Uniform Practical Nonlinear Output Regulation

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***Abstract*—In this paper, we present a solution to the problem of asymptotic and practical semiglobal regulation by output feedback for nonlinear systems. A key feature of the proposed approach is that practical regulation is achieved uniformly with respect to the dimension of the internal model and to the gain of the stabilizer near the zero error manifold. This property renders the approach interesting for a number of real cases by bridging the gap between output regulation theory and advanced engineering applications. Simulation results regarding meaningful control problems are also presented.**

***Index Terms*—Disturbance suppression, output feedback, output regulation, output tracking, robust control.**

I. Introduction

We consider the problem of semiglobal nonlinear output regulation, namely the problem, for nonlinear systems, of compensating for the effect of exogenous signals, generated by an autonomous system (the so-called exosystem), by output feedback.

Since the first seminal results for linear systems (see [9]), in which the crucial concept of internal model-based regulator has been formulated, and their first extensions to a nonlinear setting in [15], the problem has attracted a number of researchers who, in the last years, proposed even more powerful and less restrictive frameworks. In [24], [16], [27], and [3], the problem of enlarging the domain of attraction from "local" to "nonlocal" results has been addressed at different levels of generality. The important issue of robustness to plant parameter variations of internal model-based regulators has been addressed for the first time for nonlinear systems in [12]. In [28], the problem in presence of uncertainties in the exosystem structure has been proposed and solved by formulating the so-called *adaptive output regulation* problem. Recent works have focused on the identification of design procedures yielding *nonlinear internal models*. In this respect, [5] is worth mentioning, focused on an internal model constituted by a linear system having a nonlinear output map, as well as [2] and [6] (see also [7]), which have definitely focused the attention on the design of nonlinear internal models having nonlinear observability forms. All these approaches provide constructive design procedures but rely upon different forms of the so-called *immersion assumption*, which limits in a substantial way the applicability of the different theories in a nonlinear context.

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In [19], the problem of semiglobal output regulation has been addressed in a fairly general framework consisting of a class of controlled systems and exosystems required to satisfy only an appropriate minimum-phase assumption without any immersion condition. The design methodology underlying [19], induced by the approach pioneered in [3] (see, also, [2]), is based on the reformulation of the problem of semiglobal output regulation into a problem of output feedback stabilization of compact attractors. In plain words, the main achievement in [19] has been to show that the steady-state input rendering invariant a compact attractor to be stabilized by output feedback can be dynamically generated, in a robust framework, by an appropriately designed regulator without any specific condition on this input (required, on the contrary, in the past through the immersion assumption). In achieving this result, a key role has been played by the theory of nonlinear observers developed in [25] and [1].

The developments in [19] were focused on issues regarding the existence of the regulator and no special attention was given on design aspects. In this paper, we aim to fill this gap by providing a complete framework for semiglobal output regulation. More specifically, we present explicit expressions of the regulator and we address possible practical implementations. Also, we implicitly solve a problem of practical output regulation, which amounts to designing a regulator achieving arbitrarily small asymptotic regulation error. Of course, the problem of practical output regulation is not new and several attempts have been made in the past literature along this direction. One way of approaching this problem is the one pursued, besides others, in [13] and [25] (see, also, [4, Sec. 2.5]). There, the idea was to use a polynomial approximation and/or a power series expansion of the so-called regulator equations in order to identify an approximation of the desired steady-state control input, with a degree of accuracy depending on the bound of the residual error, which can be dynamically reproduced by means of a linear internal model. The main drawback in pursuing this strategy is that the dimension of the internal model is, in general, dependent on the desired bound of the regulation error and tends to grow indefinitely as the desired bound tends to zero. This, indeed, is a severe limitation for real implementations in which, in order to cope with computational limitations and real-time issues, the dimension of the regulator is required to be as small as possible.

A different control philosophy to steer the regulation error to arbitrarily small values is to use high-gain error feedback (see [14] and related literature). Techniques of this kind can be more appropriately framed into problems of practical tracking/rejection rather than practical output regulation, as the idea is to adapt the static gain by which the error (and a number of its time derivatives) are fed back, rather than to capture the essential properties of the exogenous signal into an internal model. While this kind of strategy leads to a theory that is applicable to a wide class of reference/disturbance signals (not necessarily generated by an exosystem), they present the typical problems linked to high-gain control structures, such as sensitivity-to-measurement noise and minimum-phase constraints, which substantially limit their range of applications. On the contrary, in this paper, we present practical design procedures leading to a regulator achieving practical regulation uniformly in the local gain of the stabilizer and in the dimension of the internal model.

This paper is organized as follows. In Section II, we present the framework of semiglobal output regulation without immersion proposed in [19] and a few refinements of the results of [19] regarding the design of the stabilizer. Section III complements the theory of [19] presenting two possible explicit expressions of the regulator. Then, Section IV is focused on the problem of uniform practical output regulation and related computational issues with Section V presenting two relevant numerical examples. Finally, Section VI concludes this paper by presenting final remarks. Relevant proofs of the results presented in this paper are reported in the Appendices I–V.

Notation: $\mathbb{R}$ denotes the field of real numbers and $\mathbb{C}$ the one of complex numbers. For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm and, for $\mathcal{C}$, a closed subset of $\mathbb{R}^n$, $|x|_{\mathcal{C}} = \min_{y \in \mathcal{C}} |x-y|$ denotes the distance of x from $\mathcal{C}$. For a locally Lipschitz function $f : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^m$, L_f denotes the Lipschitz constant of f on a compact set $C \subseteq \mathcal{D}$ to be specified.

II. The Framework

A. The Problem and Standing Assumptions

We consider the class of nonlinear systems with a well-defined relative degree $r \geq 1$ described in the normal form

$$\begin{aligned} \dot{z} &= f(w,z,y_1) \\ \dot{y}_i &= y_{i+1}, \qquad i = 1,\ldots,r-1 \\ \dot{y}_r &= q(w,z,y) + b(w,z,y)u \end{aligned} \tag{1}$$

with state $(z,y) \in \mathbb{R}^n \times \mathbb{R}^r$, $y = (y_1,\ldots,y_r)^{\mathrm{T}}$, control input $u \in \mathbb{R}$, measured output $y_1 \in \mathbb{R}$, and with initial conditions $(z(0), y(0))$ arbitrary in a set $Z \times Y \subset \mathbb{R}^n \times \mathbb{R}^r$. The functions f, q, and b are sufficiently smooth. The unmeasured input $w \in \mathbb{R}^s$ of (1) is an exogenous signal that is supposed to be generated by the smooth *exosystem*

$$\dot{w} = s(w) \tag{2}$$

whose initial state $w(0)$ is arbitrary in a set $W \subset \mathbb{R}^s$. Depending on the control scenario, the variable w may assume different meanings. It may represent exogenous disturbances to be rejected and/or references to be tracked. It may also contain (constant or time-varying) uncertain parameters affecting the controlled plant.

Associated with (1) and (2), there is a *regulated* output $e \in \mathbb{R}^\iota$ expressed as

$$e = h(w,z,y) \tag{3}$$

in which $h : \mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^\iota$ is a smooth function.

For system (1)–(3), the problem of *semiglobal asymptotic output regulation* is defined as follows (see [3]). Find, if possible, an output feedback controller of the form

$$\dot{\eta} = \varphi(\eta, y_1) \quad u = \vartheta(\eta, y_1) \tag{4}$$

with state $\eta \in \mathbb{R}^m$, and a compact set $M \subset \mathbb{R}^m$ such that, in the associated closed-loop system (1)–(4), the positive orbit of $W \times Z \times Y \times M$ is bounded and, for each $w(0), z(0), y(0), \eta(0) \in W \times Z \times Y \times M$

$$\lim_{t\to\infty} e(t) = 0 \tag{5}$$

uniformly with respect to $w(0), z(0), y(0), \eta(0)$.

A natural relaxation of the previous problem is the so-called *semiglobal practical output regulation problem*, which amounts, for a given $\epsilon > 0$, to identifying a regulator of the form (4) such that the same requirements as above are fulfilled with (5) relaxed to

$$\lim_{t\to\infty} \sup |e(t)| \leq \epsilon.$$

We approach the solution of the problem above under the following assumptions.

Assumptions:

A1) The sets Z, Y, and W are known and compact and the set W is (forward and backward) *invariant* for (2).

A2) There exists a compact *invariant* set $\mathcal{A}_0 \subset \mathbb{R}^{s+n}$, which is locally asymptotically stable for the system

$$\dot{w} = s(w) \quad \dot{z} = f(w,z,0) \tag{6}$$

with a domain of attraction which contains the set of initial conditions $W \times Z$.

A3) The functions f, q, b, h, and s are known. The "high-frequency gain" b and the function h satisfy, for some constant $\bar{b}$

$$\begin{aligned} &b(w,z,y) \geq \bar{b} > 0 \qquad \forall (w,z,y) \in \mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}^r \\ &h(w,z,0) = 0 \qquad \forall (w,z) \in \mathcal{A}_0. \end{aligned}$$

◁

Remark: Assumption A2 is nothing else but a reformulation of the "weak minimum-phase" assumption proposed in [3]. In particular, in [3], it has been shown that a sufficient condition for the existence of the set $\mathcal{A}_0$ which is asymptotically stable for (6), is that the positive orbit of the set $W \times Z$ under the flow of (6) is bounded and the consequent ω-limit set $\omega(W \times Z)$ of the set $W \times Z$ (see [10]) is contained in $W \times Z$. The set $\mathcal{A}_0$ in the assumption coincides, therefore, with the ω-limit set $\omega(W \times Z)$ of the set $W \times Z$. ◁

Remark: In the terminology of [3], the set $\mathcal{A}_0$ plays the role of a *steady-state locus* and the restriction of the dynamics of (6) to the *invariant* set $\mathcal{A}_0$ qualify as *steady-state dynamics*. Under this perspective, the desired steady-state behavior of the overall closed-loop system is such that y converges to 0 and (w,z) converges to $\mathcal{A}_0$. This motivates the restriction on h in A3, which, in order to satisfy the requirement that the regulated error asymptotically converges to zero, asks that the map h vanishes at the desired steady state. ◁

Remark: As discussed in [19], assumption A2 could be weakened by assuming an appropriate stabilizability property of the first dynamics of (1) and (2) by means of the "virtual" input y_1. This, in turn, would allow one to consider in our framework also certain classes of nonminimum phase systems. It must be stressed, though, that the problem of output regulation in a general robust setting is still an open problem (some results for specific classes of systems can be found in [21] and references therein). ◁

We solve the problem of semiglobal output regulation in the simplified case in which the relative degree r of the system (1) is unitary. The reason why this can be assumed without loss of generality follows from classical results about output feedback stabilization which, for sake of completeness, are briefly recalled now.

For system (1), consider the change of variables

$$\begin{aligned} &y_i \mapsto \tilde{y}_i = g^{-(i-1)} y_i, \qquad i = 1, \ldots, r-1 \\ &y_r \mapsto \tilde{y}_r := y_r + g^{r-1} a_0 y_1 + g^{r-2} a_1 y_2 + \cdots + g a_{r-2} y_{r-1} \end{aligned}$$

where $g > 1$ is a design parameter and a_i, $i = 0, \ldots, r-2$, are such that all roots of the polynomial $\lambda^{r-1} + a_{r-2}\lambda^{r-1} + \ldots + a_1\lambda + a_0 = 0$ have negative real part. This change of variables transforms system (1) and (3) into a system of the form

$$\begin{aligned} \dot{z} &= f(w, z, \tilde{y}_1) \\ \dot{\tilde{y}} &= gH\tilde{y} + B\tilde{y}_r \\ \dot{\tilde{y}}_r &= \tilde{q}(w, z, \tilde{y}, \tilde{y}_r) + \tilde{b}(w, z, \tilde{y}, \tilde{y}_r)u \\ e &= \tilde{h}(w, z, \tilde{y}, \tilde{y}_r) \end{aligned} \tag{7}$$

in which $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_{r-1})$, H is a properly defined Hurwitz matrix, and $\tilde{q}$, $\tilde{b}$, and $\tilde{h}$ are suitable smooth functions with $\tilde{b}(w, z, \tilde{y}, \tilde{y}_r) \geq \bar{b}$ and $\tilde{h}(w, z, 0, 0) = h(w, z, 0)$ for all $(w, z, \tilde{y}, \tilde{y}_r) \in \mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}^{r-1} \times \mathbb{R}$. Let $\tilde{Y} \in \mathbb{R}^{r-1}$ be a compact set such that $y \in Y \Rightarrow \tilde{y} \in \tilde{Y}$ and note that, as $g > 1$ and by definition of $\tilde{y}$, the set $\tilde{Y}$ can be taken not dependent on g. System (2) and (7), regarded as a system with input u and output $\tilde{y}_r$, has relative degree one and zero dynamics

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f(w, z, \tilde{y}_1) \\ \dot{\tilde{y}} &= gH\tilde{y}. \end{aligned} \tag{8}$$

For this system, under our assumption, classical results (see, for instance, [29]) can be invoked to show the existence of a $g^\star > 1$ such that for all $g \geq g^\star$ the set $\mathcal{A}_0 \times \{0\}$ is locally asymptotically stable for (8) [locally exponentially if $\mathcal{A}_0$ is such for system (6)] with a domain of attraction which contains $W \times Z \times \tilde{Y}$. Moreover, by definition of $\tilde{h}$ and by the fact $h(w, z, 0) = 0$ for all $(w, z) \in \mathcal{A}_0$, we have also $\tilde{h}(w, z, \tilde{y}, 0) = 0$ for all $((w, z), \tilde{y}) \in \mathcal{A}_0 \times \{0\}$. This implies that if the relative degree-r system (1)–(3), with output y_1 and input u, satisfies our assumption, also the relative degree-1 system (7), with output $\tilde{y}_r$ and input u, satisfies a similar assumption with the set $\mathcal{A}_0$ replaced by $\mathcal{A}_0 \times \{0\}$. In other words, system (7) inherits the same properties of system (1) with the "output" $\tilde{y}_r$ playing the role of measured output y. In addition, it can be shown that the solution of the problem of semiglobal output regulation for system (7), by means of an output $(\tilde{y}_r)$ feedback regulator, leads to the solution of the problem at hand for system (1) by means of a regulator of the form (4). As a matter of fact, let

$$\dot{\tilde{\eta}} = \tilde{\varphi}(\tilde{\eta}, \tilde{y}_r) \quad u = \tilde{\vartheta}(\tilde{\eta}, \tilde{y}_r) \tag{9}$$

be a regulator solving the problem for system (7) and consider the output (y_1) feedback regulator of the form[1]

$$\begin{aligned} \dot{\tilde{\eta}} &= \tilde{\varphi}(\tilde{\eta}, \hat{y}_r + \sum_{i=0}^{r-2} a_i g^{r-1-i} \hat{y}_{i+1}) \\ \dot{\hat{y}}_i &= \hat{y}_{i+1} + \ell^i c_{r-i}(y_1 - \hat{y}_1), \qquad i = 1, \ldots, r-1 \\ \dot{\hat{y}}_r &= \ell^r c_0 (y_1 - \hat{y}_1) \\ u &= \sigma_s\left(\tilde{\vartheta}\left(\tilde{\eta}, \hat{y}_r + \sum_{i=0}^{r-2} a_i g^{r-1-i} \hat{y}_{i+1}\right)\right) \end{aligned} \tag{10}$$

where the coefficients c_i, $i = 0, \ldots, r-1$, are such that all the roots of the polynomial $\lambda^r + c_{r-1}\lambda^r + \ldots + c_1\lambda + c_0 = 0$ have negative real part, $\sigma_s : \mathbb{R} \mapsto \mathbb{R}$ is a properly defined saturation function satisfying $\sigma_s(x) = s \cdot \mathrm{sgn}(x)$ for all $|x| \geq s$, and ℓ and s are positive design parameters. It can be proved (see [29]) that, under appropriate technical conditions, there exists an $s^\star > 0$ and, for all $s \geq s^\star$, an $\ell^\star > 0$ such that for all $\ell \geq \ell^\star$, the regulator (10) solves the problem of semiglobal output regulation for system (1). Details in this direction, here omitted for reasons of space, can be found in [8], [29], [20], and [17].

These reasonings and results justify the fact of focusing on the class of systems (1) with $r = 1$, which, for notational convenience, is rewritten as

$$\begin{aligned} \dot{z} &= f(w, z, y), \qquad z \in \mathbb{R}^n \\ \dot{y} &= q(w, z, y) + b(w, z, y)u, \qquad y, u \in \mathbb{R} \end{aligned} \tag{11}$$

with regulation error (3). All the forthcoming analysis will refer to system (11) and (2).

B. The Asymptotic Regulator

In this section, we present the key idea pursued in [19] to solve the problem of semiglobal output regulation and the main results, which are instrumental for the forthcoming analysis.

In order to steer the regulation error to zero, the idea in [19] is to make the set $\mathcal{A}_0 \times \{0\}$, on which the error vanishes by assumption, locally asymptotically stable for the controlled system (11) with a proper domain of attraction. What renders the problem challenging and different from a "conventional" set stabilization problem is that the set in question is not invariant for system (11) with $u = 0$ as a consequence of the fact that the term $q(w, z, 0)$ is not, in general, vanishing on $\mathcal{A}_0$. In this respect, the fundamental property required to the regulator (indeed to any regulator solving the problem at hand; see [3]) is to asymptotically reproduce the term $-q(w, z, 0)/b(w, z, 0)$, with $(w, z) \in \mathcal{A}_0$, in a robust way, namely without having access neither to the exosystem variable w (which, as previously said, may contain parametric uncertainties) nor to the state z

[1]The regulator (10) relies on the well-known "dirty derivatives observer" in order to replace the knowledge of the output time derivatives y_i, implicitly used in (9) through the variable $\tilde{y}_r$, by appropriate estimates $\hat{y}_i$ (see [8] and [29]).

but rather only using the measurable output y. This property is what, in [3], has been referred to as *internal model property*.

The regulator (4) proposed in [19] takes the specific form

$$\begin{aligned}\dot\eta &= F\eta + Gu, & \eta &\in \mathbb{R}^m\\ u &= \gamma(\eta) + v, & v &= \kappa(y)\end{aligned} \tag{12}$$

in which $m > 0$, $(F, G) \in \mathbb{R}^{m\times m}\times\mathbb{R}^{m\times 1}$ is a controllable pair with F Hurwitz, and $\gamma : \mathbb{R}^m \to \mathbb{R}$ and $\kappa : \mathbb{R} \to \mathbb{R}$ are suitable continuous maps. The initial condition of (12) is supposed to be in an arbitrary compact set $M \subset \mathbb{R}^m$.

In this framework, the problem is reduced to find a pair (F, G) and a function γ so that the regulator (12) possesses the internal model property, namely so that the overall closed-loop system (11) and (12) has an *invariant* set, whose projection on the (w, z, y) space is precisely $\mathcal{A}_0 \times \{0\}$. Then, the regulator tuning can be completed by selecting the function κ so that such a set is locally asymptotically stable with a proper domain of attraction. How this can be achieved is detailed in the following.

Consider the change of variable

$$\eta \mapsto x := \eta - \int_0^y \frac{1}{b(w,z,s)}\,ds$$

which puts the overall closed-loop system (2), (11), and (12) in the *normal form*

$$\begin{aligned}\dot w &= s(w)\\ \dot z &= f(w,z,0) + y f_1(w,z,y)\\ \dot x &= Fx - G\frac{q(w,z,0)}{b(w,z,0)} + yGg_1(w,z,y)\\ \dot y &= b(w,z,y)\left[\gamma\left(x + \int_0^y \frac{1}{b(w,z,s)}ds\right) + \frac{q(w,z,0)}{b(w,z,0)}\right.\\ &\qquad\left. +\, y q_1(w,z,x,y) + v\right]\end{aligned} \tag{13}$$

where f_1, g_1, and q_1 are suitable smooth functions. Because our ultimate idea for the design of $v = \kappa(y)$ is to adopt robust stabilization tools proposed for minimum-phase systems (see [29]), it turns out crucial to study the *zero dynamics* of system (13) with respect to the input v and output y, which are described by

$$\dot w = s(w) \quad \dot z = f(w,z,0) \quad \dot x = Fx - G\frac{q(w,z,0)}{b(w,z,0)} \tag{14}$$

or, in more compact form

$$\dot{\mathbf{z}} = \mathbf{f}_0(\mathbf{z}) \quad \dot x = Fx - G\mathbf{q}_0(\mathbf{z}) \tag{15}$$

having defined

$$\mathbf{z} := \begin{pmatrix} w\\ z\end{pmatrix} \quad \mathbf{f}_0(\mathbf{z}) := \begin{pmatrix} s(w)\\ f(w,z,0)\end{pmatrix}$$
$$\mathbf{q}_0(\mathbf{z}) := \frac{q(w,z,0)}{b(w,z,0)}.$$

The initial condition of this system ranges in the set $W \times Z \times X$ with $X \subset \mathbb{R}^m$. It turns out that, if F is chosen Hurwitz, system (14) has an asymptotically stable compact set as precisely detailed in Proposition 1 (see [19, Prop. 1 and 2]).

Proposition 1: There exists an $\ell > 0$ such that, if the eigenvalues of F are in $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < -\ell\}$, then the function $\tau : \mathcal{A}_0 \to \mathbb{R}^m$ defined as

$$\tau(\mathbf{z}) = -\int_{-\infty}^0 e^{-Fs} G\mathbf{q}_0(\mathbf{z}(s,\mathbf{z}))\,ds \tag{16}$$

in which $\mathbf{z}(t, \mathbf{z}_0)$ denotes the solution at time t of $\dot{\mathbf{z}} = \mathbf{f}_0(\mathbf{z})$ passing through $\mathbf{z}_0$ at time $t = 0$, has a C^1 extension on a neighborhood of $\mathcal{A}_0$, is the unique solution of

$$\mathfrak{L}_{\mathbf{f}_0}\tau(\mathbf{z}) = F\tau(\mathbf{z}) - G\mathbf{q}_0(\mathbf{z}) \qquad \forall\, \mathbf{z} \in \mathcal{A}_0 \tag{17}$$

where $\mathfrak{L}_{\mathbf{f}_0}$ denotes the Lie derivative along $\mathbf{f}_0$, and is such that the set

$$\mathrm{graph}(\tau) = \{(\mathbf{z}, x) \in \mathcal{A}_0 \times \mathbb{R}^m : x = \tau(\mathbf{z})\} \tag{18}$$

is locally asymptotically stable for (15) with domain of attraction $W \times \mathcal{D}$ with $\mathcal{D} \subset Z \times X$. Furthermore, the set in question is locally exponentially stable for (15) if $\mathcal{A}_0$ is such for (6). ◁

Remark: The requirement of choosing F with a certain stability margin given in Proposition 1 by the positive real number ℓ represents only a technical assumption needed to guarantee that the function τ has a C^1 extension (see [19]). In this sense, the assumption in question must be not confused with a "high gain" requirement on the choice of F. In other words, any choice of F such that (16) has a C^1 extension is an appropriate one. ◁

According to this result and with an eye to system (13), it is easy to realize that the proposed controller has the internal model property if the function γ can be chosen so that $b(w,z,0)\gamma(\tau(w,z)) + q(w,z,0)$ is vanishing for all $(w,z) \in \mathcal{A}_0$ or, equivalently, if

$$\mathbf{q}_0(\mathbf{z}) + \gamma(\tau(\mathbf{z})) = 0 \qquad \forall\, \mathbf{z} \in \mathcal{A}_0. \tag{19}$$

Indeed, in such a case, the set

$$\mathrm{graph}(\tau)\times\{0\} = \{(\mathbf{z}, x, y) \in \mathcal{A}_0\times\mathbb{R}^m\times\mathbb{R} : x = \tau(\mathbf{z}), y = 0\} \tag{20}$$

is an invariant set for (13) (with $v \equiv 0$) on which, by assumption, the regulation error e is identically zero. The crucial result which guarantees that such a γ exists is presented in Proposition 2 (see [19]).

Proposition 2: Let d be the minimal dimension of the dynamics $\dot{\mathbf{z}} = \mathbf{f}_0(\mathbf{z})$ restricted to the *invariant* set $\mathcal{A}_0$. Set

$$m \geq 2d + 2. \tag{21}$$

Let ℓ and $\tau : \mathcal{A}_0 \to \mathbb{R}^m$ be as in Proposition 1. There exists a subset $\mathcal{S} \subset \mathbb{C}$ of zero Lebesgue measure such that if the eigenvalues of F are in $\{\lambda \in \mathbb{C}, : \mathrm{Re}\lambda < -\ell\}\setminus\mathcal{S}$ and the pair (F, G) is controllable, then the function τ satisfies the *partial injectivity condition*

$$|\mathbf{q}_0(\mathbf{z}_1) - \mathbf{q}_0(\mathbf{z}_2)| \leq \varrho(|\tau(\mathbf{z}_1) - \tau(\mathbf{z}_2)|) \qquad \forall\, \mathbf{z}_1, \mathbf{z}_2 \in \mathcal{A}_0 \tag{22}$$

where ϱ is a class-$\mathcal{K}$ function and, as a consequence, there exists a continuous function γ satisfying (19). ◁

With these results at hand, the design of the regulator (12) can be completed by solving a *set stabilization problem*, namely by taking any output feedback $v = \kappa(y)$ rendering the set (20) asymptotically stable with a domain of attraction containing the set of initial conditions. To this purpose, in [19], the existence of a continuous function κ has been shown, which succeeds in this task. In particular, the stabilizer can be taken of the form $v = -ky$ with k sufficiently large if the set $\mathcal{A}_0$ is locally exponentially stable for (6) and the function γ satisfying (19) is locally Lipschitz. Further results in this respect will be given in Section II-C. In general, any design procedure leading to an output feedback control law able to asymptotically stabilize (20) can be adopted to successfully complete the regulator design.

In summary, the tuning of the regulator (12) amounts to choosing an arbitrary controllable pair (F, G), with F a Hurwitz matrix with a suitable gain margin according to the previous remark and of appropriate dimension [see (21)], to designing the function γ so that (19) is fulfilled in terms of a function τ computed as in (16) or, equivalently, as a solution of the partial differential equation (PDE) (17), and finally, to selecting the function κ solving the set stabilization problem. It turns out that one of the key issue is the selection of γ, which is merely guaranteed to exist by Proposition 2 but whose expression is an issue left open in [19]. Its design, in an exact and approximate sense, is the key topic addressed in the following.

C. Design of a Universal Stabilizer and a Total Stability Result

In general, the design of κ depends on the specific expression of γ chosen to implement (12). In this section, we refine the results of [19] by showing that a unique κ succeeds not for only one but for a whole family of γ satisfying (19) (see the forthcoming Theorem 1). Furthermore, in order to pave the way for the analysis presented in Section IV regarding practical output regulation, we enrich the stabilization tools proposed in [19] with a *total stability result* for compact attractors roughly claiming that practical stability of the set (20) can be achieved if the function γ in (12) satisfies (19) modulo an approximation error.

To make all this precise, to any class-$\mathcal{K}$ function α and nonnegative real number δ, we associate the set

$$\begin{aligned}\Gamma_\alpha(\delta) := \{\gamma \in C^0 : &|\gamma(x) + \mathbf{q}_0(\mathbf{z})| \\ &\leq \alpha(|x - \tau(\mathbf{z})|) + \delta \quad \forall x \in \mathbb{R}^m, \mathbf{z} \in \mathcal{A}_0\}.\end{aligned} \quad (23)$$

Clearly, any $\gamma \in \Gamma_\alpha(0)$ satisfies (19). Furthermore, under the conditions of Proposition 2, there exists at least one function α such that the set $\Gamma_\alpha(0)$ is not empty. With these considerations and notations in mind, the first aforementioned result can be formulated as follows.

Theorem 1: Let α be a class-$\mathcal{K}$ function such that the set $\Gamma_\alpha(0)$ is not empty. There exists a continuous function κ (dependent on α) such that for any $\gamma \in \Gamma_\alpha(0)$ the set $\mathrm{graph}(\tau) \times \{0\}$ is asymptotically stable for (13) with $v = \kappa(y)$ with a domain of attraction containing $W \times Z \times X \times Y$.

Furthermore, if α is also locally Lipschitz and $\mathcal{A}_0$ is locally exponentially stable for (6), then κ can be taken linear, namely there exists a $k^\star > 0$ such that for all $k \geq k^\star$ the choice $v = -ky$ renders the set (20) locally asymptotically stable for (13).

Proof: The proof of the theorem easily follows from standard high-gain arguments (see [19, Th. 2 and 3]) once proved that the term $\gamma(x + \int_0^y ds/b(w,z,s)) + q(w,z,0)/b(w,z,0)$ in (13) is bounded by class-$\mathcal{K}$ functions of $|y|$ and of the distance $|(x,\mathbf{z})|_{\mathrm{graph}(\tau)}$, depending on α but not on the specific $\gamma \in \Gamma_\alpha(0)$. To this end, because the function τ is continuous and the set $\mathcal{A}_0$ is closed, for $(x,\mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^{s+n}$, let $\mathbf{z}_p \in \mathcal{A}_0$ be such that

$$|(x,\mathbf{z})|_{\mathrm{graph}(\tau)} = |(x,\mathbf{z}) - (\tau(\mathbf{z}_p), \mathbf{z}_p)|$$

and note that, by definition of the distance, we have

$$\begin{aligned}|x - \tau(\mathbf{z}_p)| &\leq |(x,\mathbf{z})|_{\mathrm{graph}(\tau)} \\ |\mathbf{z} - \mathbf{z}_p| &\leq |(x,\mathbf{z})|_{\mathrm{graph}(\tau)}.\end{aligned}$$

Then, by bearing in mind the notation introduced in (15), it follows that for all $\gamma \in \Gamma_\alpha(0)$

$$\begin{aligned}|\gamma(\eta) + \mathbf{q}_0(\mathbf{z})| &\leq |\gamma(\eta) + \mathbf{q}_0(\mathbf{z}_p)| + |\mathbf{q}_0(\mathbf{z}) - \mathbf{q}_0(\mathbf{z}_p)| \\ &\leq \alpha(|\eta - \tau(\mathbf{z}_p)|) + L_{\mathbf{q}_0}|\mathbf{z} - \mathbf{z}_p|\end{aligned} \quad (24)$$

where $L_{\mathbf{q}_0}$ is the Lipschitz constant of $\mathbf{q}_0$ on the compact domain of $\mathbf{z}$. Because α is a class-$\mathcal{K}$ function, we get

$$\alpha(|\eta - \tau(\mathbf{z}_p)|) \leq \alpha(2|x - \tau(\mathbf{z}_p)|) + \alpha\left(2\left|\int_0^y \frac{ds}{b(w,z,s)}\right|\right)$$

and, therefore

$$\begin{aligned}&|\gamma(\eta) + \mathbf{q}_0(\mathbf{z})| \\ &\quad\leq \alpha(2|(x,\mathbf{z})|_{\mathrm{graph}(\tau)}) + L_{\mathbf{q}_0}|(x,\mathbf{z})|_{\mathrm{graph}(\tau)} + \alpha(2|y|/\bar{b}) \\ &\quad:= \varrho_{h1}(|(x,\mathbf{z})|_{\mathrm{graph}(\tau)}) + \varrho_{h2}(|y|)\end{aligned}$$

where $\varrho_{h1}(s) := \alpha(s) + L_{\mathbf{q}_0}s$ and $\varrho_{h2}(s) = \alpha(2s/\bar{b})$ are class-$\mathcal{K}$ functions not dependent on the specific $\gamma \in \Gamma_\alpha(0)$. From this result, the claim of the theorem can be proved by using off-the-shelf the high-gain arguments in [19, Th. 2 and 3] to which the interested reader is referred. ∎

We consider now the total stability result that can be formulated as follows.

Theorem 2: Let κ be the continuous function associated to some class-$\mathcal{K}$ function α, as guaranteed by Theorem 1. Then, for any $\epsilon > 0$, there exists a $\delta_\epsilon > 0$ such that for any $\gamma \in \Gamma_\alpha(\delta_\epsilon)$ the trajectories of (13) with $v = \kappa(y)$ originating from $W \times Z \times X \times Y$ are bounded

$$\lim_{t\to\infty} \sup |(w(t), z(t), x(t), y(t))|_{\mathrm{graph}(\tau)\times\{0\}} \leq \epsilon \quad (25)$$

and $\lim_{t\to\infty} \sup |e(t)| \leq \epsilon$.

Proof: By the same computations presented in the proof of Theorem 1, it can be shown that the term $\gamma(x + \int_0^y ds/b(w,z,s)) + q(w,z,0)/b(w,z,0)$ in (13) with $\gamma \in \Gamma_\alpha(\delta_\epsilon)$ can be bounded as

$$\begin{aligned}&\left|\gamma(x + \int_0^y \frac{ds}{b(w,z,s)}) + \mathbf{q}_0(\mathbf{z})\right| \\ &\qquad\leq \varrho_{h1}\left(|(x,\mathbf{z})|_{\mathrm{graph}(\tau)}\right) + \varrho_{h2}(|y|) + \delta_\epsilon\end{aligned} \quad (26)$$

where ϱ_{h1} and ϱ_{h2} are defined as in the proof of Theorem 1. From this bound, by local input-to-state stability arguments (see [19, Th. 2 and 3]), we obtain the existence of a time $T > 0$, a class-$\mathcal{KL}$ function β and a class-$\mathcal{K}$ function ϖ such that

$$|p(t)|_{\text{graph}(\tau)} \leq \max\{\beta(|p(T)|_{\text{graph}(\tau)}, t-T), \varpi(\max_{\tau\in[T,t]} |y(\tau)|)\}$$

for all $t \geq T$, where $p(t) := \text{col}(w(t), z(t), x(t))$. Similarly, using the same arguments of [19. Th. 3], it is possible to conclude, by taking advantage of the estimate (26), that

$$|y(t)| \leq \max\{\exp(-(t-T))|y(T)|, \sup_{\tau\in[T,t)} \bar{\varpi}(|p(\tau)|_{\text{graph}(\tau)}), \bar{\varpi}\circ\varrho_h^{-1}(\delta_\epsilon)\}$$

for all $t \geq T$ where $\bar{\varpi}$ is a class-$\mathcal{K}$ function satisfying $\varpi \circ \bar{\varpi}(s) < s$ for all $s \in \mathbb{R}^+$. From this result, the bound in (25) follows by standard small gain arguments taking

$$\delta_\epsilon \leq \min\left\{\varrho_h\left(\frac{\epsilon}{2}\right), \varrho_h\circ\bar{\varpi}^{-1}\left(\frac{\epsilon}{2}\right)\right\}.$$

Because $h(w,z,y)$ defining the regulation error e is a continuous function that, by assumption, vanishes on the set (20), also the bound on the regulation error immediately follows. ■

III. Expressions of γ

A. Integral-Based γ

Our first expression is strongly inspired by [26] (see, in particular, Lemma 4). It can be written upon the assumption that the set $\mathcal{A}_0$ is not locally thin[2] in $\mathbb{R}^{s+n}$. In order to properly define the function, it is appropriate to associate to each $x \in \mathbb{R}^m$ a point $p_{im}(x) \in \tau(\mathcal{A}_0)$ such that

$$p_{im}(x) \in \arg\min_{\mathbf{z}\in\mathcal{A}_0} |x - \tau(\mathbf{z})|. \tag{27}$$

Precisely $p_{im}(x)$ is one of the projections of x on the image of $\mathcal{A}_0$ under τ. Furthermore, we introduce a function $\omega : \mathcal{A}_0 \times (\mathbb{R}^m \setminus \tau(\mathcal{A}_0)) \to \mathbb{R}_+$ defined as

$$\omega(\zeta, x) = \frac{1}{|x - \tau(\zeta)|^{r+1}}$$

with $r = s + n$.

Proposition 3: Let τ be a function satisfying (22) for a given function ϱ. Assume that the set $\mathcal{A}_0$ is not locally thin. Then, the function defined as

$$\gamma(x) = \begin{cases} -\dfrac{\int_{\mathcal{A}_0} \mathbf{q}_0(\zeta)\,\omega(\zeta,x)\,\mathrm{d}\mu(\zeta)}{\int_{\mathcal{A}_0} \omega(\zeta,x)\,\mathrm{d}\mu(\zeta)} & \forall x \in \mathbb{R}^m \setminus \tau(\mathcal{A}_0) \\ -\mathbf{q}_0(p_{im}(x)) & \forall x \in \tau(\mathcal{A}_0) \end{cases} \tag{28}$$

[2] $\mathcal{A}_0$ is said to be not locally thin if there exist positive constants c and ε_0 such that $\int_{\mathcal{A}_0\cap\mathcal{B}_\varepsilon(\zeta)} \mathrm{d}Z \geq c\int_{\mathcal{B}_\varepsilon(\zeta)} \mathrm{d}Z$ for all $\zeta \in \mathcal{A}_0$ and $\varepsilon \in (0, \varepsilon_0]$, in which $\mathcal{B}_\varepsilon(\zeta) := \{s \in \mathbb{R}^r : |s - \zeta| < \varepsilon\}$ (see [26]). Note that this assumption requires that $\mathcal{A}_0$ has a nonempty interior.

where μ denotes the Lebesgue measure on $\mathbb{R}^r$, is continuous, bounded by $\sup_{\mathbf{z}\in\mathcal{A}_0} |\mathbf{q}_0(\mathbf{z})|$, and belongs to the set $\Gamma_\alpha(0)$, where

$$\alpha(s) = \varrho(2s) + \varrho([s/\upsilon_1]^{\frac{1}{2(r+1)}}) + \upsilon_2^{r+1}\sqrt{s} \tag{29}$$

for some positive numbers υ_1 and υ_2.

The proof of this Proposition is in Appendix I.

B. Optimization-Based γ

We present now a result, inspired by [22], showing an alternative expression for γ. In formulating this new expression, it is argued that the class-$\mathcal{K}$ function ϱ in (22) satisfies

$$\varrho(|x_3 - x_1|) \leq \varrho(|x_3 - x_2|) + \varrho(|x_1 - x_2|) \tag{30}$$

for all $(x_1, x_2, x_3) \in \mathbb{R}^{3m}$. This can be assumed without loss of generality as shown in the first part of the proof of Proposition 4 reported in Appendix II.

Proposition 4: Let τ be a function satisfying (22) for a given function ϱ satisfying (30). Then, the function $\gamma : \mathbb{R}^m \to \mathbb{R}$ defined by

$$\gamma(x) = \inf_{\mathbf{z}\in\mathcal{A}_0} \{-\mathbf{q}_0(\mathbf{z}) + \min\{\varrho(|x - \tau(\mathbf{z})|), 2Q\}\} \tag{31}$$

where $Q = \sup_{\mathbf{z}\in\mathcal{A}_0} \mathbf{q}_0(\mathbf{z})$, is bounded by $3Q$ and belongs to $\Gamma_\alpha(0)$ with $\alpha = \varrho$.

IV. Approximate Design and Uniform Practical Regulation

The expressions (28) and (31) represent formulas to complete the regulator design, which are applicable as long as one is able to compute explicitly the steady-state locus $\mathcal{A}_0$, the function τ in (16) solution of (17), and, respectively, either the volume integrals characterizing (28) or the infimum characterizing (31). This, indeed, may be a difficult task even in simple cases. For this reason, in this section, we look for an approximate expression of γ resulting into a practical regulator. More specifically, in order to obtain a practical regulator, the idea is to focus on a regulator of the form (12) in which the pair (F, G) is fixed according to Proposition 2 and the stabilizer κ is chosen as in Theorem 1 on the basis of a fixed class-$\mathcal{K}$ function α, and then look for an approximate expression of $\gamma \in \Gamma_\alpha(\delta)$ with δ a design parameter which, according to Theorem 2, can be tuned to obtain practical regulation.

Remark: It is worth stressing that, in accordance with the statement of Theorem 2, practical regulation of the error is not achieved by modifying the stabilizer κ, which in this analysis is supposed to have been fixed according to Theorem 1, nor by acting on the dimension of the regulator, fixed to a value dictated by Proposition 2. In this respect, the solution of the practical output regulation problem is achieved *uniformly* with respect to the gain of the stabilizer near the zero error manifold and *uniformly* with respect to the dimension of the internal model. ◁

The design procedure is articulated in two steps.

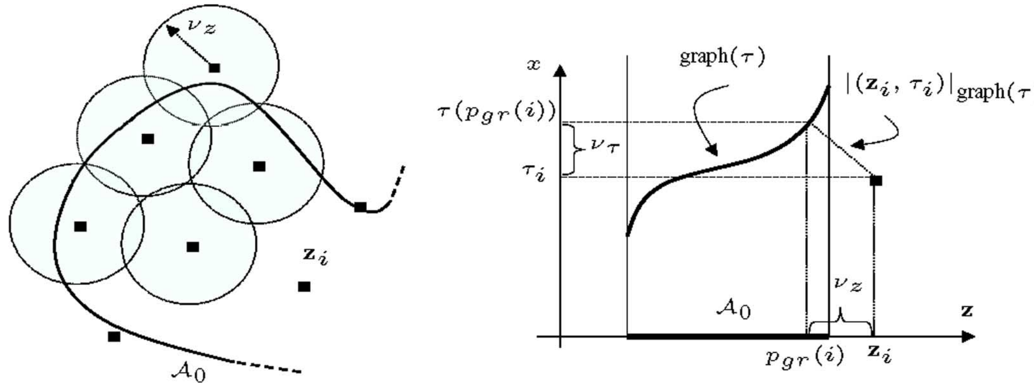

Fig. 1. Graphical sketch of the design formulas (32) and (33).

Step 1: The first step amounts to computing an approximation of the set $\mathcal{A}_0$ and of the function τ solution of (17). As far as the approximation of $\mathcal{A}_0$ is concerned, the idea is to identify a grid of points $\{\mathbf{z}_i\}$, $i \in I := \{1, 2, \ldots, N\}$, covering the set in a "sufficiently dense" way. If the set $\mathcal{A}_0$ is known, this step simply reduces to properly select a "sufficiently large" number N of points inside $\mathcal{A}_0$. On the contrary, if the set is unknown, the approximation can be accomplished by means of numerical methods, better described later, selecting points inside and, possibly, outside $\mathcal{A}_0$ covering the latter in a proper way. As far as the function τ is concerned, the idea is to identify a set $\{\tau_i\}$, $i \in I$, containing a "sufficiently precise" numerical approximation of the solution of the PDE (17) at the points $\{\mathbf{z}_i\}$, $i \in I$, as better described later. From a formal viewpoint, the degree of approximation is quantified by means of two positive design parameters, that is ν_z, which quantifies the step size of the grid $\{\mathbf{z}_i\}$, and ν_τ, which quantifies the precision of $\{\tau_i\}$ with respect to $\{\tau(\mathbf{z}_i)\}$. In particular, let $p_{gr}(i)$ be one of the projections of $(\mathbf{z}_i, \tau_i)$ on the graph (τ) satisfying

$$|\mathbf{z}_i - p_{gr}(i)| + |\tau_i - \tau(p_{gr}(i))| = |(\tau_i, \mathbf{z}_i)|_{\text{graph}(\tau)}.$$

Then, in the following, the sets $\{\mathbf{z}_i\}$ and $\{\tau_i\}$ are assumed to satisfy (see Fig. 1)

$$|\mathbf{z}_i - p_{gr}(i)| \le \nu_z \quad |\tau_i - \tau(p_{gr}(i))| \le \nu_\tau \qquad \forall i \in I \tag{32}$$

and

$$\forall \mathbf{z} \in \mathcal{A}_0, \qquad \exists j_z \in I : |\mathbf{z}_{j_z} - \mathbf{z}| \le \nu_z. \tag{33}$$

Clearly, smaller values of (ν_z, ν_τ) lead to a better approximation of the set $\mathcal{A}_0$ and of the solution of (17). In Section IV-A, possible practical algorithms to compute the sets $\{\mathbf{z}_i\}$ and $\{\tau_i\}$ given (ν_z, ν_τ) are presented. It must be observed, though, that this task is strongly case dependent and, in practice, it is important to exploit the specificities of the dynamics under study in order to reduce the numerical complexity of this step.

Step 2: The second step amounts to approximating the function γ to obtain a practical regulator. The idea is to use a numerical expression which, in the spirit of Theorem 2, belongs to the set $\Gamma_\alpha(\delta)$ [see (23)] with α a fixed class-$\mathcal{K}$ function and δ a design parameter to be tuned according to the desired asymptotic regulation error bound. The numerical expression of γ will be given in terms of the grids $\{\mathbf{z}_i\}$ and $\{\tau_i\}$, $i \in I$, satisfying (32) and (33) with the design parameters ν_z and ν_τ, which are directly related to the parameter δ: a smaller value of the former results in a smaller value of the latter which, in turn, results in a smaller asymptotic regulation error. In this respect, the ultimate design parameters to obtain a practical regulator are (ν_z, ν_τ), namely the degree of approximation of the set $\mathcal{A}_0$ and of the solution of (17). In obtaining the numerical expression of $\gamma \in \Gamma_\alpha(\delta)$, it is crucial that the class-$\mathcal{K}$ function α is independent of the parameter δ, namely of the design parameters (ν_z, ν_τ). As a matter of fact, according to Theorem 2, this allows the design of κ to be independent of the actual value of (ν_z, ν_τ), making possible to tune these parameters only in a final step to match the desired asymptotic error bound.

In Sections IV-B and IV-C, two possible numerical expressions of γ based on the theoretical formulas (28) and (31) are presented.

A. *Design of the Grid $\{\mathbf{z}_i\}$ and the Associated Set $\{\tau_i\}$*

1) $\mathcal{A}_0$ Known: When the set $\mathcal{A}_0$ is known, the grid $\{\mathbf{z}_i\}$ satisfying the first inequality of (32) and (33) for a given ν_z can be easily obtained by choosing N points $\mathbf{z}_i \in \mathcal{A}_0$ in such a way that the union of the N balls of radius ν_z centered at $\mathbf{z}_i$ covers $\mathcal{A}_0$. As far as the grid $\{\tau_i\}$ is concerned, let $(\mathbf{z}_a(t, \mathbf{z}_0), T_a(t, \mathbf{z}_0))$ be a numerical approximation at time t of the solution of the system

$$\dot{\mathbf{z}} = \mathbf{f}_0(\mathbf{z}) \quad \dot{T} = e^{-Ft} G\, \mathbf{q}_0(\mathbf{z}) \tag{34}$$

with initial condition $(\mathbf{z}(0), T(0)) = (\mathbf{z}_0, 0)$, and t_* a positive number such that

$$\sup_{\mathbf{z} \in \mathcal{A}_0} |e^{-Ft_*} G\, \mathbf{q}_0(\mathbf{z})| \le \frac{\nu_\tau}{2} \frac{a}{k} \tag{35}$$

with a and k positive numbers such that $|e^{-Ft}| \le k e^{at}$ for $t \ge 0$. Then, following (16), the grid $\{\tau_i\}$ satisfying the second inequality of (32) for a given ν_τ can be computed by choosing $\tau_i = T_a(-t_*, \mathbf{z}_i)$, $i \in I$, provided that the numerical error introduced by the approximate solution is bounded by $\nu_\tau/2$. In practice, the estimation of t_* can be carried out by simulative experiments by possibly replacing (35) by

$$\max_{i\in I}|e^{-Ft_*}\,G\,\mathbf{q}_0(\mathbf{z}_i)| \le \frac{\nu_\tau}{2}\frac{a}{k}. \tag{36}$$

2) $\mathcal{A}_0$ Unknown: When the set $\mathcal{A}_0$ is unknown, the computation of the grid $\{\mathbf{z}_i\}$ and $\{\tau_i\}$ can be done by assuming the knowledge of a compact set $\mathcal{A}$ containing $\mathcal{A}_0$ and contained in its domain of attraction, and by taking advantage of the fact that $\mathrm{graph}(\tau)$ uniformly attracts the trajectories of the system (15). In particular, let $(\mathbf{z}_a(t,\mathbf{z}), x_a(t,\mathbf{z}))$ be a numerical approximation of the solution of (15) with initial condition[3] $(\mathbf{z}(0), x(0)) = (\mathbf{z}, 0)$. Pick N points $\mathbf{z}_{0i} \in \mathcal{A}$, $i \in I$, and a positive number ν_{0z} such that the union of the N balls centered at $\mathbf{z}_{0i}$ with radius ν_{0z} covers $\mathcal{A}$. By uniform attractiveness of the set graph (τ) and by properly setting the precision of the numerical solution $(\mathbf{z}_a(t,\mathbf{z}_{0i}), x_a(t,\mathbf{z}_{0i}))$, it turns out that for all $\nu > 0$ there exists a $t_* > 0$ (independent of ν_{0z}) such that

$$|(\mathbf{z}_a(t,\mathbf{z}_{0i}), x_a(t,\mathbf{z}_{0i}))|_{\mathrm{graph}(\tau)} \le \nu \qquad \forall t \ge t_*.$$

From this, given ν_z and ν_τ, it follows that the constraint (32) is satisfied by taking

$$\mathbf{z}_i = \mathbf{z}_a(t_*, \mathbf{z}_{0i}) \quad \tau_i = x_a(t_*, \mathbf{z}_{0i}) \tag{37}$$

provided that ν is sufficiently small, namely t_* is sufficiently large, according to the value of (ν_z, ν_τ).

As far as (33) is concerned, let $\mathbf{z}$ be any point in $\mathcal{A}_0$ and note that, because the "preliminary" grid $\{\mathbf{z}_{0i}\}$ covers $\mathcal{A}_0$, and $\mathbf{z}(-t_*, \mathbf{z}) \in \mathcal{A}_0$, there exists $j_z \in I$ satisfying

$$|\mathbf{z}(-t_*, \mathbf{z}) - \mathbf{z}_{0j_z}| \le \nu_{0z}.$$

Hence, invoking standard arguments, we get

$$|\mathbf{z} - \mathbf{z}_a(t_*, \mathbf{z}_{0j_z})| \le 2\,e^{L_{\mathbf{f}_0} t_*}\,\nu_{0z} + \mathfrak{e}$$

where $L_{\mathbf{f}_0}$ denotes the Lipschitz constant of $\mathbf{f}_0$ on the closure of the forward reachable set of $\mathcal{A}$ and $\mathfrak{e}$ is a (small) number taking into account the integration error between the actual and the approximate numerical solution; so (33) holds by selecting the step size ν_{0z} of the preliminary grid satisfying

$$\nu_{0z} \le \frac{e^{-L_{\mathbf{f}_0} t_*}}{4}\nu_z$$

and setting the numerical precision of the integration algorithm so that $\mathfrak{e} \le \nu_z/2$. Note that, in order to fulfill (33), the integration time t_* does not play any role.

In summary, given (ν_z, ν_τ) and with the knowledge of a compact set $\mathcal{A} \supseteq \mathcal{A}_0$, the grids $\{\mathbf{z}_i\}$ and $\{\tau_i\}$ satisfying (32) and (33) can be obtained by numerically integrating (15), for a "sufficiently large" interval of time, starting from $(\{\mathbf{z}_{0i}\}, 0)$, with $\{\mathbf{z}_{0i}\}$ a "sufficiently dense" grid of points covering $\mathcal{A}$. In practice, intensive numerical simulation may have to be used in order to properly tune the value of t_* and ν_{0z}.

[3]Any initial condition for x is allowed.

B. Approximation of γ by a Finite Sum

In this section, a first approximate numerical expression for the function γ is presented. The expression is given in terms of the grids $\{\mathbf{z}_i\}$ and $\{\tau_i\}$ satisfying (32) and (33) for some (ν_z, ν_τ), and of an estimation $\hat{\varrho}$ of the function ϱ satisfying (22). In particular, the function $\hat{\varrho}$ is required to be any class-$\mathcal{K}$ function fulfilling

$$\hat{\varrho}(s) \ge \varrho(s) \qquad \forall s \ge 0 \tag{38}$$

where ϱ [possibly modified to satisfy (30)] characterizes the partial injectivity property (22) of the (exact) function τ in (16). In practice, the function $\hat{\varrho}$ can be selected as a linear function with slope[4]

$$\max_i \min_{j\ne i} \frac{|\mathbf{q}_0(\mathbf{z}_i) - \mathbf{q}_0(\mathbf{z}_j)|}{|\tau_i - \tau_j|}. \tag{39}$$

The degree-of-freedom for tuning the numerical expression of γ so that it belongs to the set $\Gamma_\alpha(\delta)$ with δ an arbitrary small number are the parameters (ν_z, ν_τ) entering in (32) and (33) and an additional parameter ϵ explicitly appearing in the expression of γ. The expression draws inspiration from (28) and is given as follows (see Appendix III for the proof).

Proposition 5: Let the function $\hat{\varrho}$ satisfy (38) and $\{\mathbf{z}_i, i \in I\}$ and $\{\tau_i, i \in I\}$ satisfy (32) and (33). For any $\delta > 0$ and $k \ge 2$, there exists $\epsilon^\star > 0$ such that, for all positive $\epsilon \le \epsilon^\star$ there exist $\nu_z^\star > 0$ and $\nu_\tau^\star > 0$ such that, for all positive $\nu_z \le \nu_z^\star$ and $\nu_\tau \le \nu_\tau^\star$, then the function

$$\gamma_\sigma(x) = -\frac{\displaystyle\sum_{i\in I} \frac{\mathbf{q}_0(\mathbf{z}_i)}{[\epsilon + \hat{\varrho}(\epsilon + 2|x - \tau_i|)]^{\aleph}}}{\displaystyle\sum_{i\in I} \frac{1}{[\epsilon + \hat{\varrho}(\epsilon + 2|x - \tau_i|)]^{\aleph}}} \tag{40}$$

where $\aleph$ is a real number satisfying $\aleph \ge 1 + \log N/\log k$, belongs to $\Gamma_\alpha((k+1)\delta)$ with

$$\alpha(s) = s + \frac{1}{s}\int_s^{2s} \rho(\sigma)\,d\sigma \tag{41}$$

and $\rho(\sigma) = (k+1)\sup_{\varepsilon \le 2\epsilon^\star} \hat{\varrho}(\varepsilon + 2\sigma) - \hat{\varrho}(\varepsilon)$. ◁

Remark: Although the approximate expression (40) is clearly inspired by (28), it is worth noting that Proposition 5 does not claim any kind of closeness between the two realizations of γ given, respectively, by (28) and (40). ◁

C. Approximation of γ by a "min" Formula

We proceed to introduce another approximate expression inspired by (31).

Proposition 6: Let the function $\hat{\varrho}$ satisfy (38) and $\{\mathbf{z}_i, i \in I\}$ and $\{\tau_i, i \in I\}$ satisfy (32) and (33). For any $\delta > 0$, there exist $\nu_z^\star > 0$ and $\nu_\tau^\star > 0$ such that, for all positive $\nu_z \le \nu_z^\star$ and $\nu_\tau \le \nu_\tau^\star$, the function

$$\gamma_m(x) = \min_{i\in I}\{-\mathbf{q}_0(\mathbf{z}_i) + \min\{\hat{\varrho}(|\tau_i - x|), 2\mathfrak{Q}\}\} \tag{42}$$

[4]Note, however, that (39) does not guarantee that (38) holds.

where $\mathfrak{Q}$ is a real number satisfying

$$\mathfrak{Q} \geq \max\{ \max_{i \in I} |\mathbf{q}_0(\mathbf{z}_i)|, Q\}$$

belongs to $\Gamma_\alpha(\delta)$ with

$$\alpha(s) = s + \frac{1}{s}\int_s^{2s} \rho(\sigma)\, d\sigma \tag{43}$$

and $\rho(\sigma) = 2\sup_{\varepsilon \leq 1} \hat{\varrho}(\varepsilon + 2\sigma) - \hat{\varrho}(\varepsilon)$. ◁

The proof of this proposition is given in Appendix IV.

D. A Summary of Possible Representative Scenarios

We conclude this section by enumerating possible representative scenarios that may be encountered in the application and implementation of the theory presented above.

Scenario 1: $\mathcal{A}_0$ is known and (17) *can be explicitly computed to obtain* $\tau(\mathbf{z})$. In this case, if one is able also to compute γ using (28) or (31), then an exact regulator can be obtained according to Theorem 1. If not, simply pick, for the grid $\{\mathbf{z}_i, i \in I\}$, N points $\mathbf{z}_i$ in $\mathcal{A}_0$ in such a way that the union of the N balls centered at $\mathbf{z}_i$ of radius ν_z covers $\mathcal{A}_0$ and take $\{\tau_i, i \in I\} = \{\tau(\mathbf{z}_i), i \in I\}$. In this case, as $p_{gr}(i) = \mathbf{z}_i$ and $\tau_i = \tau(p_{gr}(i))$, (32) is trivially satisfied. Moreover, (33) is fulfilled by construction. From this, according to Theorem 2 and Propositions 5 and 6, practical regulation can be obtained by implementing (40) or (42) with ν_z the design parameter to be ultimately tuned in order to enforce the bound on the asymptotic regulation error.

Scenario 2: $\mathcal{A}_0$ is known but the expression of τ in (17) *cannot be explicitly computed*. In this case, practical regulation can be obtained by the procedure described in Section IV-A1, which develops in the following steps.

- Select a set of points of $\mathcal{A}_0$ to obtain the grid $\{\mathbf{z}_i, i \in I\}$ so that the balls of radius ν_z centered at z_i cover $\mathcal{A}_0$.
- Compute the set $\{\tau_i, i \in I\}$ as the approximate solution of the second equation of (34), with initial condition $(\mathbf{z}_i, 0)$, at time t_* with t_* satisfying (35) [estimated through (36)].
- Implement (40) or (42).

This procedure, according to Theorem 2 and Propositions 5 and 6, yields uniform practical output regulation with the asymptotic regulation error tunable by means of the design parameters ν_z and t_* (see the example in Section V-A).

Scenario 3: Neither $\mathcal{A}_0$ nor τ can be explicitly computed. In this case, the design of the uniform practical regulator in the previous framework can be obtained, as detailed in Section IV-A2, through the following steps.

- Estimate $\mathcal{A}_0$ by means of a compact set $\mathcal{A} \supseteq \mathcal{A}_0$ contained in the region of attraction of $\mathcal{A}_0$.
- Select a grid $\{\mathbf{z}_{0i}, i \in I\}$ so that the N balls of radius ν_{0z} centered at $\mathbf{z}_{0i}$ cover $\mathcal{A}$.
- Compute the grids $\{\mathbf{z}_i\}$ and $\{\tau_i\}$ as in (37) by numerical integration of (15) from initial conditions $(\mathbf{z}_{0i}, 0)$, $i \in I$, in the time interval $[0, t_*]$ with t_* sufficiently large.
- Implement (40) or (42).

The analysis presented in Section IV-A2 has shown that the previous algorithm yields grids $\{\mathbf{z}_i\}$ and $\{\tau_i\}$ satisfying (32) and (33) with given (ν_z, ν_τ) provided that $t_\star$ is sufficiently large and, accordingly, ν_{0z} is sufficiently small. Hence, according to Theorem 2 and Propositions 5 and 6, uniform practical output regulation follows, with the asymptotic regulation error tunable by means of the ultimate design parameters ν_{0z} and t_* (see the example in Section V-B).

V. Examples

A. Robust Compensation of the Ripple Generated by Uncontrolled Diode Rectifiers

As a first control example, we consider the problem of robustly compensating for the effect of the voltage ripple generated by an uncontrolled diode rectifier typically used as "cheap" DC-power generator in several power electronic devices (see [23]). For illustrative purposes, we focus on the torque control of the DC-motor modeled by

$$J\dot{\omega} = -d\omega + KI \quad L\dot{I} = -K\omega - RI + b(w(t))u \tag{44}$$

with angular velocity ω and current I as state variables and controlled output given by the torque $T = KI$, in which J is the inertia of the motor, K is the electromotive force constant, d is the damping ratio, L and R are, respectively, the electric inductance and resistance. As usual in power-electronic devices, the control input $u \in \mathbb{R}$ modulates the "high-frequency" term $b(w(t))$ representing an "almost" DC voltage generated by an uncontrolled diode rectifier. A typical time behavior of $b(w(t))$ is shown in Fig. 2(a) (corresponding to the case of three-phase full-bridge rectifier with frequency of the main at 50 Hz), which can be thought as generated by the exosystem

$$\dot{w}_1 = w_2 \quad \dot{w}_2 = -fw_1 \quad b(w(t)) = |w_1| + c \tag{45}$$

in which $c = 540 \cdot \cos(\pi/6)$, $f = (300 \cdot \pi)^2$ (300 Hz), and $|(w_1(0), w_2(0))| = 540 \cdot (1 - \cos(\pi/6))$. Possible unpredictable fluctuations of the electric main result in fluctuations of the terms c and $(w_1(0), w_2(0))$, which, as a consequence, must be treated as uncertain parameters ranging in known compact sets. It turns out that, if not appropriately compensated, the term $b(w(t))$ is responsible for a steady-state ripple on the generated torque which, in several relevant applications, leads to undesirable effects such as generation of acoustic noise, degradation of tracking performances, mechanical wear, etc. In order to avoid these problems, the theory proposed in this paper can be successfully adopted as explained in the following. Let T_{ref} be the constant torque reference set point and consider the change of variables $\omega \mapsto z := \omega - T_{\text{ref}}/d$ and $I \mapsto e := KI - T_{\text{ref}}$ so that system (44) transforms as

$$\begin{aligned} \dot{z} &= -dz + e \\ L\dot{e} &= -T_{\text{ref}} w_3 - K^2 z - Re + Kb(w(t)))u \end{aligned}$$

with $w_3 := -(K^2/d + R)$. This system fits in the framework of Section II-A (unitary relative degree and regulated output e equals to the measured output y). In particular, in order to deal with possible uncertainties on the physical parameters of the DC motor, we consider an "extended" exosystem with state $w = (w_1, w_2, w_3)^{\mathrm{T}}$ governed by the first two equations of (45) and

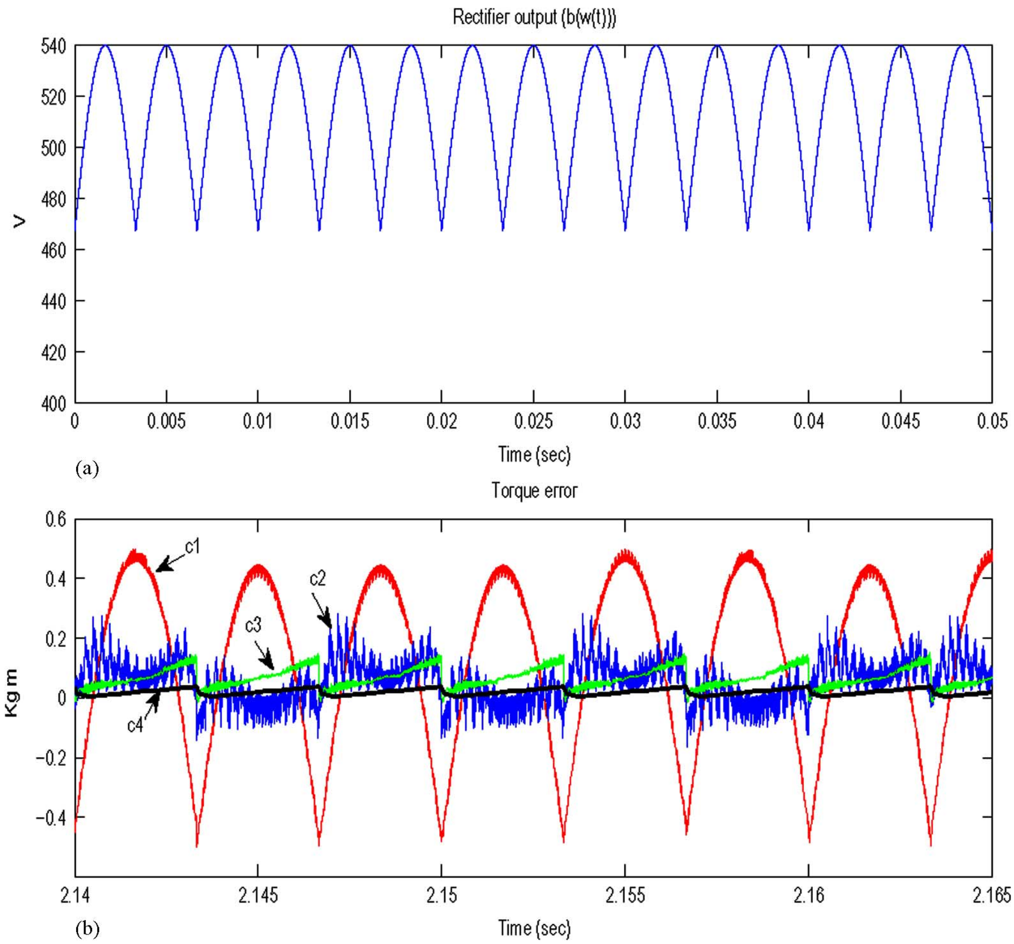

Fig. 2. Example 1. (a) Time behavior of $b(w(t))$. (b) Torque steady-state error; curve c1: no internal model (i.e., $u = -\kappa e$) with zero mean-value; curve c2: $N = 5 \cdot 10^3$; curve c3: $N = 50 \cdot 10^3$; curve c4: $N = 500 \cdot 10^3$.

$\dot{w}_3 = 0$ whose initial conditions range in a known compact set $W \subset \mathbb{R}^3$. In this case, the term $\mathbf{q}_0(\mathbf{z})$ assumes the form

$$\mathbf{q}_0(\mathbf{z}) = \frac{-T_{\rm ref} w_3 - K^2 z}{K(|w_1| + c)}$$

with c a known constant (possible uncertainties on c can be easily "translated" into uncertainties on w_1 and w_3). As $d > 0$, it turns out that the assumption in Section II-A is trivially satisfied with $\mathcal{A}_0 := \{(w, z) \in W \times \mathbb{R}, z = 0\}$. Note that this case fits in the second scenario of Section IV-D ($\mathcal{A}_0$ explicitly computable; $\tau(\cdot)$ to be approximated). According to Proposition 2, the dimension of the regulator (4) can be taken as $m = 8$. In the following simulation, we have taken $F = 10 \cdot \mathrm{blkdiag}(F_1, \ldots, F_4)$, $G = \mathrm{col}(G_1, \ldots, G_4)$ with (F_i, G_i) as

$$F_i = \begin{pmatrix} \lambda_{Ri} & -\lambda_{Ii} \\ \lambda_{Ii} & \lambda_{Ri} \end{pmatrix} \quad G_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{46}$$

with $\lambda_{Ri} = -1 + (1 - i)/2$, $\lambda_{Ii} = 1$, $i = 1, \ldots, 4$, and by implementing the approximate expression (42) with $\hat{\varrho}(s) = 30 \cdot s$. The simulation results have been obtained by assuming the following nominal values for the physical parameters: $J = 2.6 \cdot 10^{-3}$ kg m^2, $d = 14 \cdot 10^{-3}$ Nms, $K = 1$ Nm/A, $R = 2$ Ohm, and $L = 10^{-3}$ H. The compact set W have been dimensioned by assuming uncertainties up to 10% "centered" on the nominal values. According to the theory in Section IV-A, three possible regulators at increasing accurateness have been implemented by considering three grids $\{\mathbf{z}_i\}$ obtained by taking, respectively, $N = 5 \cdot 10^3$, $N = 50 \cdot 10^3$, and $N = 500 \cdot 10^3$ points uniformly distributed in W and by integrating (34) (using a standard Runge–Kutta method) with the same $t_* = 1$s. The stabilizer $\kappa(y)$ of (4) has been taken linear with $\kappa = -1$ and $T_{\rm ref} = 50$ Nm. The simulation results are shown in Figs. 2 and 3. In particular, Fig. 2(b) plots the steady-state behaviors of $e(t)$ corresponding to the three implementations and to the case of absence of internal model (i.e., with $u = -\kappa y$). In the latter case, the mean-value of the steady-state error, equal to 6.4 Kg m, has been eliminated for graphical reasons (indeed it could be eliminated by introducing a genuine integral action in the controller). Fig. 3(a) presents the desired steady-state control input (given by $-\mathbf{q}_0(\mathbf{z}(t))$) while Fig. 3(b) reports the terms $\mathbf{q}_0(\mathbf{z}(t)) + \gamma(\eta(t))$ (ideally equal to zero in steady state) in the case of the three implementations.

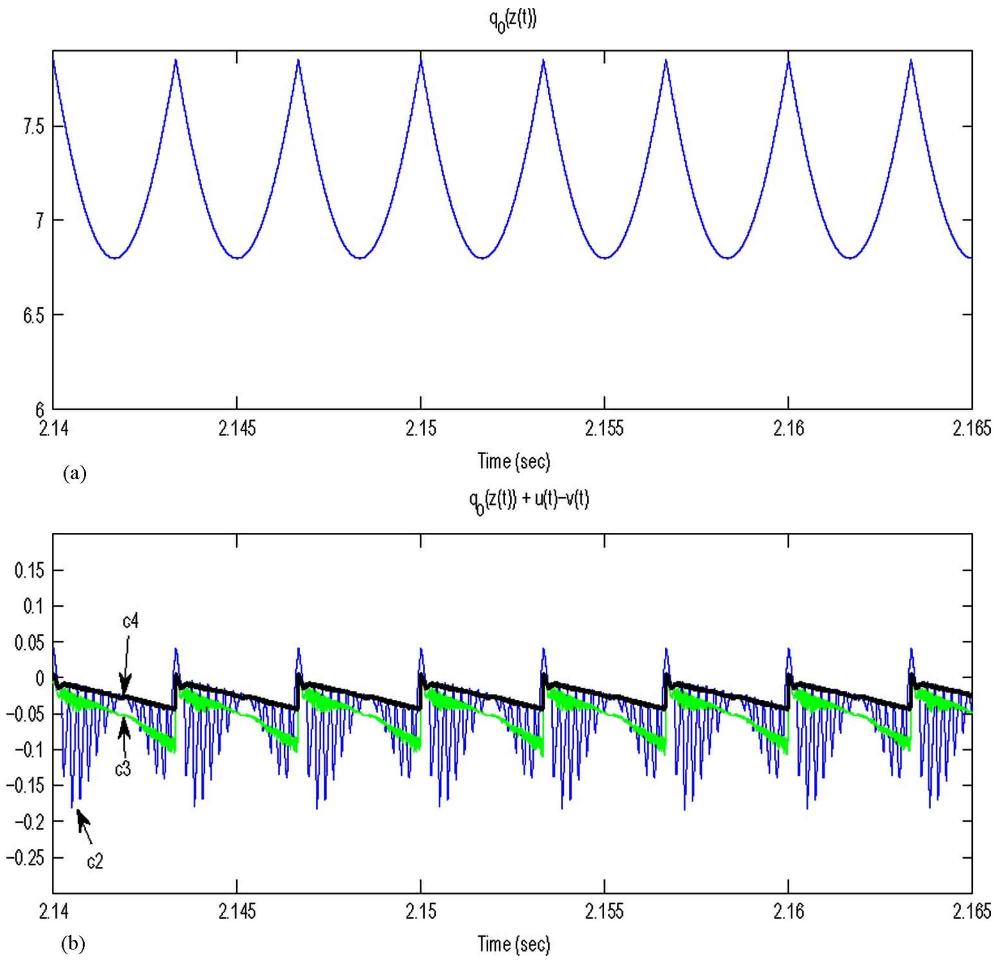

Fig. 3. Example 1. (a) Desired steady-state control input. (b) Time behavior of $\mathbf{q}_0(\mathbf{z}(t)) + \gamma(\eta(t))$: curve c2: $N = 5 \cdot 10^3$; curve c3: $N = 50 \cdot 10^3$; curve c4: $N = 500 \cdot 10^3$.

B. Lorenz-Generated Turbulence Compensation in Lateral VTOL Dynamics

As a final control example, we consider the simplified planar vertical takeoff and landing (VTOL) aircraft dynamics

$$\begin{aligned} M\ddot{x} &= -\sin\theta u_1 + q(w) \\ J\ddot{\theta} &= 2\ell u_2 \\ M\ddot{y} &= \cos\theta_1 u_1 - gM \end{aligned} \tag{47}$$

with (x, y) the lateral and vertical position of the airplane, θ the angle with respect to the horizon, (u_1, u_2) two control inputs given, respectively, by the main thrust and the force acting on the tip of the wings, ℓ the length of the wings, M and J, respectively, the mass and the inertia of the VTOL (see [18]). The inertia is assumed to be a constant *uncertain* parameter ranging in the set $J \in [\underline{J}, \bar{J}]$ with $\underline{J}$ and $\bar{J}$ known positive constants. The term $q(w)$ in (47) is a lateral force acting on the VTOL representing a wind turbulence perturbing the aircraft. It is well known (see [11]) that lateral turbulence can be accurately modeled as bandlimited white noise filtered by a first-order forming filter (the so-called Dryden wind turbulence model). In order to fit in our framework, we approximate bandlimited white noise as a state variable of the chaotic Lorenz oscillator (see [30])

$$\begin{aligned} \dot{w}_1 &= \sigma(w_2 - w_1) \\ \dot{w}_2 &= \rho w_1 - w_2 - w_1 w_3 \\ \dot{w}_3 &= -\beta w_3 + w_1 w_2 \end{aligned} \tag{48}$$

where (σ, ρ, β) are positive constants, and we model the turbulence as $q(w) = l w_4$ with w_4 governed by

$$\dot{w}_4 = -\alpha w_4 + w_1 \tag{49}$$

with α and l known positive time constants. Our goal is to stabilize the lateral and vertical position (x, y) of the VTOL to zero (or to a constant) by thus compensating for $q(w)$.

We consider the preliminary feedback (well defined if $|\theta| < \pi/2$)

$$u_1 = \frac{M}{\cos\theta}[g + u_1'] \quad u_1' = -k_1 y - k_2 \dot{y}$$

with $k_1 > 0$, $k_2 > 0$, which trivially stabilizes the vertical dynamics.[5] Moreover, we consider the following change of

[5]Pure output (y) feedback can be simply obtained by means of a dynamic controller if $\dot{y}$ is not accessible.

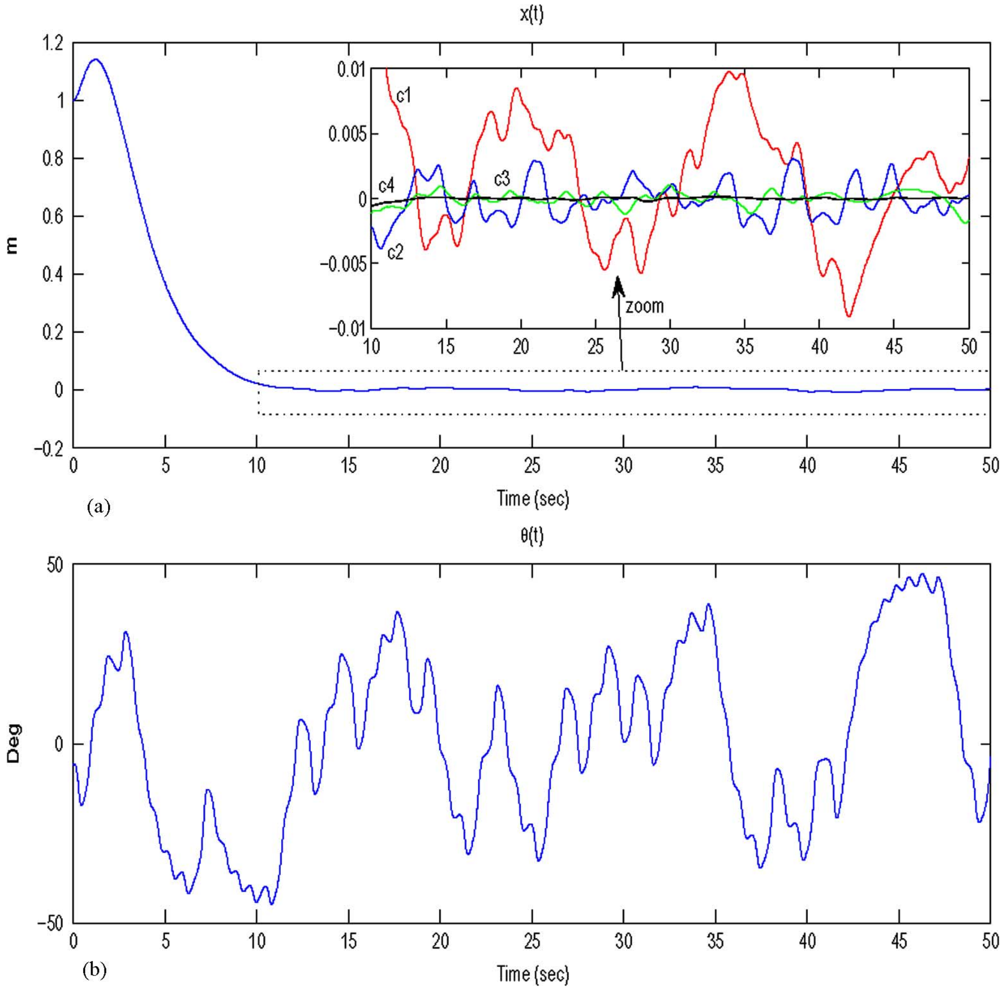

Fig. 4. Example 2. (a) Time behavior of lateral position $x(t)$ with a zoom of the steady-state error (curve c1: no internal model, i.e., $u_2 = \kappa\tilde{x}_4$; curve c2: $N = 10^4$; curve c3: $N = 10 \cdot 10^4$; curve c4: $N = 100 \cdot 10^4$. (b) Time behavior of $\theta(t)$ in the case $N = 100 \cdot 10^4$.

variable for $(\theta, \dot{\theta})$ (well defined if $|\theta| < \pi/2$)

$$x_3 := -g\tan\theta + \frac{q(w)}{M} \quad x_4 := -\frac{g}{\cos^2\theta}\dot{\theta} + \frac{\dot{q}(w)}{M}$$

which transforms the lateral–angular dynamics into (for convenience, we leave θ and $\dot{\theta}$ in the original coordinates in the second and the last equations)

$$\begin{aligned} &\dot{x}_1 = x_2 \quad \dot{x}_2 = x_3 - \tan\theta u_1' \quad \dot{x}_3 = x_4 \\ &J\dot{x}_4 = -\frac{2\ell g}{\cos^2\theta}u_2 - 2gJ\frac{\cos\theta\sin\theta}{\cos^4\theta}\dot{\theta}^2 + \frac{J}{M}\ddot{q}(w) \end{aligned} \tag{50}$$

where $x_1 = x$, $x_2 = \dot{x}$, and u_1' is a vanishing term. By following the theory in the second part of Section II-A, let $\tilde{x}_4 := x_4 + a_0x_1 + a_1x_2 + a_3x_3$ with a_i appropriate real coefficients so that system (50), (48), and (49), regarded as a system with input u_2 and output $\tilde{x}_4$, fits in the framework addressed in this paper (unitary relative degree and zero dynamics described by (48) and (49), $\dot{J} = 0$, and $\dot{x}_1 = x_2$, $\dot{x}_2 = x_3$, $\dot{x}_3 = -a_0x_1 - a_1x_2 - a_3x_3$). In particular, in this specific case, $w := \mathrm{col}(w_1, w_2, w_3, w_4, J)$, $z := \mathrm{col}(x_1, x_2, x_3)$, and

$$\mathbf{q}_0(\mathbf{z}) = \frac{\cos^2\theta_1^w}{2\ell g}\left[\frac{J}{M}\ddot{q}(w) - 2gJ\frac{\cos\theta_1^w\sin\theta_1^w}{\cos^4\theta_1^w}{\theta_2^w}^2\right]$$

where $\theta_1^w = \mathrm{atan}(q(w)/Mg)$ and $\theta_2^w = \cos^2\theta_1^w\dot{q}(w)/Mg$. Furthermore, denoting by $\mathcal{L}$ the (invariant) Lorenz compact attractor, and by $\pi : \mathbb{R}^3 \to \mathbb{R}$ the map defined as

$$\pi(w_1, w_2, w_3) := \int_{-\infty}^{0} e^{-\alpha s}w_1(s, (w_1, w_2, w_3))\,ds$$

in which $w_1(s, (w_1, w_2, w_3))$ is the solution of the first of (48) at time s with initial condition (w_1, w_2, w_3), it turns out that the (invariant) set W is, in this specific example, defined as $W := \mathrm{graph}(\pi|_{\mathcal{L}}) \times [\underline{J}, \bar{J}]$ and the assumption in Section II-A is satisfied with $\mathcal{A}_0 = W \times \{0\} \subset \mathbb{R}^4 \times \mathbb{R}^3$. Furthermore, note that boundedness of the state of (50) implies that $|\theta_1| < \pi/2$, which validates the previous change of variables. This example fits in the challenging scenario 3 of Section IV-D in which neither the set $\mathcal{A}_0$ nor the function $\tau(\cdot)$ are explicitly known. In order to run the design procedure described in Section IV-D, the first step is to compute the set $\mathcal{A} \supseteq \mathcal{A}_0$. For this purpose, note that the Lorenz attractor $\mathcal{L}$ is known to be contained in the solid ellipsoid

$$\mathcal{E} = \{(w_1, w_2, w_3) \in \mathbb{R}^3 : \rho w_1^2 + \sigma w_2^2 + \sigma(w_3 - 2\rho)^2 \le c\}$$

where c is a positive constant (see [30]). On this set, $|w_1| < \bar{w}_1 := \sqrt{c/\rho}$. Furthermore, from (49), it turns out

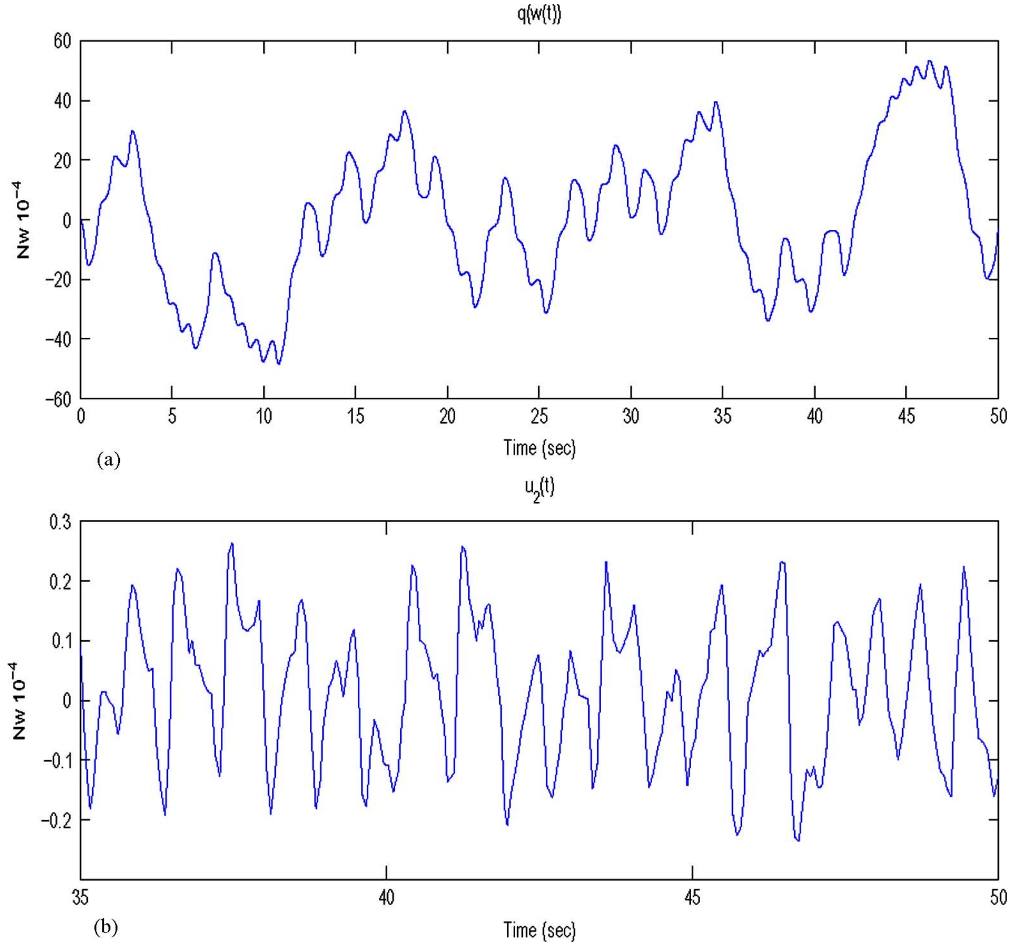

Fig. 5. Example 2. (a) Wind turbulence $q(w(t))$. (b) Control input $u_2(t)$ in the case $N = 100 \cdot 10^4$ in the time interval [35,50] s.

that $w_4 \in \pi|_{\mathcal{L}} \Rightarrow |w_4| \leq \bar{w}_1/\alpha$. Thus, as set $\mathcal{A} \supseteq \mathcal{A}_0$, it can be chosen $\mathcal{A} = \{(w,x) \in \mathbb{R}^5 \times \mathbb{R}^3 : (w_1,w_2,w_3) \in \mathcal{E}, |w_4| \leq \bar{w}_1/\alpha, J \in [\underline{J},\bar{J}], x = 0\}$, where $x := \mathrm{col}(x_1,x_2,x_3)$. The selection of the grids $\{\mathbf{z}_i\}$ and $\{\tau_i\}$ has been done by means of the procedure described in Section IV-A2 by fixing $t_* = 8$ s and by computing three different grids (at increasing densities and precision) obtained by taking, respectively, $N = 10^4$, $N = 10 \cdot 10^4$, and $N = 100 \cdot 10^4$ points uniformly distributed in $\mathcal{A}$ as initial conditions of (15). The set $\mathcal{A}$ has been computed by taking $(\sigma,\rho,\beta) = (10,28,8/3)$, $c = 3 \cdot 10^4$, $(\underline{J},\bar{J}) = (1,1.5)$ kg m^2, and $\alpha = 0.5$. A standard Runge–Kutta method has been used for numerical integration of (15).

According to the theory (see Proposition 2), the regulator (12) has been fixed of dimension $m = 10$ (since $\dim \mathbf{f}_0|_{\mathcal{A}_0} = 4$) by choosing $F = \mathrm{blkdiag}(F_1,\ldots,F_5)$, $G = \mathrm{col}(G_1,\ldots,G_5)$ with (F_i,G_i) as in (46) with $\lambda_{Ri} = -2 + (1-i)$, $\lambda_{Ii} = 1$, $i = 1,\ldots,5$, and by implementing the approximate expression (42) with $\hat{\varrho}(s) = 50 \cdot s$. Then, the controller design has been completed by choosing $(a_0,a_1,a_2) = (3,2.75,0.75)$ and $v = -\kappa\tilde{x}_4$ with $\kappa = 10^4$. In the case $q(w)$ and $\dot{q}(w)$ (and thus $\tilde{x}_4$) are not measurable, a dirty derivatives observer-based regulator can be used to obtain a pure output (x) feedback regulator (see Section II-A).

The simulation results have been obtained by choosing the values of the physical parameters of the VTOL as $M = 5 \cdot 10^4$ kg, $J = 1.25 \cdot 10^4$ kg m^2, $\ell = 4$ m, and the initial conditions at $x_1(0) = 1$, $x_2(0) = 0$, $x_3(0) = 1$, and $x_4(0) = 2$. The initial conditions of the turbulence generator (48) and (49) have been fixed at $w_1(0) = -2.5$, $w_2(0) = -4.5$, $w_3(0) = 10$, and $w_4(0) = 0$ with the parameter l fixed at $l = 8 \cdot 10^4$ so that the maximum turbulence force is of about the same entity as the VTOL weight force (equal to $49 \cdot 10^4$ kg m/s). The profile of the resulting turbulence force is shown in the upper part of Fig. 5. The time behavior of the lateral position x of the VTOL is shown in Fig. 4(a), in which the steady-state error can be observed in the three cases corresponding to the three choices of the grids $\{\mathbf{z}_i\}$ and $\{\tau_i\}$ obtained as discussed above and in the case of pure proportional feedback (i.e., $u_2 = -\kappa\tilde{x}_4$). Figs. 4(b) and 5(b) show, respectively, the angle $\theta(t)$ and the control input u_2 in the case of the third (highly accurate) controller.

VI. Conclusion

A complete framework for solving the problem of asymptotic and practical output regulation has been presented. The result, which relies on the nonequilibrium framework proposed in [3] and applies to a fairly general class of nonlinear minimum phase systems, is framed into the theory of semiglobal output regulation without immersion presented in [19]. Special emphasis has been given to the design of an implementable regulator, namely on the design of an internal model-based regulator yielding an arbitrary small asymptotic regulation error. In this respect, the distinguishing feature of the approach is that practical regulation is achieved uniformly with respect to the gain of the stabilizer near the zero-error manifold and the dimension of the internal model. Simulative control examples have been presented to show the effectiveness of the method.

APPENDIX I
PROOF OF PROPOSITION 3

Let us first restrict our attention to the set $\mathcal{A}_0$. For any x in $\tau(\mathcal{A}_0)$, we have, by definition, $x = \tau(p_{im}(x))$. Note that $p_{im}(x)$ may not be the only point of $\mathcal{A}_0$ such that its image under τ is x. Nevertheless, because of (22), if $\mathbf{z}$ is another such point, we have $\gamma(x) = -\mathbf{q}_0(p_{im}(x)) = -\mathbf{q}_0(\mathbf{z})$. This proves that γ given in (28) is well defined on $\tau(\mathcal{A}_0)$ and satisfies (19). In addition, if we pick two points x_1 and x_2 in $\tau(\mathcal{A}_0)$, there exist $\mathbf{z}_1$ and $\mathbf{z}_2$ in $\mathcal{A}_0$ such that we have $x_1 = \tau(\mathbf{z}_1)$, $\gamma(x_1) = -\mathbf{q}_0(\mathbf{z}_1)$, $x_2 = \tau(\mathbf{z}_2)$, and $\gamma(x_2) = -\mathbf{q}_0(\mathbf{z}_2)$; so with (22), we get

$$\begin{aligned}|\gamma(x_1) - \gamma(x_2)| &= |\mathbf{q}_0(\mathbf{z}_1) - \mathbf{q}_0(\mathbf{z}_2)| \\ &\le \varrho(|\tau(\mathbf{z}_1) - \tau(\mathbf{z}_2)|) = \varrho(|x_1 - x_2|). \quad (51)\end{aligned}$$

This proves that γ is continuous on $\tau(\mathcal{A}_0)$.

On the other hand, from standard results on integration theory, γ is also continuous on $\mathbb{R}^m \setminus \tau(\mathcal{A}_0)$ where $\int_{\mathcal{A}_0} \omega(\zeta, x)\mathrm{d}\mu(\zeta)$ does not vanish. It remains to be proven that it is continuous on the boundary of $\tau(\mathcal{A}_0)$. Given a positive real number σ, for each $x \in \mathbb{R}^m \setminus \tau(\mathcal{A}_0)$, let $\mathcal{A}_\sigma(p_{im}(x))$ be the set defined as

$$\mathcal{A}_\sigma(p_{im}(x)) = \{\mathbf{z} \in \mathcal{A}_0 : |\mathbf{q}_0(\mathbf{z}) - \mathbf{q}_0(p_{im}(x))| < \sigma\} \subset \mathcal{A}_0.$$

For x not in $\tau(\mathcal{A}_0)$, the term $|\gamma(x) + \mathbf{q}_0(p_{im}(x))|$ can be bounded as follows:

$$\begin{aligned}&|\gamma(x) + \mathbf{q}_0(p_{im}(x))| \\ &\quad\le \frac{\int_{\mathcal{A}_\sigma(p_{im}(x))} |\mathbf{q}_0(p_{im}(x)) - \mathbf{q}_0(\zeta)|\omega(\zeta, x)\mathrm{d}\mu(\zeta)}{\int_{\mathcal{A}_0} \omega(\zeta, x)\,\mathrm{d}\mu(\zeta)} \\ &\qquad + \frac{\int_{\mathcal{A}_0\setminus\mathcal{A}_\sigma(p_{im}(x))} |\mathbf{q}_0(p_{im}(x)) - \mathbf{q}_0(\zeta)|\,\omega(\zeta, x)\,\mathrm{d}\mu(\zeta)}{\int_{\mathcal{A}_0} \omega(\zeta, x)\,\mathrm{d}\mu(\zeta)}\end{aligned} \quad (52)$$

where we have

$$\begin{aligned}&\frac{\int_{\mathcal{A}_0\setminus\mathcal{A}_\sigma(p_{im}(x))} |\mathbf{q}_0(p_{im}(x)) - \mathbf{q}_0(\zeta)|\,\omega(\zeta, x)\,\mathrm{d}\mu(\zeta)}{\int_{\mathcal{A}_0} \omega(\zeta, x)\,\mathrm{d}\mu(\zeta)} \\ &\qquad\qquad \le \sigma\frac{\int_{\mathcal{A}_\sigma(p_{im}(x))} \omega(\zeta, x)\,\mathrm{d}\mu(\zeta)}{\int_{\mathcal{A}_0} \omega(\zeta, x)\,\mathrm{d}\mu(\zeta)} \le \sigma.\end{aligned}$$

Concerning the left-hand side of (52), we first observe that the definition of $p_{im}(x)$ gives

$$\begin{aligned}|x - \tau(\mathbf{z})| &\ge |\tau(\mathbf{z}) - \tau(p_{im}(x))| - |x - \tau(p_{im}(x))| \\ |x - \tau(\mathbf{z})| &\ge |x - \tau(p_{im}(x))|.\end{aligned}$$

By summation, this gives

$$|x - \tau(\mathbf{z})| \ge \frac{1}{2}|\tau(\mathbf{z}) - \tau(p_{im}(x))|$$

and therefore, with (22)

$$\mathbf{z} \in \mathcal{A}_0 \setminus \mathcal{A}_\sigma(p_{im}(x)) \Rightarrow |x - \tau(\mathbf{z})| \ge \frac{\varrho^{-1}(\sigma)}{2}. \quad (53)$$

Hence, we have obtained

$$\begin{aligned}&\int_{\mathcal{A}_0\setminus\mathcal{A}_\sigma(p_{im}(x))} |\mathbf{q}_0(p_{im}(x)) - \mathbf{q}_0(\zeta)|\,\omega(\zeta, x)\,\mathrm{d}\mu(\zeta) \\ &\qquad \le d\left[\frac{2}{\varrho^{-1}(\sigma)}\right]^{r+1} \int_{\mathcal{A}_0\setminus\mathcal{A}_\sigma(p_{im}(x))} \mathrm{d}\mu(\zeta) \le \frac{dc_0}{[\varrho^{-1}(\sigma)]^{r+1}}\end{aligned}$$

in which $d := \max_{\zeta\in\mathcal{A}_0} |\mathbf{q}_0(\zeta) - \mathbf{q}_0(p_{im}(x))|$ is finite, as $\mathcal{A}_0$ is bounded and $\mathbf{q}_0$ is continuous, and c_0 is a positive constant as a consequence of the fact that $\mathcal{A}_0$ is bounded.

Finally, for any $\varsigma > 0$ and any $\zeta \in \mathcal{A}_0 \cap \mathcal{B}_\varsigma(p_{im}(x))$, we have

$$\begin{aligned}\omega(\zeta, x) &\ge \frac{1}{[|x - \tau(p_{im}(x))| + |\tau(p_{im}(x)) - \tau(\zeta)|]^{r+1}}, \\ &\ge \frac{1}{[2\max\{|x - \tau(p_{im}(x))|, L_\tau\varsigma\}]^{r+1}}\end{aligned}$$

in which L_τ is the Lipschitz constant of τ. From this, using the fact that $\mathcal{A}_0$ is not locally thin, it follows that for any $\varsigma > 0$

$$\begin{aligned}\int_{\mathcal{A}_0} \omega(\zeta, x)\,\mathrm{d}\mu(\zeta) &\ge \int_{\mathcal{A}_0\cap\mathcal{B}_\varsigma(p_{im}(x))} \omega(\zeta, x)\,\mathrm{d}\mu(\zeta), \\ &\ge \frac{c_1\varsigma^r}{[2\max\{|x - \tau(p_{im}(x))|, L_\tau\varsigma\}]^{r+1}}\end{aligned}$$

where c_1 is a positive constant. Thus, from (52), we have established, for any positive real numbers σ and ς

$$\begin{aligned}&|\gamma(x) + \mathbf{q}_0(p_{im}(x))| \\ &\quad \le \sigma + \frac{dc_0}{[\varrho^{-1}(\sigma)]^{r+1}} \frac{[2\max\{|x - \tau(p_{im}(x))|, L_\tau\varsigma\}]^{r+1}}{c_1\varsigma^r}.\end{aligned} \quad (54)$$

In particular, letting $\varsigma = |x - \tau(p_{im}(x))|/L_\tau$ and $\sigma = \varrho(\varsigma^{(1/2(r+1))})$, we have

$$|\gamma(x) + \mathbf{q}_0(p_{im}(x))| \le \psi(|x - \tau(p_{im}(x))|) \quad (55)$$

where ψ is a class-$\mathcal{K}$ function defined as

$$\psi(s) = \varrho\left(\left[\frac{s}{L_\tau}\right]^{\frac{1}{2(r+1)}}\right) + \frac{2^{r+1}dc_0L_\tau^{r+\frac{1}{2}}}{c_1}\sqrt{s}.$$

Hence, for x_1 in $\tau(\mathcal{A}_0)$ and x_2 in $\mathbb{R}^m \setminus \tau(\mathcal{A}_0)$, we obtain

$$\begin{aligned}&|\gamma(x_2) - \gamma(x_1)| \\ &\quad \le |\gamma(x_2) + \mathbf{q}_0(p_{im}(x_2))| + |\mathbf{q}_0(p_{im}(x_2)) - \mathbf{q}_0(\mathbf{z}_1)| \\ &\quad \le \psi(|x_2 - \tau(p_{im}(x_2))|) + \varrho(|\tau(p_{im}(x_2)) - x_1|)\end{aligned}$$

where $\mathbf{z}_1$ is any $\mathbf{z} \in \mathcal{A}_0$ satisfying

$$x_1 = \tau(\mathbf{z}_1).$$

Because $x_1 \in \tau(\mathcal{A}_0)$ and $P_{im}(x_2) \in \arg\min_{z\in\mathcal{A}_0} |x_2 - \tau(z)|$, it follows successively

$$\begin{aligned} |x_2 - \tau(p_{im}(x_2))| &\leq |x_2 - x_1| \\ |\gamma(x_2) - \gamma(x_1)| &\leq \psi(|x_2 - x_1|) + \varrho(2|x_2 - x_1|). \end{aligned}$$

In conjunction with (51), this establishes continuity of γ on the boundary of $\tau(\mathcal{A}_0)$.

Finally, using (55), (22), and the fact that $|x - \tau(p_{im}(x))| \leq |x - \tau(\mathbf{z})|$ for all $x \in \mathbb{R}^m$ and all $\mathbf{z} \in \mathcal{A}_0$, similar computations can be used to show that

$$|\gamma(x) + \mathbf{q}_0(\mathbf{z})| \leq \psi(|x - \tau(p_{im}(x))|) + \varrho(2|x - \tau(p_{im}(x))|)$$

for any $x \in \mathbb{R}^m \setminus \tau(\mathcal{A}_0)$ and $\mathbf{z} \in \mathcal{A}_0$, and that

$$|\gamma(x) + \mathbf{q}_0(\mathbf{z})| \leq \varrho(2|x - \tau(p_{im}(x))|)$$

for any $x \in \tau(\mathcal{A}_0)$ and $\mathbf{z} \in \mathcal{A}_0$, from which the class-$\mathcal{K}$ function α in (29) is obtained. This concludes the proof of Proposition 3.

APPENDIX II
PROOF OF PROPOSITION 4

We begin by showing that, if the function ϱ characterizing (22) does not satisfy (30), it is possible to define a function ϱ_c enjoying this latter property and such that

$$|\mathbf{q_0}(\mathbf{z}_1) - \mathbf{q_0}(\mathbf{z}_2)| \leq \varrho_c(|\tau(\mathbf{z}_1) - \tau(\mathbf{z}_2)|) \qquad \forall\, \mathbf{z}_1, \mathbf{z}_2 \in \mathcal{A}_0. \tag{56}$$

To this end, let

$$s_0 = \varrho(\max\{1, 2 \sup_{\mathbf{z}\in\mathcal{A}_0} |\tau(\mathbf{z})|\}). \tag{57}$$

Note that s_0 is strictly positive and belongs to the image of ϱ. It follows that $\varrho^{-1}(s_0)$ is well defined, and the following definition makes sense:

$$\varrho_c^{-1}(s) = \begin{cases} \frac{1}{s_0}\int_0^s \varrho^{-1}(\tau)d\tau & \forall\, s \leq s_0 \\ \varrho_c^{-1}(s_0) + \frac{\varrho^{-1}(s_0)}{s_0}(s - s_0) & \forall s_0 < s. \end{cases} \tag{58}$$

Because ϱ^{-1} is increasing and continuous on $[0, s_0]$, the function ϱ_c^{-1} is convex, of class $\mathcal{K}$ and C^1 on $\mathbb{R}_+$. Consequently, the function ϱ_c is of class $\mathcal{K}$ with derivative ϱ_c' defined and nonincreasing on $\mathbb{R}_+ \setminus \{0\}$. This implies

$$\frac{\varrho_c(a) - \varrho_c(b)}{a - b} \leq \frac{\varrho_c(a - b) - \varrho_c(0)}{a - b} \tag{59}$$

for all $0 \leq b < a$. Because $\varrho_c(0) = 0$, this yields

$$\varrho_c(|x_3 - x_1|) \leq \varrho_c(|x_3 - x_2|) + \varrho_c(|x_1 - x_2|)$$

for all $(x_1, x_2, x_3) \in \mathbb{R}^{3m}$. In addition, the inequalities

$$\varrho_c^{-1}(s) \leq \frac{s}{s_0}\varrho^{-1}(s) \leq \varrho^{-1}(s) \qquad \forall\, s\, \leq\, s_0 \tag{60}$$

imply that

$$\varrho_c(s) \geq \varrho(s) \qquad \forall\, s \leq 2 \sup_{\mathbf{z}\in\mathcal{A}_0} |\tau(\mathbf{z})| \leq\, \varrho^{-1}(s_0) \tag{61}$$

which yields (56). Hence, in the following, we assume the function ϱ in (31) satisfies (30) and (22) and we prove the proposition. Continuity of $\mathbf{q}_0$ and ϱ and compactness of $\mathcal{A}_0$ imply the existence of $\mathbf{z}_x \in \mathcal{A}_0$ satisfying

$$\gamma(x) = -\mathbf{q}_0(\mathbf{z}_x) + \min\{\varrho(|\tau(\mathbf{z}_x) - x|), 2Q\}. \tag{62}$$

We remark also that for $x = \tau(\mathbf{z})$ we can pick $\mathbf{z}_x = \mathbf{z}$. Indeed, with (22), we have

$$-\mathbf{q}_0(\mathbf{z}) \leq -\mathbf{q}_0(\mathbf{z}_x) + \varrho(|\tau(\mathbf{z}) - \tau(\mathbf{z}_x)|) \leq -\mathbf{q}_0(\mathbf{z}_x) + 2Q.$$

This implies

$$\gamma(\tau(\mathbf{z})) = -\mathbf{q}_0(\mathbf{z}) \qquad \forall \mathbf{z} \in \mathcal{A}_0. \tag{63}$$

Also, with the help of (30), we get successively, for any pair (x_1, x_2), satisfying $\gamma(x_1) \geq \gamma(x_2)$

$$\begin{aligned} \gamma(x_1) &\leq -\mathbf{q}_0(\mathbf{z}_{x_2}) + \min\{\varrho(|\tau(\mathbf{z}_{x_2}) - x_1|), 2Q\} \\ \gamma(x_2) &= -\mathbf{q}_0(\mathbf{z}_{x_2}) + \min\{\varrho(|\tau(\mathbf{z}_{x_2}) - x_2|), 2Q\} \\ 0 \leq \gamma(x_1) - \gamma(x_2) &\leq \min\{\varrho(|\tau(\mathbf{z}_{x_2}) - x_1|), 2Q\} \\ &\quad - \min\{\varrho(|\tau(\mathbf{z}_{x_2}) - x_2|), 2Q\} \leq \varrho(|x_1 - x_2|). \end{aligned}$$

Hence, γ is a continuous function. Furthermore, for any $\mathbf{z} \in \mathcal{A}_0$ and $x \in \mathbb{R}^m$, the previous inequality specialized for $x_1 = x$ and $x_2 = \tau(\mathbf{z})$ yields [bearing in mind (63)]

$$|\gamma(x) + \mathbf{q}_0(\mathbf{z})| \leq \varrho(|x - \tau(\mathbf{z})|).$$

This concludes the proof.

APPENDIX III
PROOF OF PROPOSITION 5

Because $\hat{\varrho}$ is a continuous function and $\varepsilon \neq 0$, the function γ_σ is continuous. Also, we have clearly

$$\gamma_\sigma(x) \leq \sup_{i\in I} |\mathbf{q}_0(\mathbf{z}_i)| \leq \sup_{\mathbf{z}\in\mathcal{A}_0} \mathbf{q}_0(\mathbf{z}) + L_{\mathbf{q}_0}\nu_z$$

and so γ_σ is bounded.

To establish that γ_σ belongs to $\Gamma_\alpha((k+1)\delta)$, we need some preliminary technicalities. Together with (38), the partial injectivity condition gives

$$|\mathbf{q}_0(\mathbf{z}_1) - \mathbf{q}_0(\mathbf{z}_2)| \leq \hat{\varrho}(|\tau(\mathbf{z}_1) - \tau(\mathbf{z}_2)|) \qquad \forall \mathbf{z}_1, \mathbf{z}_2 \in \mathcal{A}_0. \tag{64}$$

Because $\mathbf{q}_0$ and τ are locally Lipschitz (see Proposition 1) and $\mathcal{A}_0$ is compact, we obtain

$$|\tau(\mathbf{z}_1) - \tau(\mathbf{z}_2)| \le L_\tau |\mathbf{z}_1 - \mathbf{z}_2| \qquad \forall \mathbf{z}_1, \mathbf{z}_2 \in \mathcal{A}_0$$
$$|\mathbf{q}_0(\mathbf{z}_1) - \mathbf{q}_0(\mathbf{z}_2)| \le L_{\mathbf{q}_0}|\mathbf{z}_1 - \mathbf{z}_2| \qquad \forall \mathbf{z}_1, \mathbf{z}_2 \in \mathcal{A}_0. \quad (65)$$

As in (27), let $p_{im}(x)$ be such that $|x - p_{im}(x)| \le |x - \tau(\mathbf{z})|$, $\forall \mathbf{z} \in \mathcal{A}_0$. From (33), there exists $j_{p_{im}(x)}$ in I such that $|\mathbf{z}_{j_{p_{im}(x)}} - p_{im}(x)| \le \nu_z$, which, together with (32) and (65), yields

$$\begin{aligned}|\tau(p_{gr}(j_{p_{im}(x)})) - \tau(p_{im}(x))| &\\ \le L_\tau \left(\left|p_{gr}(j_{p_{im}(x)}) - \mathbf{z}_{j_{p_{im}(x)}}\right| + \left|\mathbf{z}_{j_{p_{im}(x)}} - p_{im}(x)\right|\right) &\\ \le 2L_\tau \nu_z. & \qquad (66)\end{aligned}$$

Now let us associate to each x the index $i_x \in I$ so that T_{i_x} is the closest τ_i of x, i.e.,

$$|x - T_{i_x}| \le |x - \tau_i| \qquad \forall i \in I. \quad (67)$$

Precisely, T_{i_x} is one of the projections of x on the approximation by the grid $\{\tau_i\}$ of the image by τ of $\mathcal{A}_0$. For any $\mathbf{z}$ in $\mathcal{A}_0$, (32), (66), and (67) yield

$$\begin{aligned}|x - T_{i_x}| &\le |x - T_{j_{p_{im}(x)}}| \\ &\le |x - \tau(p_{gr}(j_{p_{im}(x)}))| + |\tau(p_{gr}(j_{p_{im}(x)})) - T_{j_{p_{im}(x)}}| \\ &\le |x - \tau(p_{im}(x))| + |\tau(p_{im}(x)) - \tau(p_{gr}(j_{p_{im}(x)}))| + \nu_\tau \\ &\le |x - \tau(\mathbf{z})| + 2L_\tau \nu_z + \nu_\tau \qquad \forall \mathbf{z} \in \mathcal{A}_0. \quad (68)\end{aligned}$$

Hence, we can get an estimation for $|\gamma_\sigma(x) + \mathbf{q}_0(\mathbf{z})|$ as follows. Bearing in mind (64), (32), (33), and (67), it turns out that

$$\begin{aligned}|\mathbf{q}_0(p_{gr}(i_x)) - \mathbf{q}_0(\mathbf{z}_i)| &\\ < |\mathbf{q}_0(p_{gr}(i_x)) - \mathbf{q}_0(p_{gr}(i))| + |\mathbf{q}_0(p_{gr}(i)) - \mathbf{q}_0(\mathbf{z}_i)| &\\ < \varrho(|\tau(p_{gr}(i_x)) - \tau(p_{gr}(i))|) + L_{\mathbf{q}_0}\nu_z &\\ < \varrho(|\tau(p_{gr}(i_x)) - T_{i_x}| + |T_{i_x} - x| + |x - \tau_i| &\\ \quad + |\tau_i - \tau(p_{gr}(i))|) + L_{\mathbf{q}_0}\nu_z &\\ < \varrho(\nu_\tau + |x - \tau_i| + |x - \tau_i| + \nu_\tau) + L_{\mathbf{q}_0}\nu_z. &\end{aligned}$$

Hence, along with (38), for all $\nu_z \le \nu_{z1}^\star$ and $\nu_\tau \le \nu_{\tau 1}^\star$ with $\nu_{z1}^\star$ and $\nu_{\tau 1}^\star$ satisfying

$$L_{\mathbf{q}_0}\nu_{z1}^\star \le \epsilon \quad 2\nu_{\tau 1}^\star \le \epsilon$$

we have

$$|\mathbf{q}_0(p_{gr}(i_x)) - \mathbf{q}_0(\mathbf{z}_i)| \le \epsilon + \hat{\varrho}(\epsilon + 2|x - \tau_i|).$$

This yields, by the Lemma 1 in Appendix V

$$\begin{aligned}|\gamma_\sigma(x) + \mathbf{q}_0(p_{gr}(i_x))| &\le \frac{\sum_{i\in I} \frac{1}{[\epsilon + \hat{\varrho}(\epsilon + 2|x - \tau_i|)]^{\aleph - 1}}}{\sum_{i\in I} \frac{1}{[\epsilon + \hat{\varrho}(\epsilon + 2|x - \tau_i|)]^{\aleph}}} \\ &\le k\,[\epsilon + \hat{\varrho}(\epsilon + 2|x - T_{i_x}|)].\end{aligned}$$

On the other hand, from (68), we have

$$|x - T_{i_x}| \le |x - \tau(\mathbf{z})| + \epsilon/2$$

for all $\nu_z \le \nu_{z2}^\star$ and $\nu_\tau \le \nu_{\tau 2}^\star$ with $\nu_{z2}^\star$ and $\nu_{\tau 2}^\star$ satisfying $\nu_{\tau 2}^\star + 2L_\tau \nu_{z2}^\star \le \epsilon/2$. From this, (64), and (32), we get for any $\mathbf{z} \in \mathcal{A}_0$, any $x \in \mathbb{R}^m$, any $\epsilon > 0$, $\nu_z \le \min\{\nu_{z1}^\star, \nu_{z2}^\star\}$, and $\nu_\tau \le \min\{\nu_{\tau 1}^\star, \nu_{\tau 2}^\star\}$

$$\begin{aligned}|\gamma_\sigma(x) + \mathbf{q}_0(\mathbf{z})| &\\ \le \gamma_\sigma(x) + \mathbf{q}_0(p_{gr}(i_x))| + |\mathbf{q}_0(p_{gr}(i_x)) - \mathbf{q}_0(\mathbf{z})| &\\ \le k\ [\epsilon + \hat{\varrho}(\epsilon + 2|x - T_{i_x}|)] + \varrho(|\tau(p_{gr}(i_x)) - \tau(\mathbf{z})|) &\\ \le k\ [\epsilon + \hat{\varrho}(2\epsilon + 2|x - \tau(\mathbf{z})|)] + \varrho(|\tau(p_{gr}(i_x)) - \tau(\mathbf{z})|). & \qquad (69)\end{aligned}$$

The second term on the right-hand side of the previous expression can be bounded as

$$\begin{aligned}\varrho(|\tau(p_{gr}(i_x)) - \tau(\mathbf{z})|) &\\ \le \varrho\left(|\tau(p_{gr}(i_x)) - T_{i_x}| + |T_{i_x} - x| + |x - \tau(\mathbf{z})|\right) &\\ \le \varrho(\nu_\tau + |x - \tau(\mathbf{z})| + 2L_\tau\nu_z + \nu_\tau + |x - \tau(\mathbf{z})|). & \qquad (70)\end{aligned}$$

Hence, for all $\nu_z \le \nu_{z3}^\star$ and $\nu_\tau \le \nu_{\tau 3}^\star$ with $\nu_{z3}^\star$ and $\nu_{\tau 3}^\star$ satisfying $\nu_{\tau 3}^\star + L_\tau \nu_{z3}^\star \le \epsilon$

$$\begin{aligned}\varrho(|\tau_\nu(p_{gr}(i_x)) - \tau(\mathbf{z})|) &\le \varrho(2\epsilon + 2|x - \tau(\mathbf{z})|) \\ &\le \hat{\varrho}(2\epsilon + 2|x - \tau(\mathbf{z})|).\end{aligned}$$

This, along with (69), yields

$$\begin{aligned}|\gamma_\sigma(x) + \mathbf{q}_0(\mathbf{z})| &\\ \le (k+1)[\epsilon + \hat{\varrho}(2\epsilon + 2|x - \tau(\mathbf{z})|)] &\\ \le (k+1)[\hat{\varrho}(2\epsilon + 2|x - \tau(\mathbf{z})|) - \hat{\varrho}(2\epsilon)] + (k+1)[\epsilon + \hat{\varrho}(2\epsilon)] &\\ \le (k+1)[\hat{\varrho}(2\epsilon + 2|x - \tau(\mathbf{z})|) - \hat{\varrho}(2\epsilon)] + (k+1)\delta &\end{aligned}$$

for any $\epsilon \le \epsilon^\star$, with $\epsilon^\star$ such that $2\epsilon^\star \le \min\{\delta, \hat{\varrho}^{-1}(\delta/2)\}$, and $\nu^\star = \min\{\nu_1^\star, \nu_2^\star, \nu_3^\star\}$.

We conclude the proof by showing that the function $\beta_\epsilon(s) := (k+1)[\hat{\varrho}(2\epsilon + 2s) - \hat{\varrho}(2\epsilon)]$ is such that $\beta_\epsilon(s) \le \alpha(s)$ for all $\epsilon \le \epsilon^\star$ and $s \in \mathbb{R}_+$ with α as in (41). To this end, note that the function $\rho(\sigma)$ in (41) is increasing, $\rho(0) = 0$, and $\rho(s) \ge \beta_\epsilon(s)$ for all $s \ge 0$ and $\epsilon \le 2\epsilon^\star$. Furthermore, it is possible to prove that it is continuous at $\sigma = 0$. Suppose that it is not continuous, namely as ρ is increasing and $\rho(0) = 0$, suppose that there exists a $\rho^\star > 0$ such that $\lim_{\sigma \to 0^+} \rho(\sigma) = \rho^\star$. This implies that there exist sequences $\{\varepsilon_n\}$ and $\{r_n\}$, with $\varepsilon_n \le 2\epsilon^\star$ and $r_n \le 1/n$, such that $(k+1)(\hat{\varrho}(\varepsilon_n+2r_n)-\hat{\varrho}(\varepsilon_n)) \geq \rho^\star/2$ for all $n \in I\!N$. As $\{\varepsilon_n\}$ is bounded, there exists a subsequence converging to $\varepsilon_\rho \leq 2\epsilon^\star$ yielding

$$\lim_{o\to 0^+}(k+1)(\hat{\varrho}(\varepsilon_\rho+o)-\hat{\varrho}(\varepsilon_\rho)) \geq \rho^\star/2$$

which violates continuity of $\hat{\varrho}$ at $\varepsilon^\star$. Hence, $\rho(\sigma)$ is continuous at $\sigma=0$. Now note that, by definition of α in (41)

$$\alpha(s) \geq \rho(s) \geq \beta_\epsilon(s)$$

for all $s \geq 0$ and $\epsilon \leq \epsilon^\star$. By construction, this function is continuous for all $s>0$ and, as $\rho(s)$ is continuous at $s=0$ and since $\alpha(s) \leq s+\rho(2s)$, it is also continuous at $s=0$. This completes the proof.

Appendix IV
Proof Proposition 6

To each x, we associate the index m_x in I such that

$$\gamma_m(x) = -\mathbf{q}_0(\mathbf{z}_{m_x}) + \min\{\hat{\varrho}(|T_{m_x}-x|), 2\mathfrak{Q}\}.$$

Clearly, we have $|\gamma_m(x)| \leq 3\,\mathfrak{Q}$. Also γ_m is continuous because using (30) one obtains

$$\begin{aligned}
&\gamma_m(x_1)-\gamma_m(x_2)\\
&\quad\leq -\mathbf{q}_0(\mathbf{z}_{m_{x_2}}) + \min\{\hat{\varrho}(|T_{m_{x_2}}-x_1|), 2\mathfrak{Q}\}\\
&\qquad + \mathbf{q}_0(\mathbf{z}_{m_{x_2}}) - \min\{\hat{\varrho}(|T_{m_{x_2}}-x_2|), 2\mathfrak{Q}\}\\
&\quad\leq \min\{\hat{\varrho}(|T_{m_{x_2}}-x_2|)+\hat{\varrho}(|x_2-x_1|), 2\mathfrak{Q}\}\\
&\qquad - \min\{\hat{\varrho}(|T_{m_{x_2}}-x_2|), 2\mathfrak{Q}\}\\
&\quad\leq \hat{\varrho}(|x_2-x_1|).
\end{aligned}$$

Moreover, with i_x defined as in (67), we have $\gamma_m(x) \leq -\mathbf{q}_0(\mathbf{z}_{i_x}) + \hat{\varrho}(|T_{i_x}-x|)$. Hence, for any $\mathbf{z}$ in $\mathcal{A}_0$, we have, with (64), (32), (68), and (70)

$$\begin{aligned}
&\gamma_m(x)+\mathbf{q}_0(\mathbf{z})\\
&\quad\leq \mathbf{q}_0(\mathbf{z})-\mathbf{q}_0(\mathbf{z}_{i_x})+\hat{\varrho}(|T_{i_x}-x|)\\
&\quad\leq \mathbf{q}_0(\mathbf{z})-\mathbf{q}_0(p_{gr}(i_x))+\mathbf{q}_0(p_{gr}(i_x))-\mathbf{q}_0(\mathbf{z}_{i_x})\\
&\qquad+\hat{\varrho}(|x-\tau(\mathbf{z})|+2L_\tau\nu_z+\nu_\tau)\\
&\quad\leq \varrho(|\tau(\mathbf{z})-\tau(p_{gr}(i_x))|)+L_{\mathbf{q}_0}\nu_z\\
&\qquad+\hat{\varrho}(|x-\tau(\mathbf{z})|+2L_\tau\nu_z+\nu_\tau)\\
&\quad\leq \varrho(2\nu_\tau+2|x-\tau(\mathbf{z})|+2L_\tau\nu_z)+L_{\mathbf{q}_0}\nu_z\\
&\qquad+\hat{\varrho}(|x-\tau(\mathbf{z})|+2L_\tau\nu_z+\nu_\tau)\\
&\quad\leq 2\hat{\varrho}(2\nu_\tau+2|x-\tau(\mathbf{z})|+2L_\tau\nu_z)+L_{\mathbf{q}_0}\nu_z.
\end{aligned}$$

Now, note that because

$$\begin{aligned}
&|\mathbf{q}_0(\mathbf{z}_{m_x})-\mathbf{q}_0(\mathbf{z})|\\
&\quad\leq Q+\mathfrak{Q}\leq 2\,\mathfrak{Q}\\
&\quad\leq |\mathbf{q}_0(p_{gr}m_x)-\mathbf{q}_0(\mathbf{z})|+|\mathbf{q}_0(\mathbf{z}_{m_x})-\mathbf{q}_0(p_{gr}m_x)|\\
&\quad\leq \varrho(|\tau(\mathbf{z})-\tau(p_{gr}m_x)|)+L_{\mathbf{q}_0}\nu_z
\end{aligned}$$

using (32) and (38), one obtains

$$\begin{aligned}
&-(\gamma_m(x)+\mathbf{q}_0(\mathbf{z}))\\
&\quad= \mathbf{q}_0(\mathbf{z}_{m_x})-\mathbf{q}_0(\mathbf{z})-\min\{\hat{\varrho}(|T_{m_x}-x|), 2\mathfrak{Q}\}\\
&\quad\leq \min\{\varrho(|\tau(\mathbf{z})-\tau(p_{gr}m_x)|)+L_{\mathbf{q}_0}\nu_z,\, 2\mathfrak{Q}\}\\
&-\min\{\hat{\varrho}(|T_{m_x}-x|), 2\mathfrak{Q}\}\\
&\quad\leq L_{\mathbf{q}_0}\nu_z-\min\{\hat{\varrho}(|T_{m_x}-x|), 2\mathfrak{Q}\}\\
&+\min\left\{\varrho(|\tau(\mathbf{z})-x|+|x-T_{m_x}|+|T_{m_x}-\tau(p_{gr}m_x)|, 2\mathfrak{Q}\right\}\\
&\quad\leq L_{\mathbf{q}_0}\nu_z-\min\{\hat{\varrho}(|T_{m_x}-x|), 2\mathfrak{Q}\}\\
&+\varrho(|\tau(\mathbf{z})-x|+\nu_\tau)+\min\{\varrho(|x-T_{m_x}|), 2\mathfrak{Q}\}\\
&\quad\leq L_{\mathbf{q}_0}\nu_z+\hat{\varrho}(|\tau(\mathbf{z})-x|+\nu_\tau).
\end{aligned}$$

As a result

$$|\gamma_m(x)+\mathbf{q}_0(\mathbf{z})| \leq 2\,\hat{\varrho}(2|\tau(\mathbf{z})-x|+2\nu_\tau+2L_\tau\nu_z)+L_{\mathbf{q}_0}\nu_z.$$

From this, the result follows by the same arguments used at the end of the proof of Proposition 5.

Appendix V
A Technical Lemma

Lemma 1: For any set of $N+1$ real numbers r_i in $(0,R]$, any real number k and any integer $\aleph$ satisfying

$$k \geq 2, \quad \aleph \geq 1+\frac{\log(N)}{\log(k)} \tag{71}$$

we have

$$\sum_{i=1}^{N+1}\frac{1}{r_i^{\aleph-1}} \leq k\inf_i r_i \sum_{i=1}^{N+1}\frac{1}{r_i^{\aleph}}.$$

Proof: Without loss of generality, let $r=r_{N+1}=\inf_i r_i$. We have to establish

$$\frac{k-1}{r^{\aleph-1}}+\sum_{i=1}^{N}\frac{kr-r_i}{r_i^{\aleph}} \geq 0.$$

Because the r_i are mutually independent, we are led to study the function $x \in [r, R] \mapsto f(x) = (kr - x)/(x^{\aleph})$. Note that

$$f'(x) = \frac{(\aleph - 1)x - \aleph kr}{x^{\aleph+1}}.$$

Hence, f reaches its minimum at R if $R \leq (\aleph)/(\aleph - 1)kr$ or at $(\aleph)(\aleph - 1)kr$, otherwise. Namely, we have

$$\begin{aligned} f(x) = \frac{kr - x}{x^{\aleph}} &\geq \frac{kr - R}{R^{\aleph}}, \qquad \text{if} \quad R \leq \frac{\aleph}{\aleph - 1}kr \\ &\geq -\frac{1}{(kr)^{\aleph-1}}\frac{1}{\aleph - 1}\left(\frac{\aleph - 1}{\aleph}\right)^{\aleph}, \qquad \text{if} \quad \frac{\aleph}{\aleph - 1}kr \leq R. \end{aligned}$$

However, the same argument gives

$$\frac{kr - R}{R^{\aleph}} \geq -\frac{1}{(kr)^{\aleph-1}}\frac{1}{\aleph - 1}\left(\frac{\aleph - 1}{\aleph}\right)^{\aleph}$$

for all $R \in [r, kr\aleph/(\aleph - 1)]$, which implies

$$\begin{aligned} &\frac{k - 1}{r^{\aleph-1}} + \sum_{i=1}^{N} \frac{kr - r_i}{r_i^{\aleph}} \\ &\quad \geq \frac{k - 1}{r^{\aleph-1}} - N\frac{1}{(kr)^{\aleph-1}}\frac{1}{\aleph - 1}\left(\frac{\aleph - 1}{\aleph}\right)^{\aleph} \\ &\quad \geq \frac{1}{r^{\aleph-1}}\left[k - 1 - N\frac{1}{k^{\aleph-1}}\frac{1}{\aleph - 1}\left(\frac{\aleph - 1}{\aleph}\right)^{\aleph}\right]. \end{aligned}$$

This yields finally, with (71)

$$\frac{1}{r^{\aleph-1}} + \sum_{i=1}^{N} \frac{kr - r_i}{r_i^{\aleph}} \geq \frac{1}{r^{\aleph-1}}\left[k - 1 - \frac{1}{\aleph - 1}\left(\frac{\aleph - 1}{\aleph}\right)^{\aleph}\right] \geq 0.$$

∎

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