathcal{D} \supset Z$. □

The resulting robustness properties of the internal-model based controller are described in the next corollary that is proved in Appendix B. In the proposition we refer to the function $u_\varepsilon^\star(\mathbf{w})$ defined as

$$u_\varepsilon^\star(\mathbf{w}) = -\frac{q(\mathbf{w}, \pi_\varepsilon(\mathbf{w}), 0)}{b(\mathbf{w}, \pi_\varepsilon(\mathbf{w}), 0)}.$$

Corollary 1. *Let $\mathbf{W}_\varepsilon$ be invariant for (18) and let the dynamics (19) satisfy Assumption 1-bis. Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz function and $\nu : \mathbf{W}_\varepsilon \to \mathbb{R}$ a continuous function such that (16) holds with $\mathbf{W}$ and $u^\star(\mathbf{w})$ replaced, respectively, by $\mathbf{W}_\varepsilon$ and $u_\varepsilon^\star(\mathbf{w})$. Finally, let $\phi_c : \mathbb{R}^d \to \mathbb{R}$ be any locally Lipschitz bounded function that agrees with ϕ on $\tau(\mathbf{W}_\varepsilon)$ where τ is defined as in (13) with $u^\star(\mathbf{w})$ replaced by $u_\varepsilon^\star(\mathbf{w})$.*

Then, there exist a continuous function $\varrho : \mathbf{W}_\varepsilon \times \mathbb{R} \to \mathbb{R}$, a $g^\star > 0$, a $c > 0$, and a continuous function $\kappa : \mathbb{R} \to \mathbb{R}$ such that for all $g \geq g^\star$ the trajectories of the system (8), with $s(\mathbf{w})$ replaced by $s(\mathbf{w}) + \varepsilon\delta(\mathbf{w})$, in closed-loop with the regulator (11), (14), (15) and $\gamma(\xi) = \xi_0$ originating from $\mathbf{W}_\varepsilon \times Z \times E \times \Xi$ are bounded and such that

$$\lim_{t\to\infty} \sup |e(t)| \leq \frac{c}{g^{d+1}} \max_{\mathbf{w}\in\mathbf{W}_\varepsilon} (|\nu(\mathbf{w})| + \varepsilon|\varrho(\mathbf{w}, \varepsilon)|).$$

4. Internal models of uncertain oscillators

4.1. The framework

In this work the results summarized above will be applied in the special case in which the function $u^\star(\mathbf{w}(t))$ is a linear combination of m_0 non-zero harmonics with uncertain frequencies, amplitudes and phases (see Serrani et al., 2001). In this context, by letting $\mathbf{w} = \text{col}(\varpi, w)$, we can redefine the exosystem dynamics $\dot{\mathbf{w}} = s(\mathbf{w})$ as

$$\begin{aligned} \dot{\varpi} &= 0 \quad \varpi \in \varpi \subset \mathbb{R}^{m_0} \\ \dot{w} &= S(\varpi)w \quad w \in W \subset \mathbb{R}^{2m_0} \end{aligned} \qquad (20)$$

where $w = (w_1 \cdots w_{m_0})^T$ with $w_i \in \mathbb{R}^2$,

$$S(\varpi) = \text{blkdiag}(S_1(\varpi_1), \dots, S_{m_0}(\varpi_{m_0}))$$
$$S_i(\varpi_i) = \begin{pmatrix} 0 & \varpi_i \\ -\varpi_i & 0 \end{pmatrix}, \qquad (21)$$

with $u^\star(\mathbf{w})$ which reads as

$$u^\star(\mathbf{w}) = \Gamma(\varpi)w \quad \Gamma(\varpi) = \begin{pmatrix} \Gamma_1(\varpi) & \cdots & \Gamma_{m_0}(\varpi) \end{pmatrix} \qquad (22)$$

with $\Gamma_i(\varpi) \in \mathbb{R}^{1\times 2}$ and the pair $(S(\varpi), \Gamma(\varpi))$ assumed, without loss of generality, observable for all $\varpi \in \varpi$. The unknown values of ϖ and w are supposed to range on a known compact set $\mathbf{W} = \varpi \times W \subset \mathbb{R}^{m_0} \times \mathbb{R}^{2m_0}$. It is assumed that $W = W_1 \times \cdots \times W_{m_0}$, with $W_i = \{w_i \in \mathbb{R}^2 : \|w_i\| \in [a_i, b_i]\}$ for some positive $a_i < b_i$, $i = 1, \dots, m_0$. Note that $\mathbf{W}$ is invariant for (20). It is also assumed that the frequencies ϖ_i are such that $\varpi_i \neq \varpi_j$, for all $i, j = 1, \dots, m_0$, $i \neq j$, and that $\varpi_i \neq 0$ for all $i = 1, \dots, m_0$. The signal $u^\star(\mathbf{w}(t))$, hence, is the superposition of m_0 harmonic components of unknown frequencies, dependent on (the initial condition of) $\varpi = (\varpi_1 \cdots \varpi_{m_0})^T$, of unknown amplitudes and phases dependent on the initial condition of w.

Within the previous framework, we are interested to develop a design methodology not relying upon the exact knowledge of the number m_0 of non-zero harmonics but rather on an estimate m of m_0, with m not necessarily equal to m_0. If m happens to be larger or equal to the number m_0 of such components, asymptotic regulation is achieved. On the other hand, if $m < m_0$, i.e. if the dimension of exosysyem is under-estimated, only practical regulation is obtained with a residual regulation error that – however – can be arbitrarily decreased by properly tuning a high-gain parameter (in the light of Theorem 2).

4.2. Design of the internal model

The proposed regulator is a $4m$-dimensional system, with state

$$\xi = \text{col}(\xi_0, \xi_1, \dots, \xi_{4m-1})$$

whose dynamics are constructed (see Proposition 2) starting from the vector

$$\ell(\xi) := (\xi_{2m}, \ldots, \xi_{4m-1})^T$$

and matrices $A_i(\xi) \in \mathbb{R}^{2i\times 2i}$, $i = 1, \ldots, m$, defined as the $2i \times 2i$ upper diagonal block of the following matrix

$$A(\xi) = \begin{pmatrix} \xi_0 & \xi_1 & \cdots & \xi_{2m-1} \\ \xi_1 & \xi_2 & \cdots & \xi_{2m} \\ \vdots & \vdots & & \vdots \\ \xi_{2m-1} & \xi_{2m} & \cdots & \xi_{4m-2} \end{pmatrix}.$$

Note that $A_m(\xi) = A(\xi)$.

The construction of the regulator relies on a special "regularity" assumption concerning the matrix $A_{\min\{m,m_0\}}$, specified as follows. Let $\tau(\mathbf{w})$ be defined as in (13) with $d = 4m$, that is

$$\tau(\mathbf{w}) = \begin{pmatrix} u^\star(\mathbf{w}) & L_s u^\star(\mathbf{w}) & \cdots & L_s^{4m-1} u^\star(\mathbf{w}) \end{pmatrix}^T, \tag{23}$$

with $u^\star(\mathbf{w})$ defined as in (22) and $s(\mathbf{w})$ defined as in (20)–(21). The assumption in question is the following one.

Assumption 2. There exists $\epsilon > 0$ such that

$$|\det(A_{\min\{m,m_0\}}(\tau(\mathbf{w})))| \geq \epsilon \quad \forall \mathbf{w} \in \mathbf{W}. \quad \square \tag{24}$$

Remarks about the role of this assumption are given after the forthcoming proposition.

The value of ϵ is used in the design of the regulator. Specifically, we let $A^{-1}_{i,\mathrm{sat}}(\xi)$, for $i = 1, \ldots, m$, be any (at least locally Lipschitz) bounded matrix that agrees with $A_i^{-1}(\xi)$ for all ξ such that $|\det A_i(\xi)| \geq \epsilon$ and $b_i(\xi)$ any (at least locally Lipschitz) function satisfying

$$b_i(\xi) = \begin{cases} 1 & \text{if } |\det A_i(\xi)| \geq \epsilon \\ 0 & \text{if } |\det A_i(\xi)| \leq \dfrac{\epsilon}{2}. \end{cases}$$

With the previous notations in hand, we are in the position of presenting the main result of the section (proved in Appendix C) that, in conjunction with Theorem 2, yields the required regulator.

Proposition 2. *Let condition (24) be fulfilled. Let*

$$\phi(\xi) = \sum_{i=1}^{m} \alpha_i(\xi)\phi_i(\xi) \tag{25}$$

where, for $i = 1, \ldots, m$, the ϕ_i's are defined as[2]

$$\phi_i(\xi) = \lfloor \ell(\xi) \rfloor_i^T A^{-1}_{i,\mathrm{sat}}(\xi) \lfloor \ell(\xi) \rfloor_i,$$

and the α_i's are recursively defined as

$$\alpha_m(\xi) = b_m(\xi)$$

$$\alpha_i(\xi) = b_i(\xi) \prod_{j=i+1}^{m} (1 - \alpha_j(\xi)) \quad i = 1, \ldots, m-1.$$

There exists a continuous function $\nu : \mathbf{W} \to \mathbb{R}$, with the property that $\nu \equiv 0$ if $m \geq m_0$, such that relation (16) holds with $d = 4m$ for all $\mathbf{w} \in \mathbf{W}$.

We conclude the section with some remarks about the basic requirement (24). Clearly, the fulfillment of this condition is influenced by the shape of $\mathbf{W}$. In what follows, it will be shown that if $m \geq m_0$ no extra hypotheses are required, while if $m < m_0$ condition (24) is fulfilled if, among the m_0 harmonics characterizing the signal $u^\star(\mathbf{w}(t))$, at least $m_0 - m$ components have "sufficiently small" amplitudes.

[2] We use the notation $\lfloor v \rfloor_j$ to denote a vector in $\mathbb{R}^{2j}$ obtained by extracting the first $2j$ components from the vector $v \in \mathbb{R}^{2d}$, $d \geq j$.

Proposition 3. *Let $\underline{a} \leq \bar{a}$ be fixed positive numbers and ϖ be a fixed compact set of $\mathbb{R}^{m_0}$.*

If $m \geq m_0$, condition (24) holds with $\mathbf{W}$ a set of the form $\mathbf{W} = \varpi \times W$ in which

$$W = W_{\underline{a},\bar{a}} \times \cdots \times W_{\underline{a},\bar{a}} \in \mathbb{R}^{2m_0} \tag{26}$$

with $W_{\underline{a},\bar{a}} = \{w \in \mathbb{R}^2 : |w| \in [\underline{a}, \bar{a}]\}$.

If $m < m_0$, there exists a $\bar{\sigma} > 0$ such that for all positive $\sigma \leq \bar{\sigma}$ condition (24) holds with $\mathbf{W}$ a set of the form $\mathbf{W} = \varpi \times W$ with

$$W = \underbrace{W_{\underline{a},\bar{a}} \times \cdots \times W_{\underline{a},\bar{a}}}_{m \text{ times}} \underbrace{W_{\sigma\underline{a},\sigma\bar{a}} \times \cdots \times W_{\sigma\underline{a},\sigma\bar{a}}}_{m_0 - m \text{ times}}.$$

The proof of this proposition is deferred to Appendix D.

Remark 2. Note that, if the actual exosystem consists of m_0 harmonic oscillators and $m \geq m_0$, and hence our internal model is over-dimensioned, the combination of Theorem 2 and Proposition 2 leads to the claim that the proposed regulator guarantees *exponential* convergence of the error to zero, so long as the set graph(π) is locally exponentially stable for (4). If specialized to the class of *linear* minimum-phase regulated systems (in which the requirement of exponential stability of the zero dynamics is indeed fulfilled), this fact means that the proposed controller guarantees exponential regulation even if some of the harmonic oscillators that characterize the exosystem are not "excited". This kind of asymptotic behavior of the error is not necessarily guaranteed if the internal model is designed as a fixed bench of oscillators whose frequencies are adaptively tuned, as proposed in Serrani et al. (2001) and more recently in Marino and Tomei (2011) for linear minimum-phase systems. In this case, in fact, just asymptotic (and not exponential) regulation is guaranteed.

4.3. Fine tuning of the internal model dimension

A possible drawback of the design solution presented in the previous section is related to the dimension of the regulator, which is equal to $4m$. This fact, in turn, leads to implement a high-gain controller of the form (11), (14), (15) with the value of the gain g raised to the 4 m power (see the expression of G), opening the doors to some problem in the implementation phase if m is large. In order to mitigate this problem, a first wise design improvement of the proposed solution is to adopt strategies that adjust the value of the gain online as suggested, for instance, in Ahrens and Khalil (2009), Boizot, Busvelle, and Gauthier (2010), and Sanfelice and Praly (2010). Furthermore, it turns out useful to look for strategies leading to keep m as small as possible according to the effective number of exogenous harmonics (i.e. m_0). In this respect, it is interesting to note that the design framework presented in this paper lends itself to the possibility of "fine-tuning" the dimension of the internal model, so as to avoid over-dimensioned, and hence redundant, controllers. This can easily be accomplished if an upper bound $\bar{m}_0$ for the number of harmonic components of the exogenous input is available. In this case, in fact, it is easy to implement – at least in principle – a simple procedure by means of which the actual value of m_0 is identified, and accordingly, the appropriate value of m can be set.

Bearing in mind the notation of the previous section and the proof of Proposition 2, we know that for all $\mathbf{w}(t) \in \mathbf{W}$,

$$\begin{aligned} & m = m_0 \Rightarrow \det A_m(\tau(\mathbf{w}(t))) \geq \epsilon \\ & m > m_0 \Rightarrow \det A_{m_0}(\tau(\mathbf{w}(t))) \geq \epsilon, \\ & \det A_{m_0+1}(\tau(\mathbf{w}(t))) = \cdots = \det A_m(\tau(\mathbf{w}(t))) = 0. \end{aligned} \tag{27}$$

We also know that, if $m \geq m_0$, a property of the form (16) holds with $\nu(\mathbf{w}) = 0$ and, from the proof of Proposition 2, it follows that, if system (8) is controlled via (11)–(14)–(15) with $\gamma(\xi) = \xi_0$ and suitable $g > 0$ and $\kappa(\cdot)$, then

$$\lim_{t\to\infty} |\xi(t) - \tau(\mathbf{w}(t))| = 0. \tag{28}$$

Suppose an upper bound $\bar{m}_0$ for the actual value of m_0 is known, set $m = \bar{m}_0$, let $\phi(\xi)$ be defined as in Proposition 2 and let the controller be designed as indicated in Proposition 2. The controller in question is a system of fixed structure

$$\begin{aligned} \dot{\xi} &= A\xi + B\phi(\xi) - gD_g \Lambda\kappa(e) \\ u &= \xi_0 - \kappa(e), \end{aligned} \tag{29}$$

with A, B, D_g, Λ defined as in the proof of Theorem 2. The function $\phi(\xi)$ only depends on the value of m (which is set equal to $\bar{m}_0$) and on the value of ϵ.

From the proof of Theorem 2, we know that for every $\delta > 0$, there is a time T_δ, that – for a fixed plant (8) – depends on the actual values of $\bar{m}_0, \epsilon$, on the choices of $\Lambda, g, \kappa(\cdot)$ in (29), and on the admissible set of initial conditions, such that

$$|\xi(t) - \tau(\mathbf{w}(t))| \leq \delta \quad \text{for all } t \geq T_\delta,$$

uniformly in the initial conditions (as long as the latter range over a fixed compact set). By continuity of $\det A_i(\xi)$, it is known that for each θ there is a δ_θ such that

$$|\xi - \tau(\mathbf{w})| \leq \delta_\theta \Rightarrow |\det A_i(\xi) - \det A_i(\tau(\mathbf{w}))| \leq \theta$$

for all $i \leq \bar{m}_0$. Thus, we deduce that

$$\begin{aligned} t \geq T_{\delta_\theta} \Rightarrow & |\det A_i(\tau(\mathbf{w}(t)))| - \theta \leq |\det A_i(\xi(t))| \\ & \leq |\det A_i(\tau(\mathbf{w}(t)))| + \theta, \end{aligned}$$

uniformly in the initial condition. From this, taking $\theta = \epsilon/4$, we conclude that (by taking into account (27))

$$\bar{m}_0 = m_0 \Rightarrow |\det A_{\bar{m}_0}(\xi(t))| \geq 3\epsilon/4 \quad \text{for all } t \geq T_{\delta_\theta}$$

and

$$\bar{m}_0 > m_0 \Rightarrow \begin{cases} |\det A_{\bar{m}_0 - i}(\xi(t))| \leq \epsilon/4 & \text{for all } t \geq T_{\delta_\theta}, \\ \quad 0 \leq i \leq \bar{m}_0 - m_0 - 1 \\ |\det A_{m_0}(\xi(t))| \geq 3\epsilon/4 & \text{for all } t \geq T_{\delta_\theta}. \end{cases}$$

This suggests a simple procedure to identify the actual value of m_0, provided that an upper bound $\bar{m}_0$ is known. It suffices, in fact, "to wait" a time T_{δ_θ}, with $\theta = \epsilon/4$, and then to determine the value of m_0 as the largest $i \leq \bar{m}_0$ such that $|\det A_i(\xi(t))| \geq 3\epsilon/4$ for all $t \geq T_{\delta_\theta}$.

4.4. *Robustness to slightly varying frequencies*

In this part the robustness properties of the internal model design proposed in Section 4.2 to possibly time-varying frequencies are investigated. It is shown that if the m_0 frequencies ϖ_i are varying, the resulting steady state regulation error is bounded with an upper bound linearly dependent on the rate of change of the ϖ_i's.

Specifically, within the framework set up in Sections 4.1 and 4.2, we consider an exosystem of the form (18), with $s(\mathbf{w})$ of the form (20). As in Corollary 1, we assume that there exists a set $\mathbf{W}_\varepsilon$ that is invariant for (18) and that the dynamics (19) satisfy Assumption 1-bis. Then, Corollary 1 specializes as follow.

Corollary 2. *Let $\mathbf{W}_\varepsilon$ be invariant for (18) and let the dynamics (19) satisfy Assumption 1-bis. Let $\tau(\mathbf{w})$ be defined as in (23) with $s(\mathbf{w})$ replaced by $s(\mathbf{w}) + \varepsilon\delta(\mathbf{w})$, and suppose Assumption 2, with $\mathbf{W}$ replaced by $\mathbf{W}_\varepsilon$, is fulfilled for some ϵ. Then, there exist bounded functions $\nu : \mathbf{W}_\varepsilon \to \mathbb{R}$, with $\nu(\mathbf{w}) = 0$ if $m \geq m_0$, and $\varrho : \mathbf{W}_\varepsilon \times \mathbb{R} \to \mathbb{R}$, a continuous function $\kappa : \mathbb{R} \to \mathbb{R}$, and a $g^\star > 0$ and a $c > 0$, and such that for all $g \geq g^\star$ the trajectories of the system (8) and (18), in closed-loop with the regulator (11), (14), (15) and $\gamma(\xi) = \xi_0$ are bounded and such that*

$$\lim_{t\to\infty} \sup |e(t)| \leq \frac{c}{g^{4m+1}} \max_{\mathbf{w}\in\mathbf{W}_\varepsilon} (|\nu(\mathbf{w})| + \varepsilon|\varrho(\mathbf{w}, \varepsilon)|).$$

5. Simulation results

We consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \mu_1 x_2 - x_2^3 - \mu_2 x_1 x_2^2 + u \end{aligned}$$

that is a Van der Pol oscillator perturbed by $-\mu_2 x_1 x_2^2$ and forced by the control input u. The vector $\mu = (\mu_1, \mu_2)$ is a vector of constant uncertain parameters whose values range in a compact set. In the simulation below it is assumed that $|\mu_1| \leq 2$ and $|\mu_2| \leq 2$. The goal is to make the variable x_1 tracking a reference signal $x_1^\star$ generated as

$$x_1^\star = \Gamma^\star w^\star, \qquad \dot{\varpi}^\star = 0, \qquad \dot{w}^\star = S^\star(\varpi^\star)w^\star \quad w^\star \in \mathbb{R}^{2m_0^\star}$$

with $m_0^\star > 0$, $\varpi^\star = (\varpi_1^\star, \ldots, \varpi_{r^\star}^\star)$ in which $S^\star$ is of the form (21) with S, ϖ and m_0 replaced by $S^\star$, $\varpi^\star$ and $m_0^\star$. The values of $m_0^\star$, $\varpi^\star(0)$ and $w^\star(0)$ are unknown but supposed to range on known compact sets. By defining the error variables $e_1 := x_1 - \Gamma^\star w^\star$ and $e_2 = x_2 - \Gamma^\star S^\star w^\star$, the error system can be put in the form (1) where the relative degree $r = 2$, the z-dynamics are absent, $\mathbf{w} = \mathrm{col}(\mu, \varpi^\star, w^\star)$ with the $\mathbf{w}$ dynamics given by $\dot{\mu} = 0$, $\dot{\varpi}^\star = 0$ and $\dot{w}^\star = S^\star(\varpi^\star)w^\star$ and where $u^\star(\mathbf{w})$ is given by

$$\begin{aligned} u^\star(\mathbf{w}) = & \Gamma^\star w^\star - \mu_1 \Gamma^\star S^\star w^\star + (\Gamma^\star S^\star w^\star)^2 \\ & + \mu_2 (\Gamma^\star w^\star)(\Gamma^\star S^\star w^\star)^2 + \Gamma^\star S^{\star 2} w^\star. \end{aligned} \tag{30}$$

The desired steady state input is thus a polynomial function of the state $\mathbf{w}$. By Huang (1995) it turns out that system $\dot{\mu} = 0$, $\dot{\varpi}^\star = 0$ and $\dot{w}^\star = S^\star(\varpi^\star)w^\star$ with "output" (30) is immersed into a system of the form (20)–(21)–(22). For example if $m_0^\star = 1$, all the desired steady state inputs of the form (30) can be generated by means of an exosystem of the form (20)–(21)–(22) with $m_0 = 2$, $\varpi(0) = (\varpi^\star(0), 3\,\varpi^\star(0))$ and with initial conditions of $w \in \mathbb{R}^4$ dependent on the initial conditions of $(\varpi^\star, w^\star)$ and on the value of μ. The internal model based regulator can be thus designed according to the theory presented in the paper. In particular, by following the theory in Section 4.2, we have chosen $m = 2$ (resulting in an internal model of dimension 8). After a few numerical tests obtained by taking $W = \{w \in \mathbb{R}^4 : |w| \leq 5\}$ and $\varpi = \{\varpi \in \mathbb{R}^2 : |\varpi_1| \leq 4, |\varpi_2| \leq 10\}$, the internal model unit has been tuned with $\epsilon = 5 \cdot 10^{-3}$ (see (24)), $g = 10$ and the λ_i's so that the 8 roots of $\lambda^8 + \lambda_7\lambda^7 + \cdots \lambda_1\lambda + \lambda_0 = 0$ are half in -1 and half in -2. By following the theory presented in the final part of Section 2, a stabilizer has been then designed with $k = 1$, $a_0 = 1$, and $\kappa = 10^2$ (see (11)). The designed controller has been simulated in different scenarios in order to check all the features of the proposed internal model unit design. In particular, the simulated reference signal $x_1^\star$ is shown in Fig. 1. In a first time period $[0, 25]$ s (labeled (a) in the figure) a reference signal $x_1^\star$ composed by two harmonics (at frequencies $\varpi_1^\star = 1$ and $\varpi_2^\star = 3$, both with amplitude 1) has been implemented in order to simulate a scenario in which the internal model is under-dimensioned with respect to the exosystem generating the ideal steady state control input (in this case $m_0 = 8$ and $m = 2$). According to Theorem 2 the regulator guarantees a bounded steady state tracking error that can be observed in Fig. 2. According to the value of g the error is of magnitude 10^{-4}. At time $t = 25$ the second harmonics of the reference signal is disconnected so that the reference signal becomes a pure unitary harmonic at frequency $\varpi_1^\star = 1$ (in this case $m_0 = m = 2$). This reference signal is kept unchanged in the

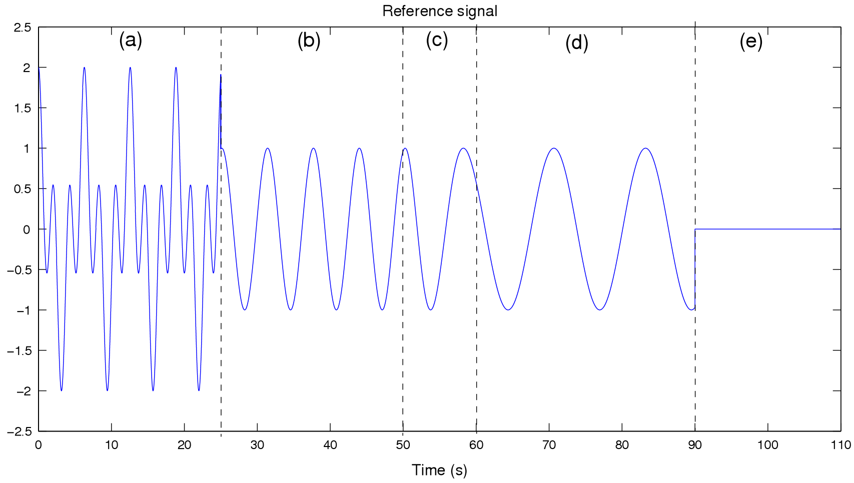

Fig. 1. Reference signal $x_1^\star(t)$.

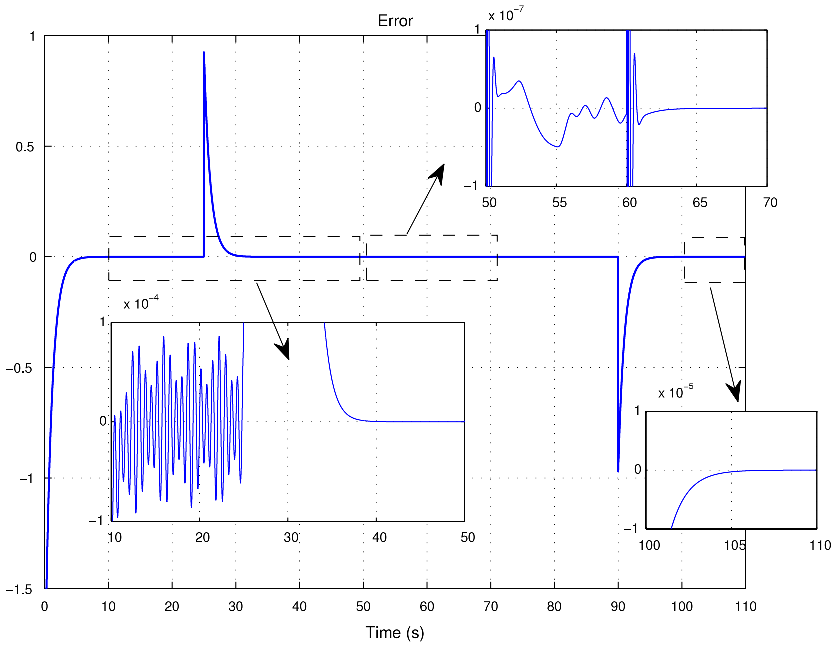

Fig. 2. Tracking error $e_1(t) = x_1(t) - x_1^\star(t)$.

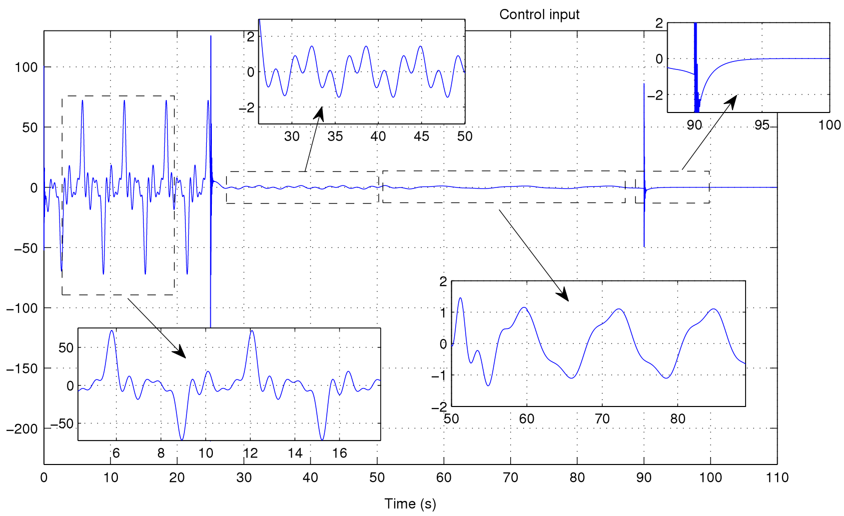

Fig. 3. Control input $u(t)$.

time interval [25, 50] s, labeled (b) in Fig. 1. In this case the internal model embedded in the regulator has the right dimension and, as predicted by the theory, the error converges to zero (see Fig. 2). At time $t = 50$ s, the frequency of the single harmonic of $x_1^\star$ is steadily decreased in order to reach the value $\varpi_1^\star = 0.5$ at time $t = 60$ s. As predicted by Corollary 2 the regulator still guarantees a bounded tracking error that can be observed in Fig. 2. In the subsequent time interval [60, 90] s, labeled (d) in Fig. 1, the frequency of the reference signal is kept constant at $\varpi_1^\star = 0.5$ and, as shown by Fig. 2, the internal model adapts to the new frequency and steers the tracking error to zero. Finally, in the time interval [90, 110] s (labeled (e) in Fig. 1), the reference signal is set to zero in order to simulate a lack in the persistence of excitation, namely an over-dimensioned internal model (in this case $m_0 = 0 < m = 2$). As shown by Fig. 2 even in this case the error converges to zero. The behavior of the control input $u(t)$ is shown in Fig. 3.

6. Conclusions

This paper presented a new approach to design adaptive internal model-based regulators for a class of minimum-phase nonlinear systems. With respect to existing approaches, the proposed method does not rely upon an explicit adaptation method of the control law. The new method has been developed in a general framework handling both the case of over- and under-dimensioned internal models. In the case of under-dimensioned internal models we showed how the proposed controller ensures a bounded steady state regulation error that can be arbitrarily decreased by acting on a design parameter. Future works on this subject will focus on extending the design methodology also to the class of nonlinear but linearly parametrized uncertain exosystems and on numerical validation of the proposed approach and comparison with existing methods.

Appendix A. Proof of Theorem 2

We consider the change of variable $\xi \to \tilde{\xi} := \xi - \tau(\mathbf{w})$, transforming system (8), (11), with F as in (14) and $\gamma(\xi) = \xi_0$, as

$$\begin{aligned}
\dot{\mathbf{w}} &= s(\mathbf{w})\\
\dot{z} &= f(\mathbf{w}, z, e)\\
\dot{\tilde{\xi}} &= A\tilde{\xi} + B\left(\tilde{\phi}(\tilde{\xi}, \mathbf{w}) - \nu(\mathbf{w})\right) + Gv\\
\dot{e} &= q(\mathbf{w}, z, e) + b(\mathbf{w}, z, e)(\tau_0(\mathbf{w}) + \tilde{\xi}_0 + v)
\end{aligned} \tag{A.1}$$

where A is the "shift" matrix (all entries are zeros except those on the superdiagonal which are all ones), $B = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}^T$,

$$\tilde{\phi}(\tilde{\xi}, \mathbf{w}) = \phi_c(\tilde{\xi} + \tau(\mathbf{w})) - \phi(\tau(\mathbf{w})).$$

Since ϕ is locally Lipschitz and ϕ_c is locally Lipschitz and bounded and agrees with ϕ on $\tau(\mathbf{W})$, there exist two real numbers c_1 and c_2 so that we have

$$|\tilde{\phi}(\tilde{\xi}, \mathbf{w})| \le \min\{c_1|\tilde{\xi}|, c_2\} \le c_1|\tilde{\xi}| \quad \forall(\tilde{\xi}, \mathbf{w}) \in \mathbb{R}^d \times \mathbf{W}.$$

Note also that $q(\mathbf{w}, z, 0) + b(\mathbf{w}, z, 0)\tau_0(\mathbf{w}) = 0$ for all $(\mathbf{w}, z) \in \text{graph}(\pi)$. By the further change of variable (meant to put system (A.1) in normal form)

$$\tilde{\xi} \to \chi := \tilde{\xi} - G\int_0^e \frac{1}{b(\mathbf{w}, z, s)}ds,$$

system (A.1) transforms as

$$\begin{aligned}
\dot{\mathbf{w}} &= s(\mathbf{w})\\
\dot{z} &= f(\mathbf{w}, z, e)\\
\dot{\chi} &= A\chi + B\left(\tilde{\phi}(\chi, \mathbf{w}) - \nu(\mathbf{w})\right)\\
&\quad - G\left(\chi_0 + \tau_0(\mathbf{w}) + \frac{q(\mathbf{w}, z, 0)}{b(\mathbf{w}, z, 0)}\right) + L(\mathbf{w}, z, \chi, e)\\
\dot{e} &= q(\mathbf{w}, z, e) + b(\mathbf{w}, z, e)\\
&\quad \times \left(\tau_0(\mathbf{w}) + \chi_0 + g\lambda_0\int_0^e \frac{1}{b(\mathbf{w}, z, s)}ds + v\right)
\end{aligned} \tag{A.2}$$

where

$$\begin{aligned}
L &= AG\int_0^e \frac{1}{b(\mathbf{w}, z, s)}ds\\
&\quad + B\left(\tilde{\phi}\left(\chi + G\int_0^e \frac{1}{b(\mathbf{w}, z, s)}ds, \mathbf{w}\right) - \tilde{\phi}(\chi, \mathbf{w})\right)\\
&\quad - G\left(\frac{q(\mathbf{w}, z, e)}{b(\mathbf{w}, z, e)} - \frac{q(\mathbf{w}, z, 0)}{b(\mathbf{w}, z, 0)} + g\lambda_0\int_0^e \frac{1}{b(\mathbf{w}, z, s)}ds\right.\\
&\quad \left. + \int_0^e \frac{\partial}{\partial \mathbf{w}}\frac{1}{b(\mathbf{w}, z, s)}ds\dot{\mathbf{w}} + \int_0^e \frac{\partial}{\partial z}\frac{1}{b(\mathbf{w}, z, s)}ds\,\dot{z}\right).
\end{aligned}$$

Note that $L(\mathbf{w}, z, \chi, 0) = 0$ for all $(\mathbf{w}, z, \chi) \in \mathbf{W} \times \mathbb{R}^n \times \mathbb{R}^d$. By following the high-gain observer theory (see Esfandiari & Khalil, 1992, Teel & Praly, 1995), we finally re-scale the χ variable as

$$\chi = D_g\tilde{\chi} \quad \text{with } D_g = \text{diag}(1, g, \ldots, g^{d-1})$$

in this way transforming the χ and e dynamics in (A.2) as

$$\begin{aligned}
\dot{\tilde{\chi}} &= gH\tilde{\chi} + \frac{1}{g^{d-1}}B\left(\tilde{\phi}(D_g\tilde{\chi}, \mathbf{w}) - \nu(\mathbf{w})\right) - g\Lambda y_z(\mathbf{w}, z)\\
&\quad + D_g^{-1}L(\mathbf{w}, z, D_g\tilde{\chi}, e)\\
\dot{e} &= q(\mathbf{w}, z, e) + b(\mathbf{w}, z, e)\\
&\quad \times \left(\tau_0(\mathbf{w}) + \tilde{\chi}_0 + g\lambda_0\int_0^e \frac{1}{b(\mathbf{w}, z, s)}ds + v\right)
\end{aligned}$$

where $\Lambda = \begin{pmatrix} \lambda_0 & \cdots & \lambda_{d-1} \end{pmatrix}^T$, $y_z = \tau_0(\mathbf{w}) + \frac{q(\mathbf{w},z,0)}{b(\mathbf{w},z,0)}$, and H is a Hurwitz matrix, independent of g, whose eigenvalues are the roots of the polynomial $p(s) := s^d - \sum_{i=0}^{d-1}\lambda_i s^{d-1-i}$. The overall closed-loop system, regarded as a system with input v and output e, has relative degree one and zero dynamics given by

$$\begin{aligned}
\dot{\mathbf{w}} &= s(\mathbf{w})\\
\dot{z} &= f(\mathbf{w}, z, 0)\\
\dot{\tilde{\chi}} &= gH\tilde{\chi} + \frac{1}{g^{d-1}}B\left(\tilde{\phi}(D_g\tilde{\chi}, \mathbf{w}) - \nu(\mathbf{w})\right) - g\Lambda y_z(\mathbf{w}, z).
\end{aligned} \tag{A.3}$$

Standard ISS arguments, using the fact that H is Hurwitz and that $\tilde{\phi}(\chi, \mathbf{w})$ is globally Lipschitz in χ, uniformly in $\mathbf{w}$, can be used to show that there exists a $g^\star > 0$ such that for all $g \ge g^\star$ the $\tilde{\chi}$-subsystem, regarded as a system with inputs (y_z, ν), is ISS. In particular there exists a positive c' such that the following asymptotic estimate holds

$$\begin{aligned}
&\limsup_{t\to\infty} |\tilde{\chi}(t)|\\
&\quad \le c' \max\left\{\frac{1}{g^d}\limsup_{t\to\infty} |\nu(\mathbf{w}(t))|, \limsup_{t\to\infty} |y_z(t)|\right\}.
\end{aligned}$$

Seeing (A.3) as the subsystem (4) driving the $\tilde{\chi}$ subsystem through the coupling function $y_z(\mathbf{w}, z)$, using the minimum-phase assumption and the fact that $y_z(\mathbf{w}, z) = 0$ for all $(\mathbf{w}, z) \in$

graph(π), standard arguments can be used to conclude that system (A.3), also regarded as a system with inputs ν, is ISS with an asymptotic estimate of the form

$$\lim_{t\to\infty}\sup|(\mathbf{w}(t),z(t),\tilde\chi(t))|_{\mathrm{graph}(\pi)\times\{0\}}\le\frac{c'}{g^d}\lim_{t\to\infty}\sup|\nu(\mathbf{w}(t))|.$$

From this, the small gain arguments of Marconi et al. (2007) lead to the conclusion that there exists a continuous $\kappa(\cdot)$ such that the claim of Theorem 2 holds for some positive c. Furthermore, $\kappa(\cdot)$ is linear if graph(π) is also locally exponentially stable for (4).

The second part of the claim, namely exponential convergence of the error when $\nu=0$ and graph(π) is locally exponentially stable for (4), follows from Byrnes and Isidori (2004).

Appendix B. Proof of Corollary 1

The result immediately follows by the fact the $\phi(\cdot)$ is locally Lipschitz and by Theorem 2. As a matter of fact note that for all $k\ge 0$

$$L^k_{s(\mathbf{w})+\varepsilon\delta(\mathbf{w})}u^\star_\varepsilon(\mathbf{w})=L^k_{s(\mathbf{w})}u^\star_\varepsilon(\mathbf{w})+\varepsilon\rho_k(\mathbf{w},\varepsilon)\tag{B.1}$$

where $\rho_k(\mathbf{w},\varepsilon)$ is a continuous function. By using the previous expression, it turns out that

$$\begin{aligned}L^d_{s(\mathbf{w})+\varepsilon\delta(\mathbf{w})}u^\star_\varepsilon(\mathbf{w})&\\=\phi(u^\star_\varepsilon(\mathbf{w}),L_{s(\mathbf{w})}u^\star_\varepsilon(\mathbf{w}),\ldots,L^{d-1}_{s(\mathbf{w})}u^\star_\varepsilon(\mathbf{w}))+\nu(\mathbf{w})+\varepsilon\rho_d(\mathbf{w},\varepsilon)&\\=\phi(u^\star_\varepsilon(\mathbf{w}),L_{s(\mathbf{w})}u^\star_\varepsilon(\mathbf{w}),\ldots,L^{d-1}_{s(\mathbf{w})}u^\star_\varepsilon(\mathbf{w}))&\\\quad-\phi(u^\star_\varepsilon(\mathbf{w}),L_{s(\mathbf{w})+\varepsilon\delta(\mathbf{w})}u^\star_\varepsilon(\mathbf{w}),\ldots,L^{d-1}_{s(\mathbf{w})+\varepsilon\delta(\mathbf{w})}u^\star_\varepsilon(\mathbf{w}))&\\\quad+\phi(u^\star_\varepsilon(\mathbf{w}),L_{s(\mathbf{w})+\varepsilon\delta(\mathbf{w})}u^\star_\varepsilon(\mathbf{w}),\ldots,L^{d-1}_{s(\mathbf{w})+\varepsilon\delta(\mathbf{w})}u^\star_\varepsilon(\mathbf{w}))&\\\quad+\nu(\mathbf{w})+\varepsilon\rho_d(\mathbf{w},\varepsilon)&\\=\phi(u^\star_\varepsilon(\mathbf{w}),L_{s(\mathbf{w})+\varepsilon\delta(\mathbf{w})}u^\star_\varepsilon(\mathbf{w}),\ldots,L^{d-1}_{s(\mathbf{w})+\varepsilon\delta(\mathbf{w})}u^\star_\varepsilon(\mathbf{w}))&\\\quad+\nu(\mathbf{w})+\varepsilon\rho_d(\mathbf{w},\varepsilon)+\Delta(\mathbf{w},\varepsilon)&\end{aligned}$$

where

$$\begin{aligned}\Delta(\mathbf{w},\varepsilon):=&\ \phi(u^\star_\varepsilon(\mathbf{w}),L_{s(\mathbf{w})}u^\star_\varepsilon(\mathbf{w}),\ldots,L^{d-1}_{s(\mathbf{w})}u^\star_\varepsilon(\mathbf{w}))\\&-\phi(u^\star_\varepsilon(\mathbf{w}),L_{s(\mathbf{w})+\varepsilon\delta(\mathbf{w})}u^\star_\varepsilon(\mathbf{w}),\ldots,L^{d-1}_{s(\mathbf{w})+\varepsilon\delta(\mathbf{w})}u^\star_\varepsilon(\mathbf{w})).\end{aligned}$$

From this, the result immediately follows, by compactness arguments, using (B.1), the fact that $\phi(\cdot)$ is locally Lipschitz and Theorem 2.

Appendix C. Proof of Proposition 2

Let

$$p_{m_0}(\lambda)=\lambda^{2m_0}+a_{2m_0-1}\lambda^{2m_0-1}+\cdots+a_1\lambda+a_0$$

be the characteristic polynomial of the block-diagonal matrix $S(\varpi)$. By the Cayley–Hamilton theorem, it turns out that $p_{m_0}(S(\varpi))=0$ and thus

$$\Gamma(\varpi)\,S^k(\varpi)\,p_{m_0}(S(\varpi))w=0\tag{C.1}$$

for any $(\varpi,w)\in\mathbf{W}$ and any $k\ge 0$. If $m_0>m$ we introduce the coefficients $c_i=a_i/a_{2m}$, $i=0,\ldots,2m_0-1$, and $c_{2m_0}=1/a_{2m}$, which are well defined as the coefficient $a_{2m}\neq 0$ (as all "even" coefficients of the characteristic polynomial of a set of oscillators), and we note that relation (C.1) implies that

$$\begin{aligned}L^{k+2m}_s u^\star=&-c_0L^k_su^\star-\cdots-c_{2m-1}L^{k+2m-1}_su^\star\\&-c_{2m+1}L^{k+2m+1}_su^\star-\cdots-c_{2m_0}L^{k+2m_0}_su^\star\end{aligned}\tag{C.2}$$

for all $k\ge 0$, with $u^\star(\mathbf{w})$ defined as in (22) and $s(\mathbf{w})$ defined as in (20)–(21). On the other hand, if $m_0\le m$, we let $c_i=0$, $i=0,\ldots,2(m-m_0)-1$, and $c_i=a_{i-2(m-m_0)}$, $i=2(m-m_0),\ldots,2m-1$, and we note that relation (C.1) implies that

$$L^{k+2m}_su^\star=-c_0L^k_su^\star-\cdots-c_{2m-1}L^{k+2m-1}_su^\star.\tag{C.3}$$

By collecting the $2m$ relations obtained by evaluating (C.2) and (C.3) for $k=0,\ldots,2m-1$, one obtains

$$\ell(\tau(\mathbf{w}))=-A_m(\tau(\mathbf{w}))\,c+Q(\mathbf{w})\tag{C.4}$$

where $c=(c_0,\ldots,c_{2m-1})^T$ and

$$Q(\mathbf{w})=-\begin{pmatrix}L^{2m+1}_su^\star&\cdots&L^{2m_0}_su^\star\\\vdots&\ddots&\vdots\\L^{4m}_su^\star&\cdots&L^{2m+2m_0-1}_su^\star\end{pmatrix}\begin{pmatrix}c_{2m+1}\\\vdots\\c_{2m_0}\end{pmatrix}$$

if $m_0>m$, while $Q(\mathbf{w})=0$ otherwise. Furthermore, relations (C.2) and (C.3) evaluated for $k=2m$ yield

$$L^{4m}_su^\star(\mathbf{w})=-\ell^T(\tau(\mathbf{w}))\,c+q(\mathbf{w})\tag{C.5}$$

where

$$q(\mathbf{w})=c_{2m+1}L^{4m+1}_su^\star-\cdots-c_{2m_0}L^{2(m+m_0)}_su^\star$$

if $m_0>m$, while $q(\mathbf{w})=0$ otherwise.

Suppose $m\le m_0$. In this case, by Assumption 2, the matrix $A_m(\tau(\mathbf{w}))$ is nonsingular for all $\mathbf{w}\in\mathbf{W}$. Hence, (C.4) can be uniquely solved for c as

$$c=A_m^{-1}(\tau(\mathbf{w}))[Q(\mathbf{w})-\ell(\tau(\mathbf{w}))].$$

Replacing this into (C.5) yields

$$\begin{aligned}L^{4m}_su^\star(\mathbf{w})&=-\ell^T(\tau(\mathbf{w}))\,A_m^{-1}(\tau(\mathbf{w}))[Q(\mathbf{w})-\ell(\tau(\mathbf{w}))]+q(\mathbf{w})\\&=\ell^T(\tau(\mathbf{w}))A_m^{-1}(\tau(\mathbf{w}))\ell(\tau(\mathbf{w}))+\nu(\mathbf{w}),\end{aligned}\tag{C.6}$$

in which

$$\nu(\mathbf{w})=-\ell^T(\tau(\mathbf{w}))\,A_m^{-1}(\tau(\mathbf{w}))Q(\mathbf{w})+q(\mathbf{w}).$$

In the special case $m=m_0$, we have $Q(\mathbf{w})=0$ and $q(\mathbf{w})=0$ and hence $\nu(\mathbf{w})=0$.

Consider now the case $m>m_0$, in which we know that $Q(\mathbf{w})=0$ and $q(\mathbf{w})=0$. Observe that for all $i=1,\ldots,m$, the matrix $A_i(\tau(\mathbf{w}))$ can be factored as

$$A_i(\tau(\mathbf{w}))=\mathcal{O}_i(\varpi)\quad\mathcal{C}_i(\varpi,w)\quad i=1,\ldots,m$$

in which $\mathcal{O}_i\in\mathbb{R}^{2i\times 2m_0}$ and $\mathcal{C}_i\in\mathbb{R}^{2m_0\times 2i}$ are the matrices

$$\begin{aligned}\mathcal{O}_i(\varpi)&=\begin{pmatrix}\Gamma^T(\varpi)&S^T(\varpi)\Gamma^T(\varpi)&\cdots&(S^{2i-1}(\varpi))^T\Gamma^T(\varpi)\end{pmatrix}^T\\\mathcal{C}_i(\varpi,w)&=\begin{pmatrix}w&S(\varpi)w&\cdots&S^{2i-1}(\varpi)w\end{pmatrix}.\end{aligned}$$

From this, we observe that rank$A_i(\tau(\mathbf{w}))\le 2m_0$ for all i. Since, by Assumption 2, rank$A_{m_0}(\tau(\mathbf{w}))=2m_0$ for all $\mathbf{w}\in\mathbf{W}$, we conclude that the matrix $A_m(\tau(\mathbf{w}))$ has rank $2m_0$, with the first $2m_0$ columns of $A_m(\tau(\mathbf{w}))$ being linearly independent for all $\mathbf{w}\in\mathbf{W}$.

Any solution of (C.4) can be written as $c=c_\star+c_k$ where $c_\star$ is a solution of (C.4) and $c_k\in\mathrm{Ker}A_m(\tau(\mathbf{w}))$. From the properties of $A_m(\tau(\mathbf{w}))$ just described, it turns out that a possible solution of (C.4) is given by $c_\star=\mathrm{col}(c'_\star,0)$ with (recall that $Q(\mathbf{w})=0$)

$$c'_\star=-A_{m_0}^{-1}(\tau(\mathbf{w}))^{-1}\lfloor\ell(\tau(\mathbf{w}))\rfloor_{m_0}.$$

Hence, using $c=c_\star+c_k$, with $c_k\in\mathrm{Ker}A_m(\tau(\mathbf{w}))$, in (C.5), we have (recall that $q(\mathbf{w})=0$)

$$L^{4m}_su^\star(\mathbf{w})=-\ell^T(\tau(\mathbf{w}))\left[\begin{pmatrix}-A_{m_0}^{-1}(\tau(\mathbf{w}))\lfloor\ell(\tau(\mathbf{w}))\rfloor_{m_0}\\0\end{pmatrix}+c_k\right].$$

From (C.4), it is seen that

$$\ell^T(\tau(\mathbf{w}))\, c_k = -c^T A_m^T(\tau(\mathbf{w}))\, c_k = 0$$

because $A_m(\tau(\mathbf{w})) = A_m^T(\tau(\mathbf{w}))$ and $c_k \in \mathrm{Ker} A_m(\tau(\mathbf{w}))$. Hence

$$L_s^{4m} u^\star(\mathbf{w}) = \lfloor \ell^T(\tau(\mathbf{w}))\rfloor_{m_0} A_{m_0}^{-1}(\tau(\mathbf{w}))\lfloor \ell(\tau(\mathbf{w}))\rfloor_{m_0}. \qquad \text{(C.7)}$$

Letting $i^* := \min\{m_0, m\}$, and using the fact that Assumption 2 implies $A_{i^\star}^{-1}(\tau(\mathbf{w})) = A_{i^\star,\mathrm{sat}}^{-1}(\tau(\mathbf{w}))$, the two formulas (C.6) and (C.7) can be rewritten together as

$$\begin{aligned} L_s^{4m} u^\star(\mathbf{w}) &= \lfloor \ell^T(\tau(\mathbf{w}))\rfloor_{i^\star} A_{i^\star,\mathrm{sat}}^{-1}(\tau(\mathbf{w}))\lfloor \ell(\tau(\mathbf{w}))\rfloor_{i^\star} + \nu(\mathbf{w}) \\ &= \phi_{i^\star}(\tau(\mathbf{w})) + \nu(\mathbf{w}) \end{aligned}$$

with $\phi_{i^\star}(\xi)$ defined as in the proposition and $\nu(\mathbf{w}) = 0$ whenever $m_0 \le m$. From this, from the definition of ϕ and the fact that, by the definition of the α_i's, $\alpha_{i^\star}(\tau(\mathbf{w})) = 1$ and $\alpha_i(\tau(\mathbf{w})) = 0$ for all $i \ne i^\star$, the result of Proposition 2 follows.

Appendix D. Proof of Proposition 3

First, note that in both cases $\mathbf{W}$ is clearly invariant, since the norm of the state of each oscillator is kept constant by the flow of S.

Consider first the case $m \ge m_0$ and note that

$$A_r(\tau(\mathbf{w})) = \mathcal{O}_r(\varpi)\mathcal{C}_r(\varpi, w)$$

with $\mathcal{O}_{m_0} \in \mathbb{R}^{2m_0\times 2m_0}$ and $\mathcal{C}_{m_0} \in \mathbb{R}^{2m_0\times 2m_0}$ defined as

$$\begin{aligned} &\mathcal{O}_{m_0}(\varpi) \\ &= \begin{pmatrix}\Gamma^T(\varpi) & S^T(\varpi)\Gamma^T(\varpi) & \cdots & (S^{2m_0-1}(\varpi))^T\Gamma^T(\varpi)\end{pmatrix}^T \\ &\mathcal{C}_{m_0}(\varpi, w) = \begin{pmatrix} w & S(\varpi)w & \cdots & S^{2m_0-1}(\varpi)w\end{pmatrix}. \end{aligned}$$

Since, by hypothesis, $\varpi_i \ne \varpi_j$ for $i \ne j$ and no ϖ_i is zero, the latter matrices are indeed non singular if $(S_i(\varpi), \Gamma_i(\varpi))$ is observable for all i and if W has the structure indicated in (26). Hence (24) holds for some $\epsilon > 0$ dependent on $\underline{a}$ and ϖ.

Consider now the case $m < m_0$ and let $\varpi \in \mathbb{R}^{m_0}$ be partitioned as $\varpi = (\varpi', \varpi'') \in \mathbb{R}^m \times \mathbb{R}^{m_0-m}$ with $\varpi' = (\varpi_1 \cdots \varpi_m)^T$ and $\varpi'' = (\varpi_{m+1} \cdots \varpi_{m_0})^T$. If $w \in \mathbb{R}^{2m_0}$ is partitioned consistently as $w = (w', w'') \in \mathbb{R}^{2m} \times \mathbb{R}^{2(m_0-m)}$, then $\dot w' = S'(\varpi')w'$ and $\dot w'' = S''(\varpi'')w''$ where $S'(\varpi') = \mathrm{blkdiag}(S_1(\varpi_1), \ldots, S_m(\varpi_m))$ and $S''(\varpi'') = \mathrm{blkdiag}(S_{m+1}(\varpi_{m+1}), \ldots, S_{m_0}(\varpi_{m_0}))$. Let ϖ be a set of the form $\varpi = \varpi' \times \varpi''$ and W a set of the form $W = W' \times W''$ with $W' = W_{\underline{a},\bar a} \times \cdots \times W_{\underline{a},\bar a} \in \mathbb{R}^{2m}$ and $W'' = W_{\sigma\underline{a},\sigma\bar a} \times \cdots \times W_{\sigma\underline{a},\sigma\bar a} \in \mathbb{R}^{2(m_0-m)}$. Finally, let $\mathbf{w}' = (\varpi', w')$ and $\mathbf{w}'' = (\varpi'', w'')$.

Since the function $u^\star(\mathbf{w})$ can be decomposed as $u^\star(\mathbf{w}) = u^{\star\prime}(\mathbf{w}') + u^{\star\prime\prime}(\mathbf{w}'')$ with $u^{\star\prime} = \sum_{i=1}^m \Gamma_i(\varpi_i)w_i$, and $u_u^{\star\prime\prime} = \sum_{i=m+1}^{m_0} \Gamma_i(\varpi_i)w_i$, it turns out that $\tau(\mathbf{w})$ and $A_m(\tau(\mathbf{w}))$ can be expressed as $\tau(\mathbf{w}) = \tau'(\mathbf{w}') + \tau''(\mathbf{w}'')$ and

$$A_m(\tau(\mathbf{w})) = A_m(\tau'(\mathbf{w}')) + A_m(\tau''(\mathbf{w}''))$$

where τ' and τ'' are defined as τ with $(S, u^\star)$ replaced by $(S', u^{\star\prime})$ and $(S'', u^{\star\prime\prime})$, respectively. By the definition of τ,

$$\det(A_m(\tau(\mathbf{w}))) = \det(A_m(\tau'(\mathbf{w}_1)) + \sigma A_m(\tau''(\mathbf{w}_2)))$$

with $\mathbf{w}_1 \in \varpi_1 \times W_1$ and $\mathbf{w}_2 \in \varpi_2 \times (1/\sigma)W_2$. By the discussion above, there exists $\eta > 0$ such that $|\det(A_m(\tau'(\mathbf{w}_1)))| \ge \eta$ for all $\mathbf{w}_1 \in \varpi_1 \times W_1$. From this the result of the proposition follows by the continuity of the determinant in the parameter σ.

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