Stabilization by output feedback for systems with ISS inverse dynamics

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Abstract: We consider the subclass of the set of systems which admit a global normal form where only the output and not its time derivatives appear in the nonlinearities. We prove that, when the inverse dynamics are "input-to-state stable" (ISS) and a finite gain condition is satisfied, global asymptotic stability can be achieved by dynamic output feedback.

Keywords: Stabilization; dynamic feedback; output feedback; minimum phase systems; "input-to-state stability"

1. Definitions and notations

• $| \cdot |$ denotes the Euclidian norm and, for a vector $\omega = (\omega_i)_{i \in \{1, \ldots, r\}}$, we denote $\|\omega\| = \sup_{i \in \{1, \ldots, r\}} \{ |\omega_i| \}$.

• For a real function $f(x)$, we denote by $f'$ its first derivative $df/dx$.

• A function $\gamma: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is said to be of class $\mathcal{K}$ if it is continuous, nondecreasing and satisfies $\gamma(0) = 0$.

• A function $\beta: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is said to be of class $\mathcal{KL}$ if, for each $t$ in $\mathbb{R}^+$, $\beta(\cdot, t)$ is a function of class $\mathcal{K}$ and, for each $s$ in $\mathbb{R}^+$, the function $\beta(s, \cdot)$ is nonincreasing and we have

$$\lim_{t \to +\infty} \beta(s, t) = 0.$$  \hfill (1)

• Given a strictly convex $C^1$ function $\varphi$ of class $\mathcal{K}$, we denote by $\ell \varphi$ its Legendre–Fenchel transform, i.e.

$$\ell \varphi(x) = \int_0^x (\varphi')^{-1}(s) \, ds,$$  \hfill (2)

where the function $(\varphi')^{-1}$ is the inverse function of the first derivative of $\varphi$. The interest of this definition is in Young's inequality [5, Theorem 156], i.e.

$$xy \leq \varphi(|x|) + \ell \varphi(|y|) \quad \forall (x, y) \in \mathbb{R}^2.$$  \hfill (3)

• A function $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is said to be of class $\mathcal{LKL}$ if it is $C^1$, strictly convex, of class $\mathcal{K}$, has a Legendre–Fenchel transform $\ell \varphi$ as defined above and the function $\ell \varphi(|x|)/x$ is of class $C^m$ and zero at zero.

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1 Usually, one requires strictly increasing function.
2. Problem statement and main result

Consider a single input $u$, single output $y$ dynamical system which can be described globally in the following normal form:

$$
\begin{align*}
\dot{z} &= h(z, x_1), \\
\dot{x}_i &= x_{i+1} + f_i(z, x_1) + g_i(x_1), \quad i \in \{1, \ldots, r-1\}, \\
\dot{x}_r &= u + f_r(z, x_1) + g_r(x_1), \\
y &= x_1,
\end{align*}
$$

where $z$ is in $\mathbb{R}^n$ and $f_i$'s, $g_i$'s and $h$ are $C^\infty$ functions, with

$$
\begin{align*}
f_i(0, 0) &= 0, \\
g_i(0) &= 0, \\
h(0, 0) &= 0.
\end{align*}
$$

In the following, it will be useful to denote by $\Phi(z, y)$ the following vector:

$$
\Phi(z, y) = (f_1(z, y), \ldots, f_r(z, y))^T.
$$

Remarks.

(1) Note the restriction that the $f_i$'s and $h$ are not allowed to depend on $x_i$, $i > 1$.

(2) The system

$$
\dot{z} = h(z, y)
$$

corresponds to the minimal dynamics of an inverse of (4) (see [6]).

For such a system, our problem is to design an output feedback making the origin a globally asymptotically stable equilibrium point.

Conditions under which a nonlinear system can be written in a form close to the one in (4) are discussed in [2, 11]. In the case where the system is linear, if it is also minimum phase, then a $(r - 1)$-dimensional output feedback with high gain is sufficient to guarantee global asymptotic stabilization. This result has been extended to the nonlinear case in [9] when the functions $f_i$'s and $h$ are globally Lipschitz-continuous and the zero solution of

$$
\dot{z} = h(z, 0)
$$

is globally exponentially stable. When this Lipschitz condition is not satisfied, one obstruction to global stabilization by output feedback is due to the fact that even a vanishing time function $y(t)$ may be destabilizing for the system (see [19, 15, 16]):

$$
\dot{z} = h(z, y(t)).
$$

To overcome this difficulty, we strengthen the global asymptotic stability of (8) into the property of being input to state, as defined in [17]. Namely, we assume\(^2\) the following.

Assumption IIS (ISS-inverse system). There exist two real functions $\beta$, of class $\mathcal{K}$, and $\gamma$, of class $\mathcal{K}$, such that for any initial condition $z_0$, and any $C^0$ time function $y: [0, T) \to \mathbb{R}$, there exists a unique solution $z(t)$ of

$$
\dot{z} = h(z, y(t)), \quad z(0) = z_0.
$$

It is right maximally defined on $[0, T)$ and satisfies, for all $t$ in $[0, T)$,

$$
|z(t)| \leq \beta(|z_0|, t) + \gamma \left( \sup_{0 \leq s \leq t} \{|y(s)|\} \right).
$$

\(^2\) This definition, although written differently, is strictly equivalent to the one given in [17] but restricted to $C^0$ controls.
An algorithm to check if a system is ISS and to obtain the corresponding functions $\beta$ and $\gamma$ can be found in [17, Proof of Theorem 1].

With assumption IIS, the output feedback stabilization problem has been solved in [11, 8, 12] when (4) has the following special form:

\[
\dot{z} = Hz + G(x_1),
\]

\[
\dot{x}_i = x_{i+1} + g_i(x_1), \quad i \in \{1, \ldots, r-1\},
\]

\[
\dot{x}_r = u + Fz + g_i(x_1),
\]

\[
y = x_1,
\]

where $F$ and $H$ are matrices of appropriate dimensions, $H$ being strictly Hurwitz, and the $g_i$'s and $G$ are $C^\infty$ functions. When $z$ does not appear linearly as is the case in (4), we have the following key intermediate result.

**Lemma 2.1.** Under assumption IIS, for any strictly positive real number $\varepsilon$, we can find a continuous dynamic output feedback depending on $\varepsilon$ which makes all the solutions of (4) bounded and converging to a compact set $\Gamma_{\varepsilon}$.

This result is not completely satisfactory. Indeed, it will appear from its proof (see (91)) that, as $\varepsilon$ goes to zero, coordinates depending on $\varepsilon$ go to 0. But, coordinates independent of $\varepsilon$, like $(x_1, \ldots, x_r)$, may go to infinity, except for $x_1 = y$ which is guaranteed to go to 0. This incompleteness of the result is due to a singularity occurring at the equilibrium point. A sufficient condition for this singularity not to be an obstruction can be written in terms of a local small-gain condition. Indeed, we have the following theorem.

**Theorem 2.2.** Suppose that IIS assumption is satisfied. Assume also that, for some invertible proper function $\theta$ of class $\mathcal{K}$ and some strictly positive real numbers $\kappa, \chi$ and $C$, with $\kappa \leq 1$, we have for all $x \in [0, \chi]$,

\[
\gamma_y(x) + \gamma_z(2\gamma(x)) \leq \begin{cases} C x & \text{if } r > 1 \\ C x^\kappa & \text{if } r = 1, \end{cases}
\]

where

\[
\gamma_y(x) = \sup_{\{(z, y)\mid \|z\| \leq \gamma(y(x)) \leq x\}} |\Phi(z, y)|, \quad \gamma_z(x) = \sup_{\{(z, y)\mid \|\Phi(z, y)\| \leq \gamma(z) \leq x\}} |\Phi(z, y)|.
\]

Under these conditions, the origin can be made a globally asymptotically stable equilibrium point by a continuous dynamic output feedback.

**Remark.** Since $\Phi$ is a $C^\infty$ function, by taking

\[
\theta(|z|) = |z|,
\]

we can guarantee that $\gamma_y$ and $\gamma_z$ are at least linear locally. In this case, the first inequality in (14) can be reduced to

\[
\frac{1}{C} \gamma_z(2\gamma(x)) \leq x.
\]

This condition is nothing but the small-gain condition of [10]. Also, it follows from [20], for example, that, if $\partial h/\partial z(0, 0)$ is a strictly Hurwitz matrix, i.e. the zero dynamics (8) are locally exponentially stable, the function $\gamma$ is also linear locally. Therefore, in this case, condition (13) is satisfied.
3. Proofs

To prove Theorem 2.2 and, more precisely, to design explicitly a dynamic output feedback solving our problem, we follow the same procedure as in [8, 12]:

Step 1: we choose the following "observer":

\[
\begin{align*}
\dot{x}_1 &= \dot{x}_2 + g_1(y) + k_1(y - \dot{x}_1), \\
\dot{x}_2 &= \dot{x}_3 + g_2(y) + k_2(y - \dot{x}_1), \\
&\vdots \\
\dot{x}_r &= u + g_r(y) + k_r(y - \dot{x}_1),
\end{align*}
\]

(17)

where the gains \(k_i\)'s are appropriately chosen so that the observation error

\[
\tilde{e} = \tilde{x} - x
\]

satisfies

\[
\dot{\tilde{e}} = h(z, y), \quad \dot{\tilde{x}} = A\tilde{x} - \Phi(z, y),
\]

(19)

where \(A\) is a strictly Hurwitz matrix. From assumption IIS, system (19) is ISS. Indeed, the matrix \(A\) being strictly Hurwitz, there exist strictly positive real numbers \(K_1, K_2\) and \(\lambda\) such that, for any initial condition \((\tilde{x}_0, z_0)\) and any \(C^0\) time function \(y : [0, T) \to \mathbb{R}\), there exists a unique solution \((\tilde{x}(t), z(t))\) of (19) which is defined on \([0, T)\) and satisfies, for all \(t\) in \([0, T]\).

\[
|\tilde{e}(t)| \leq K_1 \exp(-\lambda t)|\tilde{x}_0| + K_2 \int_0^t \exp(-\lambda(t - s))|\Phi(z(s), y(s))| \, ds,
\]

(20)

\[
|z(t)| \leq \beta(|z_0|, t) + \gamma \left( \sup_{0 \leq s \leq t} \{|y(s)|\} \right).
\]

(21)

On the other hand, we have

\[
|\Phi(z, y)| \leq \gamma_1(|y|) + \gamma_2(|z|),
\]

(22)

where the functions \(\gamma_1\) and \(\gamma_2\) are defined in (14). These functions being nondecreasing, by using (22) with [17, equation (12)], (20) becomes

\[
|\tilde{e}(t)| \leq \beta_{\lambda}\left( \left(\begin{array}{c} \tilde{x}_0 \\ z_0 \end{array}\right), t \right) + \frac{K_2}{\lambda} (\gamma_1 + \gamma_2 \cdot 2\gamma) \left( \sup_{0 \leq s \leq t} \{|y(s)|\} \right),
\]

(23)

where \(\beta_{\lambda}\), defined as follows, is of class \(\mathcal{KL}\):

\[
\beta_{\lambda}\left( \left(\begin{array}{c} \tilde{x}_0 \\ z_0 \end{array}\right), s \right) = \exp(-\lambda s) \left[ K_1 \left(\begin{array}{c} \tilde{x}_0 \\ z_0 \end{array}\right) + \frac{K_2}{\lambda} \gamma_1 \left( 2\beta\left( \left(\begin{array}{c} \tilde{x}_0 \\ z_0 \end{array}\right), 0 \right) \right) \right] \\
+ K_2 \int_0^s \exp(-\lambda(s - \tau))\gamma_2 \left( 2\beta\left( \left(\begin{array}{c} \tilde{x}_0 \\ z_0 \end{array}\right), \tau \right) \right) \, d\tau.
\]

(24)

\(^3\) We consider here a full-order observer. This will lead to an \(r\)-dimensional output feedback. But exactly the same procedure could be applied with a reduced-order observer leading to an \((r - 1)\)-dimensional output feedback as in [9, 12].
Note that the \( \dot{x} \) and \( z \) subsystems driven by \( y \) being time-invariant, inequalities (21) and (23) hold also for any initial condition at any starting time, i.e.

\[
|z(t)| \leq \beta(|z(s)|, (t-s)) + \gamma\left(\sup_{s \leq t} \{|y(\tau)|\}\right) \quad \forall s \in [0, t].
\]

Step 2: We design a state feedback for the "observer" (17) or, more precisely, for

\[
\begin{align*}
\dot{y} & = \dot{y}_2 + g_1(y) + \delta_1(\dot{x}, z, y), \\
\dot{y}_2 & = \dot{y}_3 + g_2(y) + \delta_2(\dot{x}, z, y), \\
\vdots \\
\dot{y}_r & = u + g_r(y) + \delta_r(\dot{x}, z, y),
\end{align*}
\]

where the \( \delta_i \)'s are given by

\[
\delta_1(\dot{x}, z, y) = x - \dot{x} + f_1(y), \quad \delta_i(\dot{x}, z, y) = k_i(y - \dot{x}_i), \quad 2 \leq i \leq r.
\]

If the \( f_i \)'s were zero, according to [18] it would be sufficient to design \( u \) to make (26) with input \( \delta_i \)'s ISS. But when the \( f_i \)'s are not zero, a more specific result is needed. For this, we remark on the existence of a positive real number \( K_3 \) such that, for all \( i \) in \( \{ 1, \ldots, r \} \),

\[
|\delta_i(\dot{x}, z, y)| \leq K_3|\dot{x}| + \gamma_y(|y|) + \gamma_z(|z|).
\]

Then it follows from (25) that, for all \( i \) in \( \{ 1, \ldots, r \} \), we have

\[
|\delta_i(\dot{x}(t), z(t), y(t))| \leq \beta_i\left(\left(\begin{array}{c} \dot{x}(s) \\ z(s) \end{array}\right)\right), (t-s) + \gamma_i\left(\sup_{s \leq t} \{|y(\tau)|\}\right) \quad \forall s \in [0, t],
\]

where \( \beta_i \) of class \( \mathcal{KL}^\infty \) and \( \gamma_i \) of class \( \mathcal{AL} \) are defined as follows:

\[
\beta_i(s, t) = K_3\beta_x(s, t) + \gamma_x(2\beta(s, t)), \quad \gamma_i(s) = \left[1 + \frac{K_2K_3}{\lambda}\right]\gamma_x + \gamma_x \circ 2\gamma(s).
\]

Inequality (29) can be viewed as showing the existence of a "nonlinear \( L^\infty \)-gain" for the operator \( y \mapsto (\delta_i)_{i=1, \ldots, r} \). This motivates our interest for the following lemma about the stability of composite systems which is in the spirit of the small-gain theorem [10] (see [7] for a more general statement).

**Lemma 3.1.** Consider the following composite system:

\[
\begin{align*}
\dot{\zeta} & = \psi_1(\zeta, \omega), \\
\omega & = \psi_2(\zeta, \sigma), \\
\dot{\chi} & = \psi_3(\omega, \chi), \\
\sigma & = \psi_4(\chi),
\end{align*}
\]

where \( \zeta \in \mathbb{R}^n, \chi \in \mathbb{R}^n, \psi_1, \psi_2 \) and \( \psi_3, \psi_4 \) are \( C^1 \) functions, with \( \psi_4(0) = 0 \), and \( \psi_3 \) is a \( C^0 \) function. Assume the following:

1. For the \( \zeta \)-subsystem with input \( \sigma \) and output \( \omega \), there exist functions \( \beta_1 \) and \( \beta_2 \) of class \( \mathcal{KL}^\infty \), \( \gamma_1 \) and \( \gamma_2 \) of class \( \mathcal{AL} \) such that, for any initial condition \( \zeta(0) \) and any \( C^0 \) time function \( \sigma : [0, T] \to \mathbb{R}^n \), there exists a unique solution. It is defined on \([0, T]\) and satisfies, for all \( t \in [0, T] \),

\[
\begin{align*}
|\zeta(t)| & \leq \beta_1(|\zeta(s)|, (t-s)) + \gamma_1\left(\sup_{s \leq t} \{|\sigma(\tau)|\}\right) \quad \forall s \in [0, t], \\
\|\omega(t)\| & \leq \beta_2(|\zeta(s)|, (t-s)) + \gamma_2\left(\sup_{s \leq t} \{|\sigma(\tau)|\}\right).
\end{align*}
\]
For the χ-subsystem with input ω and output m, there exist a positive-definite proper C¹ function $U$ and a convex function $\phi$ of class $\mathcal{K}$ such that

$$\frac{\partial U}{\partial \chi}(\chi)\psi_3(\omega, \chi) \leq - U(\chi) + \phi(\|\omega\|) \quad \forall (\omega, \chi) \quad (33)$$

and, for some positive real number $\eta$,

$$\phi(2\gamma_2(|\psi_4(\chi)|)) \leq U(\chi) + \eta \quad \forall \chi. \quad (34)$$

Under these conditions, system (31) has well-defined solutions for any initial conditions. They are bounded and converge to a compact set $\Gamma_\eta$ and, in particular, we have

$$\limsup_{t \to +\infty} U(\chi(t)) \leq \eta. \quad (35)$$

Moreover, if $\eta = 0$ in (34) then $\Gamma_\eta$ is reduced to the origin.

Proof of Lemma 3.1. We first remark that $U$ being positive-definite and proper, there exists an invertible proper function $\gamma_3$ of class $\mathcal{K}$ such that [14, Theorem 7.13]

$$\gamma_3(|\chi|) \leq U(\chi). \quad (36)$$

Moreover, $\psi_4$ being a C¹ function with $\psi_4(0) = 0$, there exists a function $\gamma_3$ of class $\mathcal{K}$ such that

$$|\psi_4(\chi)| \leq \gamma_3(|\chi|) \quad \forall \chi. \quad (37)$$

Then, the system (31) having a continuous right-hand side, for each initial condition, there exist (may be nonunique) solutions right maximally defined on $[0, T)$, for some strictly positive real number $T$. Along each solution, we have from (33), for all $t$ in $[0, T)$,

$$\dot{\chi} \leq - U(\chi(t)) + \phi(\|\omega(t)\|). \quad (38)$$

But the function $\phi$ being convex, the function $\phi(2\gamma_2)$ being nondecreasing and the function $\psi_4(\chi(t))$ being $C^0$ on $[0, T)$, (32) and (34) yield, for $0 \leq s \leq t < T$,

$$\phi(\|\omega(t)\|) \leq \frac{1}{2}\phi(2\beta_2(|\zeta(s)|, (t - s))) + \frac{1}{2}\phi\left(2\gamma_2\left(\sup_{s \leq t \leq \tau} \{|\psi_4(\chi(\tau))|\}\right)\right) \quad (39)$$

$$\leq \frac{1}{2}\phi(2\beta_2(|\zeta(s)|, (t - s))) + \frac{1}{2}\sup_{s \leq \tau \leq t} \phi(2\gamma_2(|\psi_4(\chi(\tau))|)) \quad (40)$$

$$\leq \frac{1}{2}\phi(2\beta_2(|\zeta(s)|, (t - s))) + \frac{1}{2}\sup_{s \leq \tau \leq t} \{U(\chi(\tau))\} + \frac{1}{2}\eta. \quad (41)$$

Therefore, we have for $0 \leq s \leq t < T$,

$$\dot{\chi} \leq - U(\chi(t)) + \frac{1}{2}\sup_{s \leq \tau \leq \tau} \{|U(\chi(\tau))| + \frac{1}{2}\phi(2\beta_2(|\zeta(s)|, (t - s))) + \frac{1}{2}\eta. \quad (42)$$

From this inequality, boundedness of $U$ and consequently of $\chi$ and $\zeta$ can be proved by invoking the small-gain theorem [10]. Here is a more specific proof.

By letting

$$v_0 = \max\{U(\chi(0)), \phi(2\beta_2(|\zeta(0)|, 0)) + \eta\}. \quad (43)$$
we get with Gronwall inequality
\[
U(x(t)) - v_0 \leq \frac{1}{2} \sup_{0 \leq s \leq t} \{U(x(s)) - v_0\}. \tag{44}
\]
It follows readily that \(U(x)\) is bounded by \(v_0\) on \([0, T)\). But, with (36) and (32), this yields
\[
|\langle \xi(t), \chi(t) \rangle \|^2 \leq \beta_1(|\langle \xi(0), \chi(t) \rangle|, t) + (\text{Id} + \gamma_1 \circ \gamma_3)(\gamma_v^{-1}(v_0)) \quad \forall t \in [0, T),
\tag{45}
\]
with \(\text{Id}\) denoting the identity function. This implies with a contradiction argument that \(T = + \infty\) and that the corresponding solution is defined and bounded on \([0, + \infty)\). And, in particular, we have
\[
|\langle \xi(t) \rangle| \leq \xi_0 \overset{\text{def}}{=} \beta_1(|\langle \xi(0), \chi(t) \rangle|, 0) + \gamma_1 \circ \gamma_3 \circ \gamma_v^{-1}(v_0) \quad \forall t \in [0, + \infty). \tag{46}
\]
Let us now prove that
\[
\limsup_{t \to +\infty} U(x(t)) \leq \eta. \tag{47}
\]
For this, with \(v_0\) defined above, let us define \(t_0 = 0\). If \(v_0 \leq \eta\), we augment it so that \(v_0 > \eta\). Then, by induction, we assume the existence of a positive real number \(v_n\) and a time instant \(t_n\) such that
\[
v_n > \eta, \quad U(x(t)) \leq v_n, \quad \forall t \geq t_n. \tag{48}
\]
In this condition, (42) yields, for all \(t\) larger than \(t_n\),
\[
\dot{U} \leq -U(x(t)) + \frac{1}{2} v_n + \frac{1}{2} \phi(2\beta_2(|\langle \xi(t_n) \rangle|, (t - t_n))) + \frac{1}{2} \eta. \tag{49}
\]
But, the function \(\beta_2\) being of class \(\mathcal{K}\mathcal{L}\) and the function \(\phi\) of class \(\mathcal{K}\), there exists a time instant \(s_n \geq t_n\) such that
\[
\phi(2\beta_2(|\langle \xi(t_n) \rangle|, (t - t_n))) \leq \frac{1}{2}(v_n - \eta) \quad \forall t \geq s_n. \tag{50}
\]
This implies that
\[
\dot{U} \leq -U(x(t)) + \frac{1}{2} v_n + \frac{1}{2} \eta \quad \forall t \geq s_n. \tag{51}
\]
Therefore, let
\[
t_{n+1} = s_n + \log 2, \quad v_{n+1} = \frac{3}{2} v_n + \frac{1}{2} \eta. \tag{52}
\]
We have established that
\[
v_{n+1} > \eta, \quad U(x(t)) \leq v_{n+1}, \quad \forall t \geq t_{n+1}. \tag{53}
\]
Property (47) follows since the sequence \(v_n\) converges to \(\eta\).

We have established, for each positive integer number \(n\),
\[
|\langle \xi(t), \chi(t) \rangle \|^2 \leq \beta_1(|\langle \xi(t_n) \rangle|, (t - t_n)) + (\text{Id} + \gamma_1 \circ \gamma_3)(\gamma_v^{-1}(v_n)) \quad \forall t \geq t_n. \tag{54}
\]
But \(\beta_1\) being of class \(\mathcal{K}\mathcal{L}\), with (46), we have
\[
\beta_1(|\langle \xi(t_n) \rangle|, (t - t_n)) \leq \beta_1(\xi_0, t_n) \quad \forall t \geq 2t_n. \tag{55}
\]
Therefore, we have finally, for each positive integer number \(n\),
\[
|\langle \xi(t), \chi(t) \rangle \|^2 \leq \beta_1(\xi_0, t_n) + (\text{Id} + \gamma_1 \circ \gamma_3)(\gamma_v^{-1}(v_n)) \quad \forall t \geq 2t_n, \tag{56}
\]
where the sequence \(\beta_1(\xi_0, t_n)\) converges to 0 since the sequence \(t_n\) goes to infinity.

The conclusion of the lemma follows by defining the compact set \(\Gamma_\eta\) as the closed ball centered at the origin with radius \(\rho(\eta)\), where the function \(\rho\) of class \(\mathcal{K}\) is given by
\[
\rho(\eta) = (\text{Id} + \gamma_1 \circ \gamma_3 \circ \gamma_v^{-1}(\eta)). \tag{57}
\]
As mentioned above our proofs of Lemma 2.1 and Theorem 2.2 will be complete if we can design a control law for the system (26) so that (33) and (34) are satisfied. To achieve this objective, we note that the system (26) is in feedback form, a form which has received a lot of attention (see [3] and the references therein for the unperturbed case and [4,13] for the perturbed case). We follow here a mixture with some extension of the ideas contained in these studies. Indeed, our interpretation of the results we are aware of is that they are based on the following two elementary technical lemmas.

**Lemma 3.2 (adding a perturbation).** Assume that for the system
\[
\dot{x} = f(x) + A + gu,
\]
there exist a $C^1$ control law $w$ and a $C^{1+1}$ function $V$, such that $w(0) = 0$, $L_g V(0) = 0$ and, for any solution $x(t)$ with $u = w(x)$, we have
\[
\dot{V} \leq -V(x) + \alpha
\]
for some time function $\alpha$. Consider the system
\[
\dot{x} = f(x) + A + g(u + \delta).
\]
Then, for any function $\varphi$ of class $\mathcal{L}_s^1$, the control law
\[
\tilde{w}(x) = w(x) + \frac{\varphi(|L_g V(x)|)}{L_g V(x)}
\]
is $C^1$, $\tilde{w}(0) = 0$ and, along the solutions of (61) with $u = \tilde{w}$, we get
\[
\dot{V} \leq -V(x) + \alpha + \varphi(|\delta|).
\]

**Proof.** Let us compute the time derivative of $V$ along the solutions of (60) with
\[
u = v + w(x).
\]
We get, with obvious omitted arguments,
\[
\dot{V} \leq -V + \alpha + L_g V(v + \delta).
\]
But with (3), we get
\[
L_g V(v + \delta) \leq L_g V\left(v + \frac{\varphi(|L_g V|)}{L_g V}\right) + \varphi(|\delta|).
\]
The result follows readily. □

**Lemma 3.3 (adding one integrator).** Assume that for the system
\[
\dot{x} = f(x) + A + gu,
\]
with $f(0) = 0$, there exist a $C^{1+1}$ control law $w$ and a $C^{1+1}$ function $V$ such that $w(0) = 0$, $L_g V(0) = 0$ and, for any solution $x(t)$ with $u = w(x)$, we have
\[
\dot{V} \leq -V(x) + \alpha.
\]
Then, for the system
\[
\dot{x} = f(x) + A + gy,
\]
\[
\dot{y} = h(x,y) + u,
\]

\[L_g V\] denotes as usual $(\delta V/\delta x)g$. 

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with \( h(0, 0) = 0 \) and for any function \( \varphi \) of class \( \mathcal{L} L^1 \), the control law

\[
\ddot{w}(x, y) = -\frac{1}{2} (y - w(x)) - L_y V(x) + \frac{\partial w}{\partial x} (x) (f(x) + gy) - \frac{\ell \varphi \left( |y - w(x)| \frac{\partial w}{\partial x} (x) \right)}{y - w(x)} - h(x, y) 
\]

is \( C^1 \), \( \ddot{w}(0, 0) = 0 \), the function\(^5\)

\[
U(x, y) = V(x) + \frac{1}{2} (y - w(x))^2
\]

is \( C^{1+1} \) and we have along the solutions of (68), with \( u = \ddot{w} \),

\[
\dot{U} \leq -U(x, y) + \alpha + \varphi(|\Delta|).
\]

Note that one degree of smoothness may be lost while going from \( w \) and \( V \) to \( \ddot{w} \).

**Proof.** Let us compute the time derivative of \( U \) along the solution of (68). With (67) and

\[
u = v - \frac{1}{2} (y - w(x)) - L_y V(x) + \frac{\partial w}{\partial x} (x) (f(x) + gy) - h(x, y),
\]

we get

\[
\dot{U} \leq -U + \alpha + (y - w) \left( v - \frac{\partial w}{\partial x} \Delta \right).
\]

But with (3), we get

\[
(y - w) \left( v - \frac{\partial w}{\partial x} \Delta \right) \leq (y - w) \left( v + \frac{\ell \varphi \left( |y - w| \frac{\partial w}{\partial x} \right)}{y - w} \right) + \varphi(|\Delta|).
\]

The result follows readily. \( \square \)

The main point of these lemmas is that, when going from system (58) to system (61) or from system (66) to system (68), one can find control laws so that the effect of the uncertainties is only to increment \( \alpha \) by "\( \varphi \) (uncertainties)" (see (62) and (71)).

We are now ready to solve our design problem for system (26) by following the technique of induction on adding integrators. Precisely, we apply Lemmas 3.2 and 3.3 recursively while adding integrators.

Let \( \varphi \) be a function of class \( \mathcal{L} L^{r-1} \), \( V_1(y) \) a proper positive-definite \( C^r \) function and \( u_0 \) a \( C^{r-1} \) function such that \( u_0(0) = 0 \) and

\[
V'_1(y) u_0(y) \leq -V'_1(y).
\]

These three functions will be specified later.

We consider the system

\[
y = u_{11} + g_1(y).
\]

With (5), the control law

\[
u_{11}(y) = u_0(y) - g_1(y)
\]

\(^5\) More generally, one can take \( U(x, y) = h_1(V(x)) + h_2(y - w(x)) \), where \( h_1 \) and \( h_2 \) are any smooth, positive, increasing and proper functions with \( h_2 \) quadratic on a neighborhood of 0.
is $C^{r-1}, u_1(0) = 0$ and along, the solutions of (76), we have
\[
\dot{V}_1 \leq -V_1.
\] (78)

Next we consider the system
\[
\dot{y} = u_{12} + g_1(y) + \delta_1(\bar{x}, z, y).
\] (79)
The assumptions of Lemma 3.2 are satisfied with
\[
l = r - 1, \quad V = V_1, \quad \alpha = 0.
\] (80)
Therefore, (61) gives us appropriate $C^{r-1}$ control law $u_{12}(y)$ and Lyapunov function $V_1$ for the system (79) in the sense that the assumptions of Lemma 3.3 are satisfied with
\[
\Delta = \delta_1, \quad l = r - 2, \quad V = V_1, \quad \alpha = \phi(|\delta_1|), \quad u_{12}(0) = 0, \quad V_1(0) = 0.
\] (81)

We can now consider the system
\[
\dot{y} = \dot{x}_2 + g_1(y) + \delta_1(\bar{x}, z, y), \quad \dot{x}_2 = g_2(y) + u_{21}.
\] (82)
By applying Lemma 3.3, we get an appropriate $C^{r-2}$ control law $u_{21}(y, \dot{x}_2)$ and $C^{r-1}$ Lyapunov function $V_2$ for system (83) in the sense that the assumptions of Lemma 3.2 are satisfied with
\[
\delta = \delta_2, \quad l = r - 2, \quad V_2(y, \dot{x}_2) = V_1(y) + \frac{1}{2} (\dot{x}_2 - u_{12}(y))^2, \quad \alpha = 2\phi(|\delta_1|), \quad u_{21}(0, 0) = 0.
\] (83)
Continuing this procedure, we finally get a $C^0$ control $u_{r2}$ and a $C^1$ function
\[
V_r(y, \dot{x}_2, \ldots, \dot{x}_r) = V_1(y) + \frac{1}{2} \sum_{i=1}^{r-1} (\dot{x}_{i+1} - u_{12}(y, \dot{x}_2, \ldots, \dot{x}_i))^2,
\] (84)
where each of the $u_{12}$'s is $C^{r-i}$ and is zero at zero. Moreover, along the solutions of (26), with the perturbations $\delta_i$'s generated by (19) and satisfying (29), and
\[
u = u_{r2}(y, \dot{x}_2, \ldots, \dot{x}_r),
\] (85)
we have
\[
\dot{V}_r \leq -V_r(y(t), \dot{x}_2(t), \ldots, \dot{x}_r(t)) + (2r - 1)\phi \left( \sup_{i=1, \ldots, r} |\delta_i| \right).
\] (86)
On the other hand, we know that the function $V_1$ is proper and positive-definite. It follows that the function $V_r(y, \dot{x}_2, \ldots, \dot{x}_r)$ is a positive-definite proper function.

Then, let us check now if we can apply Lemma 3.1 with the $\zeta$-subsystem given by (19) and the $\chi$-subsystem given by (26) with $u = u_{r2}$. From (86), assumption (33) is satisfied with $V_r$ playing the role of $\Omega$, $(\delta_i)_{i=1, \ldots, r}$ the role of $\omega$ and
\[
\phi(s) = (2r - 1)\phi(s).
\] (87)
We have already mentioned that (2) holds from (25) and (29) with, in particular,
\[
\gamma_2(s) = \left[ 1 + \frac{K_2 K_3}{h} \right](\gamma_r + \gamma_s 2\gamma)(s).
\] (88)
It remains to check condition (35). To meet this condition, we adjust our three functions $\phi$, $V_1$ and $u_0$. We have the following technical lemma (see the appendix).

**Lemma 3.4.** For any function $\gamma_2$ of class $\mathcal{K}$ and any strictly positive real number $\eta$, there exist a function $\phi$ of class $\mathcal{K}^r$, a proper positive-definite $C^r$ function $V_1(y)$ and a $C^{r-1}$ function $u_0$ such that $u_0(0) = 0$ and, for all $y \in \mathbb{R}$, we have
\[
-V_1(y)u_0(y) \geq V_1(y) \geq (2r - 1)\phi(2\gamma_2(|y|)) - \eta.
\] (89)
Moreover if, for some strictly positive real numbers $\kappa, \chi$ and $C$, with $\kappa \leq 1$, we have, for all $x \in [0, \chi]$,

$$\gamma_2(x) = \begin{cases} Cx & \text{if } r > 1 \\ Cx^{r} & \text{if } r = 1, \end{cases}$$

(90)

then the same result is satisfied with $\eta = 0$.

Consequently, by choosing $\eta$ strictly positive if (13) does not hold or zero if not, assumption (34) of Lemma 3.1 is satisfied from (87)-(89) with $\gamma_2$ given in (88) and $\psi_4(x) = y$. Lemma 2.1 and Theorem 2.2 follow from the conclusion of Lemma 3.1 and, in particular, we have

$$\limsup_{t \to +\infty} V_r(y(t), \dot{x}_1(t), \ldots, \dot{x}_n(t)) \leq \eta.$$  

(91)

4. Example

Consider the system

$$\dot{z} = -z^3 + x_1,$$

$$\dot{x}_1 = x_2 + zx_1,$$

$$\dot{x}_2 = u,$$

$$y = x_1.$$  

Following [17, Proof of Theorem 1], we can see that the system

$$\dot{z} = -z^3 + y$$

(93)

is ISS with

$$\beta(s, t) = \sqrt{\frac{4s^2}{4 + 7s^2t}}, \quad \gamma(s) = 2s^{1/3}.$$  

(94)

Also, by choosing $\theta(x) = x^2$ in (15), we get, for $x \in \mathbb{R}_+$,

$$\gamma_2(x) = x^{3/2}, \quad \gamma_4(x) = x^3.$$  

(95)

It follows, for $x \in \mathbb{R}_+$ that

$$\gamma_2(x) + \gamma_4(2\gamma(x)) = x^{3/2} + 64x.$$  

(96)

We conclude that the assumptions of Theorem 2.2 are satisfied. Therefore, there exists a continuous dynamic output feedback which makes the origin a globally stable equilibrium point.

Then, let us exhibit this dynamic output feedback. Following our design, we first consider the following observer:

$$\dot{\hat{x}}_1 + \dot{\hat{x}}_2 + 3(y - \hat{x}_1), \quad \dot{\hat{x}}_2 = u + 2(y - \hat{x}_1).$$  

(97)

In this case the error system is

$$\dot{\hat{x}}_1 = -3\hat{x}_1 + \hat{x}_2 - zy, \quad \dot{\hat{x}}_2 = -2\hat{x}_1.$$  

(98)

We have

$$\delta_1 = x_2 - \hat{x}_2 + zy, \quad \delta_2 = 2(y - \hat{x}_1)$$

(99)

and (29) is satisfied with

$$K_2 = 7, \quad \lambda = 1, \quad K_3 = 2.$$  

(100)
It follows that by choosing
\[\phi(x) = \frac{1}{2}x^2, \quad V_1(y) = 1350y^2(\sqrt{|y|} + 64)^2, \quad u_0(y) = -\frac{y(\sqrt{|y|} + 64)}{3\sqrt{|y|} + 128},\] (101)
inequality (89) of Lemma 3.4 is satisfied with \(\eta = 0\) and
\[\ell \phi(x) = \frac{1}{2}x^2.\] (102)

Then, by applying Lemmas 3.2 and 3.3 successively, we get
\[u_{11}(y) = \frac{-y(\sqrt{|y|} + 64)}{3\sqrt{|y|} + 128},\]
\[u_{12}(y) = u_{11}(y) - 675y(\sqrt{|y|} + 64)(3\sqrt{|y|} + 128),\]
\[u_{21}(y, \dot{y}) = -\frac{1}{2}(\dot{x}_2 - u_{12}(y)) - 675y(\sqrt{|y|} + 64)(3\sqrt{|y|} + 128)\]
\[+ u_{12}(y)\dot{x}_2 - \frac{1}{2}(\dot{x}_2 - u_{12}(y))u_{12}(y)^2,\] (103)
\[u_{22}(y, \dot{y}) = u_{21}(y, \dot{y}) - \frac{1}{2}(\dot{x}_2 - u_{12}(y)).\]

Therefore, a globally stabilizing dynamic output feedback is
\[\dot{x}_1 = \dot{x}_2 + 3(y - \dot{x}_1), \quad \dot{x}_2 = u + 2(y - \dot{x}_1), \quad u = u_{22}(y, \dot{y}).\] (104)

5. Concluding remarks

We have established that a subclass of the set of systems which admit a global normal form (4) and whose inverse dynamics are ISS can be stabilized by dynamic output feedback.

This result has been obtained by noting that it was already potentially contained in [8, 12], where systems in the form (12) are considered. The main point was to realize that what is effectively used in [8, 12] about the z-subsystem in (12) is the fact that it is BIBO system with \(y\) as input. Then, for the system (4), the natural extension is the property, for its z-subsystem, to be ISS.

The instrumental result which allowed us to do this extension is Lemma 3.1. It concerns the stability of composite systems with conditions à la small-gain theorem. Since it was very helpful here, we decided to devote the companion paper [7] to generalize it.

However, this extension asks for a knowledge of the function \(y\) characterizing the property of being ISS. In the linear context of [8, 12], this function can easily be derived from the data of the system. For our more nonlinear context, we know only the algorithm proposed in [17, proof of Theorem 1]. It is not very practical. So, for the time being, our result is more an existence result than a practical result. Another problem we have raised, but not solved, in this paper is the small-gain condition close to the equilibrium. The consequence is that only the asymptotic smallness of the output is always guaranteed.

Here we have concentrated our attention on the stabilization problem for a system already written in appropriate coordinates. The next step is to find a coordinate-free condition guaranteeing that the assumptions of Theorem 2.2 are satisfied. This has been done in [8, 12] for the special form (12).

6 See our first remark after (6).
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Appendix

A.1. Proof of Lemma 3.4

The result of this lemma is very intuitive. We provide here a proof for the sake of completeness. It has no practical interest. It is based on the following lemma established by Jean-Michel Coron.

Lemma A.1. Let \( k: [0, +\infty) \to \mathbb{R} \) be a continuous function such that, for some strictly positive real number \( \chi \), we have, for all \( x \in [0, \chi] \),

\[
k(x) \leq 0.
\]

Under this condition, there exists a proper strictly increasing \( C^\infty \) function \( W: [0, +\infty) \to [0, +\infty) \) and a strictly positive real number \( \mu \) satisfying:

(i) \( W(x) = \frac{1}{2} x^2 \quad \forall x \in [0, \mu] \),

(ii) \( W(x) \geq k(x) \quad \forall x \geq 0 \),

(iii) \( W'(x) > 0 \quad \forall x > 0 \).

Proof. For any \( i \) in \( \mathbb{Z} \), let \( \mathcal{S}_i \) be the following open set:

\[
\mathcal{S}_i = \{ x \in \mathbb{R} | 2^{i-1} < x < 2^{i+1} \}.
\]

We have

\[
\bigcup_{i \in \mathbb{Z}} \mathcal{S}_i = (0, +\infty).
\]

Then let \( \{ \chi_i \}_{i \in \mathbb{Z}} \) be the partition of unity subordinate to this open covering [1, Theorem V.4.4]. Let also the real numbers \( c_i \)'s be defined by

\[
c_i = \sup \{ \{ k(x) | x \in \mathcal{S}_i \} \cup \{ 0 \} \}.
\]

Then \( \tilde{W}_1 \), defined by

\[
\tilde{W}_1(x) = \sum_{i \in \mathbb{Z}} c_i \chi_i(x),
\]

is a \( C^\infty \) function satisfying

(i) \( \tilde{W}_1(x) = 0 \quad \forall x \in [0, \mu_1] \),

(ii) \( \tilde{W}_1(x) \geq k(x) \quad \forall x \geq 0 \),

with \( \mu_1 \) some strictly positive real number. With the same procedure but with \( k \) replaced by \( -\tilde{W}'_1 \), we get a second function \( \tilde{W}_2 \). Then the conclusion of the lemma follows by taking

\[
W(x) = \frac{1}{2} x^2 + \int_0^x \tilde{W}_2(s) \, ds + \tilde{W}_1(x).
\]
With this result, the first point of Lemma 3.4 is obtained by choosing
\[ \varphi(x) = \frac{1}{2} x^2, \]  
\[ u_0(y) = - \frac{V_1(y)}{V_1'(y)}. \]  

and
\[ V_1(y) = V_1(- y) = W(y) \quad \forall y \geq 0, \]  
where \( W \) is the function given by lemma A.1 with
\[ k(x) = (2r - 1) \varphi(2\gamma_2(x)) - \eta. \]  

Note that if, for some positive real numbers \( C \) and \( \chi > 0 \), we have
\[ 2\gamma_2(x) \leq Cx \quad \forall x \in [0, \chi], \]  
then, with \( \varphi \) as defined in (113), we get
\[ \varphi(2\gamma_2(x)) \leq \frac{C^2}{2} x^2 \quad \forall x \in [0, \chi]. \]  

In this case, we define \( u_0 \) as in (114), but with \( V_1 \) given by
\[ V_1(y) = V_1(- y) = W(y) + (2r - 1) C^2 y^2 \quad \forall y \geq 0, \]  
where now \( W \) is obtained from
\[ k(x) = (2r - 1) \varphi(2\gamma_2(x)) - (2r - 1) C^2 x^2. \]  

Similarly, when \( r = 1 \), if, for some positive real numbers \( C, \chi > 0 \) and \( \kappa \leq 1 \), we have
\[ 2\gamma_2(x) \leq Cx^\kappa \quad \forall x \in [0, \chi], \]  
then we choose \( \varphi \) as
\[ \varphi(x) = \frac{\kappa}{1 + v} |x|^{(1 + v) / \kappa}, \]  
with \( v \) some strictly positive real number. In this case, we get
\[ \ell \varphi(x) = \frac{1 + v}{1 + v - \kappa} |x|^{(1 + v) / (1 + v - \kappa)}. \]  

So, clearly \( \ell \varphi(|x|) / x \) is continuous, i.e. \( \varphi \) is of class \( \mathcal{L}^\infty \). Moreover, we get
\[ \varphi(2\gamma_2(x)) \leq \frac{\kappa C^{(1 + v) / \kappa}}{1 + v} |x|^{1 + v} \quad \forall x \in [0, \chi]. \]  

In this case, we define again \( u_0 \) by (115), with \( V_1 \) given by
\[ V_1(y) = V_1(- y) = W(y) + C^{(1 + v) / \kappa} y^{1 + v} \quad \forall y \geq 0, \]  
where now \( W \) is obtained from
\[ k(x) = \varphi(2\gamma_2(x)) - C^{(1 + v) / \kappa} x^{1 + v}. \]
References