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Sensitivity to High-Frequency Measurement Noise of Nonlinear High-Gain Observers

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Abstract: This paper deals with the characterisation of the sensitivity to high-frequency measurement noise of nonlinear high-gain observers. The proposed tool provides bounds for the steady-state estimate in presence of noise when the high-gain parameter characterising the speed of convergence of the observer is fixed. The nonlinear analysis captures the effect of the noise frequency showing the "low-pass" filtering properties of the observer.

1. INTRODUCTION

High-gain observers have been routinely used since the beginning of 90's as a tool of fast-state estimation in many control-feedback scenarios (see, for instance, the special issue in Khalil and Praly (2014) and references therein). One of the most important features is that of having a rate of convergence that can be made arbitrarily fast by augmenting one single parameter, also known as "high-gain parameter" (denoted with ℓ throughout this work). In general, this parameter has to be large enough in order to overcome the Lipschitz constant of the observed nonlinear system. Furthermore, the larger ℓ becomes, the smaller becomes the estimation error bound in presence of certain disturbances and parametric uncertainties. Nevertheless, increasing the speed of convergence may incredibly deteriorate the asymptotic estimation in presence of high-frequency measurement noise. Also, by augmenting the high-gain parameter, peaking may increase, and digital implementation is more demanding. This trade-off is very well-known in literature and many efforts have been done in order to round the above problems. As reported in Khalil and Praly (2014), "a sound strategy to achieve fast convergence while reducing the impact of measurement noise at steady state is to use a larger ℓ during the transient time and then decrease it at steady state". Most of the proposed techniques rely in varying the high-gain parameter with some scheme (see among others, Ahrens and Khalil (2009), Boizot et al. (2010), Marino and Santosuosso (2007), Prasov and Khalil (2013), Sanfelice and Praly (2011)).

In this note we focus on the effect of the measurement noise on the steady state of the high-gain observer estimate. In particular we consider the case when measurement noise is a high-frequency signal. To the best of the author's knowledge, attempts of analysis have been done but only $\mathcal{H}_\infty$ bounds have been characterised (see, for instance, Ball and Khalil (2009), or Vasiljevic and Khalil (2008)). As already noticed in Vasiljevic and Khalil (2006), the $\mathcal{H}_\infty$ analysis is too conservative and fails to catch the "low-pass" filtering characteristics of the high-gain observer when ℓ is fixed to some (eventually large) value. We propose a technique, which may be eventually applied to other frameworks, to analyse the error estimate steady-state behaviour of the high-gain observer, based on the approximation of a solution of a partial differential equation modelling the steady state of the estimate. The given bound captures both the effects of the high-gain parameter and the frequency of the measurement noise, highlighting the "low-pass" filter properties that can be extracted by a frequency analysis in the linear case.

2. PROBLEM DESCRIPTION AND PRELIMINARIES

We consider the class of nonlinear systems that, maybe after a change of coordinates (see, for instance, Gauthier and Bornard (1981)), are described in the uniform observability form

$$\begin{aligned} \dot{x} &= Ax + B\varphi(x) \\ y &= Cx + \nu(t) \end{aligned} \tag{1}$$

with state $x \in X \subseteq \mathbb{R}^n$ (X a bounded set) and measured output $y \in \mathbb{R}$, where $\varphi(\cdot)$ is a locally Lipschitz function and (A, B, C) is a triplet in prime form of dimension n, that is

$$A := \begin{pmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ 0 & 0_{1\times(n-1)} \end{pmatrix}, \quad B := \begin{pmatrix} 0_{(n-1)\times 1} \\ 1 \end{pmatrix},$$

$$C := \begin{pmatrix} 1 & 0_{1\times(n-1)} \end{pmatrix},$$

and $\nu(t)$ is a bounded measurement noise. For such nonlinear system we consider a standard high-gain observer of the form

$$
\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + \ell k_1 (y - \hat{x}_1) \\
\dot{\hat{x}}_2 &= \hat{x}_3 + \ell^2 k_2 (y - \hat{x}_1) \\
&\vdots \\
\dot{\hat{x}}_{n-1} &= \hat{x}_n + \ell^{n-1} k_{n-1} (y - \hat{x}_1) \\
\dot{\hat{x}}_n &= \varphi_s(\hat{x}) + \ell^n k_n (y - \hat{x}_1)
\end{aligned} \tag{2}
$$

where here $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)^T \in \mathbb{R}^n$ is the estimated state, k_j, $j = 1, \ldots n$, and ℓ are positive coefficients to be designed (with $\ell \geq 1$ the high-gain parameter), and $\varphi_s(\cdot)$ is a locally Lipschitz bounded function that agree with $\varphi(\cdot)$ on a bounded set $X_\delta \supset X$, namely there exists a $\bar{\varphi} > 0$ such that $|\varphi_s(x)| \leq \bar{\varphi}$ for all $x \in \mathbb{R}^n$ and $\varphi_s(x) = \varphi(x)$ for all $x \in X_\delta$.

By considering the change of coordinates

$$\hat{x}_j \mapsto e_j := \hat{x}_j - x_j \qquad j = 1, \ldots, n,$$

it turns out that system (2) transforms as

$$\dot{e} = Fe + B\Delta_\varphi(e, x) + G\nu(t) \tag{3}$$

where

$$
F := \begin{pmatrix} -\ell k_1 & 1 & \cdots & 0 \\ -\ell^2 k_2 & & 1 & \\ \vdots & & \ddots & \\ -\ell^{n-1} k_{n-1} & & & 1 \\ -\ell^n k_n & 0 & \cdots & 0 \end{pmatrix}, \quad G := \begin{pmatrix} \ell k_1 \\ \vdots \\ \ell^n k_n \end{pmatrix},
$$

and $\Delta_\varphi(e, x)$ is the locally Lipschitz function defined as

$$\Delta_\varphi(e, x) := \varphi_s(e + x) - \varphi(x).$$

It is a well-known fact that if the k_j's are chosen so that the polynomial equation

$$\lambda^n + k_1 \lambda^{n-1} + \ldots k_n = 0$$

has roots with negative real part and if ℓ is taken sufficiently large then system (3) is input-to-state stable with respect to the input ν. In particular, by means of standard Lyapunov arguments, it is possible to prove (see Khalil and Praly (2014)) the existence of positive constants c_1, c_2 and c_3 such that the j-th error component can be estimated as

$$
\begin{aligned}
\|e_j(t)\| \leq \max\{c_1\, \ell^{j-1} \exp(-c_2 \ell\, t)\, \|e(0)\|, \\
c_3\, \ell^{j-1} \|\, \nu(\cdot)\|_\infty\}
\end{aligned} \tag{4}
$$

for all $t \geq 0$, $j = 1, \ldots, n$. If the measurement noise is absent, the previous result implies that the estimation error converges to zero asymptotically with an exponential decay rate that can be arbitrarily decreased by increasing ℓ, with the term ℓ^{j-1} in factor of the exponential decay modelling the so-called "peaking" phenomenon. Otherwise, for a *generic* bounded measurement noise, the observer guarantees bounded trajectories with a linear asymptotic gain. The asymptotic gain of the j-th state estimates, in fact, depends on ℓ^{j-1} namely tends to be worst as long as "higher" components in (3) are considered.

The goal of the paper is to better characterise the asymptotic gain in presence of *high-frequency* noise with ℓ that is fixed in order to have the above mentioned ISS property. Towards this end we model the measurement noise as

$$\varepsilon \dot{w} = Sw\,, \qquad \nu = Pw\,, \tag{5}$$

where S is a neutrally stable matrix, P is a row vector, and $\varepsilon \in (0, 1)$ is parameter that will be taken small in the forthcoming analysis. System (5) can be conveniently seen as generator of $m > 0$ harmonics at frequencies $\omega_i/\varepsilon > 0$, $i = 1, \ldots, m$, namely, the matrices S and P take the form

$$S = \mathrm{blkdiag}(S_1, \ldots, S_m)\,, \qquad S_i = \begin{pmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{pmatrix}$$

and $P = ((0\ 1)\ (0\ 1)\ \cdots\ (0\ 1))$. In the following we assume that w ranges in a compact invariant set W.

As a preparatory step towards the nonlinear analysis, it is instructive to consider the linear case, namely the case in which $\varphi(x) = \Phi x$ with Φ a row vector. In this case the observer (2) can be taken [1] with $\varphi_s(\hat{x}) = \Phi\hat{x}$, thus resulting in an error system (3)-(5) given by

$$
\begin{aligned}
\varepsilon \dot{w} &= Sw \\
\dot{e} &= (F + B\Phi)e + GPw
\end{aligned}
$$

with the matrix $F + B\Phi$ that is Hurwitz for ℓ sufficiently large. Using the fact that S is neutrally stable and that $F + B\Phi$ is Hurwitz it follows that the state of the previous system reaches a steady state fully described by the state of the noise generator. In particular, denoting by Π_ε the matrix solution of the Sylvester equation

$$\Pi_\varepsilon S = \varepsilon(F + B\Phi)\Pi_\varepsilon + \varepsilon GP$$

it turns out that

$$\lim_{t\to\infty} (e(t) - \Pi_\varepsilon w(t)) = 0\,. \tag{6}$$

The solution of the previous Sylvester equation can be characterised at high-frequency (namely for small value of ε) to have more insight about how the gain between the measurement noise and the j-th estimation error is affected by ℓ. In particular, using that the fact that S is not singular, it is easy to check that

$$\Pi_\varepsilon = \varepsilon GPS^{-1} + \varepsilon^2 \bar{\Pi}_\varepsilon$$

with

$$\bar{\Pi}_\varepsilon := \sum_{k=2}^{\infty} \varepsilon^{k-2} (F + B\Phi)^{k-1} GPS^{-k}\,,$$

is a solution of the Sylvester equation. In particular, the series defining $\bar{\Pi}_\varepsilon$ is convergent as long as ε is taken sufficiently small. Namely, there exist $\varepsilon_1^\star > 0$ and $\bar{\pi} > 0$ such that $\|\bar{\Pi}_\varepsilon\| \leq \bar{\pi}$ for all positive $\varepsilon \leq \varepsilon_1^\star$ By bearing in mind how ℓ enters in G, and by denoting with Π_j the j-th row of Π_ε, $j = 1, \ldots, n$, it is thus possible to claim the existence of a positive $\varepsilon_2^\star(\ell) \leq \varepsilon_1^\star(\ell)$ such that for all positive $\varepsilon \leq \varepsilon_2^\star(\ell)$ the following holds

$$\lim_{\varepsilon\to 0} \|\Pi_j\| \leq c\, \varepsilon\, \ell^j$$

where c is a positive constant. From this, using (6) and the fact that W is compact, we can then conclude that for all positive $\varepsilon \leq \varepsilon_2^\star(\ell)$ the following holds

$$\lim_{t\to\infty} \sup |e_j(t)| \leq c\, \varepsilon\, \ell^j\, |w|_\infty\,.$$

The previous relation clearly shows the "low-pass" filtering properties of the high-gain observer, namely

[1] Because of linearity, boundedness of the function $\varphi_s(x)$ is not needed.

$$\lim_{\varepsilon\to 0}\lim_{t\to\infty}\sup|e_j(t)| = 0\,,$$

and the fact that the asymptotic gain of the j-th error component at high-frequency depends on ℓ^j (whereas the $\mathcal{H}_\infty$ bound is proportional to ℓ^{j-1}, as shown by (4)).

The main goal of this note is to present the theoretical tool that allows one to get the same kind of result also in the nonlinear setting.

3. MAIN RESULT

By compactly writing the system dynamics (1) as

$$\dot{x} = f(x)$$

the overall dynamics given by the observed system (1), the observer error dynamics (3) and the noise generator (5) read as

$$\begin{aligned} \varepsilon\dot{w} &= Sw \\ \dot{x} &= f(x) \\ \dot{e} &= Fe + B\Delta_\varphi(e,x) + GPw\,. \end{aligned} \tag{7}$$

Having tuned the parameters k_i, $i = 1,\dots,n$, and ℓ as said before, the trajectories of this system are bounded. The system in question, thus, has a well-defined steady state that can be characterised with the tools proposed in Isidori and Byrnes (2008). More specifically, the triangular structure of the system (with the x and w subsystem driving the e subsystem) implies that the existence of a possibly set-valued function $\pi_\varepsilon : X\times W \rightrightarrows \mathbb{R}^n$ such that the set

$$\mathrm{graph}(\pi_\varepsilon) = \big\{(w,x,e)\in W\times X\times\mathbb{R}^n \;:\; e\in\pi_\varepsilon(w,x)\big\}$$

is asymptotically stable for (7). Furthermore, the properties of the high-gain observer when the measurement noise is absent (*i.e.* when $w = 0$) show that

$$\pi_\varepsilon(0,x) = \{0\} \qquad \forall\, x\in X\,.$$

The following technical lemma provides an arbitrarily accurate approximation of a continuous selection of $\pi_\varepsilon(\cdot,\cdot)$.

Lemma 1. Consider system (3) and let r be an arbitrary positive number. There exist continuous functions $\bar\psi_{j,i} : X\times W\to\mathbb{R}$, $j = 1,\dots,n$, $i = 1,\dots,r$, such that having defined

$$\Psi_j(w,x) := \sum_{i=1}^{r}\ell^{j+i-1}\bar\psi_{j,i}(w,x)\,\varepsilon^i$$

$$\Psi_\varepsilon(w,x) := \mathrm{col}\Big(\,\Psi_1(w,x)\,\cdots\,\Psi_n(w,x)\Big)$$

and

$$\begin{aligned} E_\varepsilon(w,x) := {}& \frac{\partial\Psi_\varepsilon(w,x)}{\partial w}Sw + \frac{\partial\Psi_\varepsilon(w,x)}{\partial x}f(x) \\ & - F\Psi_\varepsilon(w,x) - GPw - B\Delta_\varphi(\Psi_\varepsilon(w,x),x)\,, \end{aligned}$$

the following holds

$$\lim_{\varepsilon\to 0^+}\frac{E_\varepsilon(w,x)}{\varepsilon^{r-1}} = 0\,, \qquad \lim_{w\to 0^+}E_\varepsilon(w,x) = 0\,.$$

Proof. Let

$$\psi_{j,i}(w,x) := \ell^{j+i-1}\bar\psi_{j,i}(w,x)$$

so that

$$\Psi_j(w,x) = \sum_{i=1}^{r}\psi_{j,i}(w,x)\,\varepsilon^i\,. \tag{8}$$

Since w and x range in bounded sets and the function $\psi_{j,i}(\cdot,\cdot)$ are continuous, we have that

$$\lim_{\varepsilon\to 0^+}\Psi_\varepsilon(w,x) = 0 \qquad \forall\,(w,x)\in W\times X\,.$$

Expanding $\Delta_\varphi(\Psi_\varepsilon,x)$ by Taylor around $\Psi_\varepsilon = 0$ we obtain

$$\Delta_\varphi(\Psi_\varepsilon,x) \;=\; \sum_{i=1}^{r}\varphi_i(x)[\Psi_\varepsilon]^i + \rho_r(\Psi_\varepsilon,x)$$

in which $\varphi_i(\cdot)$, $i = 1,\dots,r$, are properly defined continuous functions, $\rho_r(\cdot,\cdot)$ is a properly defined continuous remainder function, and the $[\Psi_\epsilon]^i$ are monomials of the form

$$[\Psi_\varepsilon]^i = \prod_{j=1}^{n}\Psi_j^{k_j}\,, \qquad\qquad \sum_{j=1}^{n}k_j \;=\; i\,.$$

By replacing Ψ_j with the expressions (8) and grouping the terms with the same power of ε, the Taylor expansion of $\Delta_\varphi(\cdot,\cdot)$ can be rewritten as

$$\Delta_\varphi(\Psi_\varepsilon,x) \;=\; \sum_{i=1}^{r}\varepsilon^i\phi_i(w,x) + \varepsilon^{r+1}R_\varepsilon(w,x) \tag{9}$$

where the functions $\phi_i(\cdot)$, $i = 1,\dots,r$, and $R_\varepsilon(\cdot,\cdot)$ are appropriately defined continuous functions. As far as the ϕ_i's are concerned, in particular, we observe that, because the Ψ_j are polynomials in ε and the $[\Psi_\varepsilon]^i$ are polynomials in the Ψ_j, only the coefficients of power smaller or equal to i in ε in the Ψ_j can be in ϕ_i. Namely, $\phi_i(\cdot,\cdot)$ depends only on $\psi_{j,k}$ with $k\le i$, for all $i = 1,\dots,r$ and $j = 1,\dots,n$.

Consider now the expression of $E_\varepsilon(\cdot,\cdot)$ and, denoting by $E_1(\cdot,\cdot)$, ..., $E_n(\cdot,\cdot)$ its components, note that

$$\begin{aligned} E_1(w,x) &= \dot\Psi_1 + \ell k_1\Psi_1 - \Psi_2 - \ell k_1 Pw \\ &\;\;\vdots \\ E_n(w,x) &= \dot\Psi_n + \ell^n k_n\Psi_1 - \Delta_\varphi(\Psi_\epsilon,x) - \ell^n k_n Pw \end{aligned}$$

where, for sake of compactness, we omitted the argument (w,x) from the functions Ψ_j, $j = 1,\dots,n$, and Ψ_ε. By embedding (8) and (9) in the previous expressions, the following is obtained

$$\begin{aligned} & E_j(w,x) \\ &= \sum_{i=1}^{r}\left[L_f\psi_{j,i} + \frac{1}{\varepsilon}L_S\psi_{j,i}\right]\varepsilon^i + \ell^j k_j\sum_{i=1}^{r}\psi_{1,i}\,\varepsilon^i \\ &\quad - \sum_{i=1}^{r}\psi_{j+1,i}\varepsilon^i - \ell^j k_j Pw \\ &= \left[L_S\psi_{j,1} - \ell^j k_j Pw\right] \\ &\quad + \sum_{i=1}^{r}\varepsilon^i\left[L_f\psi_{j,i} + L_S\psi_{j,i+1} + \ell^j k_j\psi_{1,i} - \psi_{j+1,i}\right] \end{aligned}$$

for $j = 1,\dots,n-1$ and

$$
\begin{aligned}
&E_n(w,x)\\
&= \sum_{i=1}^{r}\left[L_f\psi_{n,i} + \frac{1}{\varepsilon}L_S\psi_{n,i}\right]\varepsilon^i + \ell^n k_n \sum_{i=1}^{r}\psi_{1,i}\varepsilon^i\\
&\quad - \sum_{i=1}^{r}\phi_i\varepsilon^i - \varepsilon^{r+1}R_\varepsilon(w,x) - \ell^n k_n P w\\
&= [L_S\psi_{n,1} - \ell^n k_n P w] - \varepsilon^{r+1}R_\varepsilon(w,x)\\
&\quad + \sum_{i=1}^{r}\varepsilon^i\left[L_f\psi_{n,i} + L_S\psi_{n,i+1} + \ell^n k_n\psi_{1,i} - \phi_i\right]
\end{aligned}
$$

in which $\psi_{j,r+1} := 0$, $j = 1,\dots,n$, and

$$
L_f\psi_{j,i} := \frac{\partial\psi_{j,i}(w,x)}{\partial x}f(x)\,,\ L_S\psi_{j,i} := \frac{\partial\psi_{j,i}(w,x)}{\partial w}Sw\,.
$$

The idea now is to iteratively select the functions $\psi_{j,i+1}(\cdot,\cdot)$ to annihilate, in the previous expressions, the terms in ε^i, $i = 0,\dots,r-1$, $j = 1,\dots,n$. We start by considering the terms of order 0 in ε which are all annihilated by taking

$$
\psi_{j,1}(w,x) \;=\; \ell^j\, k_j\, P\, S^{-1}\, w \;:=\; \ell^j\,\bar\psi_{j,1}(w,x)\,, \qquad j = 1,\dots,n\,. \tag{10}
$$

We observe that $\bar\psi_{j,1}(w,x)$, and thus $\psi_{j,1}(w,x)$, $j = 1,\dots,n$, are polynomials in w of order 1 with constant coefficients. We proceed now by induction by assuming that all the functions $\psi_{j,k}(\cdot,\cdot)$, $k = 1,\dots,i$, $j = 1,\dots,n$ have been fixed in the form

$$
\psi_{j,k}(w,x) = \ell^{j+k-1}\,\bar\psi_{j,k}(w,x)
$$

for some continuous $\bar\psi_{j,k}(w,x)$ that are polynomials in w of order k with coefficients dependent on x, so that to annihilate the terms in ε^{k-1}. Furthermore, since $\phi_i(\cdot,\cdot)$ only depends on $\psi_{j,k}$ with $k \le i$, $j = 1\dots,n$, we assume that $\phi_i(\cdot,\cdot)$ is a polynomial in w of order i with coefficients dependent on x. In this case we see that the terms of order i in ε are annihilated if $\psi_{j,i+1}(\cdot,\cdot)$ can be chosen so that

$$
\begin{aligned}
-L_S\psi_{j,i+1} &= L_f\psi_{j,i} + \ell^j k_j\psi_{1,i} - \psi_{j+1,i}\\
&\qquad\qquad j = 1,\dots,n-1\,,\\
-L_S\psi_{n,i+1} &= L_f\psi_{n,i} + \ell^n k_n\psi_{1,i} - \phi_i\,.
\end{aligned}
$$

Using the induction assumptions on the functions $\psi_{j,i}(\cdot,\cdot)$, $j = 1,\dots,n$, and $\phi_i(\cdot,\cdot)$, and the fact that S is invertible, it is easy to see that the previous PDEs admit solutions of the form

$$
\psi_{j,i+1}(w,x) = \ell^{j+i}\,\bar\psi_{j,i+1}(w,x)
$$

$j = 1,\dots,n$, for some $\bar\psi_{j,i+1}(w,x)$ which, in turn, are polynomials in w of order $i+1$ with coefficients that are continuous functions of x. The induction iteration can be then used to choose $\psi_{j,i}(\cdot,\cdot)$, $j = 1,\dots,n$, $i = 1,\dots,r$, of the form

$$
\psi_{j,i}(w,x) = \ell^{j+i-1}\,\bar\psi_{j,i}(w,x)
$$

where $\bar\psi_{j,i}(w,x)$ are polynomial functions in w of order i with coefficients that are continuous functions of x. By embedding those functions in the expressions of $E_j(\cdot,\cdot)$, $j = 1,\dots,n$, and bearing in mind the definition of $R_\varepsilon(\cdot,\cdot)$, it is readily seen that

$$
\begin{aligned}
E_j(w,x) &= \varepsilon^r[L_f\psi_{j,r} + \ell^j k_j\psi_{1,r} - \psi_{j+1,r}]\\
E_n(w,x) &= \varepsilon^r[L_f\psi_{n,r} + \ell^n k_n\psi_{1,r} - \phi_r] + \varepsilon^{r+1}w\bar R_\varepsilon(w,x)
\end{aligned} \tag{11}
$$

where $\bar R_\varepsilon(\cdot,\cdot)$ is an appropriately defined continuous function, by which the claim of the lemma immediately follows. □

The previous lemma is instrumental to the proof of the next proposition which is the main result of the paper.

Proposition 1. Consider system (7) with $x(t) \in X$ and $w(t) \in W$ for all $t \ge 0$ with X and W bounded compact sets. Let the function $\varphi_s(\cdot)$ embedded in $\Delta_\varphi(\cdot,\cdot)$ be chosen so that it is locally Lipschitz and it agrees with $\varphi(\cdot)$ on a set $X_\delta \supset X$. Let ℓ be fixed so that system (3) is ISS with respect to the input ν. Then, there exists a $\varepsilon^\star(\ell) > 0$ such that for all positive $\varepsilon \le \varepsilon^\star(\ell)$ the following holds

$$
\lim_{t\to\infty}\sup|e_j(t)| \;\le\; c\,\varepsilon\,\ell^j\,|w|_\infty \quad j = 1,\dots,n
$$

with c a positive constant.

Proof. Let consider the change of variables

$$
e \;\mapsto\; \tilde e := e - \Psi_\varepsilon(w,x)
$$

with $\Psi_\varepsilon(\cdot,\cdot)$ introduced in the previous lemma with an $r > 1$ and observe that, by bearing in mind the definition of $E_\varepsilon(\cdot,\cdot)$,

$$
\dot\Psi_\varepsilon = F\Psi_\varepsilon + B\Delta_\varphi(\Psi_\varepsilon, x) + GPw + E_\varepsilon(w,x)\,.
$$

Furthermore, note that

$$
\begin{aligned}
\Delta_\varphi(e,x) &- \Delta_\varphi(\Psi_\varepsilon(w,x),x)\\
&= \Delta_\varphi(\tilde e + \Psi_\varepsilon(w,x),x) - \Delta_\varphi(\Psi_\varepsilon(w,x),x)\\
&= \varphi_s(\tilde e + \Psi_\varepsilon(w,x) + x) - \varphi(x)\\
&\qquad -(\varphi_s(\Psi_\varepsilon(w,x) + x) - \varphi(x))\\
&= \varphi_s(\tilde e + \Psi_\varepsilon(w,x) + x) - \varphi_s(\Psi_\varepsilon(w,x) + x)\\
&= \Delta_\varphi(\tilde e, \Psi_\varepsilon + x)\,.
\end{aligned}
$$

Note that there exists a $\varepsilon_1^\star(\ell) \in (0,1]$ such that for all positive [2] $\varepsilon \le \varepsilon_1^\star(\ell)$

$$
\Delta_\varphi(0, \Psi_\varepsilon(w,x) + x) = 0 \qquad \forall\,(w,x) \in W \times X\,.
$$

By the previous facts the error dynamics in the new coordinates can be easily computed as

$$
\dot{\tilde e} = F\tilde e + B\Delta_\varphi(\tilde e, \Psi_\varepsilon(w,x) + x) + E_\varepsilon(w,x)\,. \tag{12}
$$

Since the Lipschitz constant of $\Delta_\varphi(\cdot,\cdot)$ is not affected by the value of the arguments, the same values of ℓ that make system (3) ISS with respect to the input $\nu(t)$ make also system (12) ISS with respect to the input $E_\varepsilon(\cdot,\cdot)$. In particular, there exists a positive constant c_0 such that

$$
\begin{aligned}
\lim_{t\to\infty}\sup|\tilde e(t)| &= \lim_{t\to\infty}\sup|e(t) - \Psi_\varepsilon(w(t),x(t))|\\
&\le c_0\lim_{t\to\infty}\sup|E_\varepsilon(w(t),x(t))|\\
&\le c_0|E_\varepsilon(w,x)|_\infty
\end{aligned}
$$

Using the fact that, for any $r \ge 1$, $E_\varepsilon(w,x)$ is a term in ε^r, it follows that there exists a positive constant c_1 such that

[2] The value of $\varepsilon^\star$ depends, besides other, on the choice of the set X_δ on which $\varphi_s(\cdot)$ coincides with $\varphi(\cdot)$.

$$\lim_{t\to\infty} \sup |\tilde{e}(t)| \le c_1 \varepsilon^r |w|_\infty .$$

Consider now the the expressions of the components $\Psi_j(\cdot,\cdot)$, $j = 1,\ldots,n$, of $\Psi_\varepsilon(\cdot,\cdot)$ introduced in the previous lemma. It turns out that there exist a positive $\varepsilon_2^\star(\ell) \le \varepsilon_1^\star(\ell)$ and a positive constant c_2 such that

$$|\Psi_j(w,x)| \le c_2\, \varepsilon\, \ell^j\, |w|$$

for all $j = 1,\ldots,n$, for all positive $\varepsilon \le \varepsilon_2^\star(\ell)$ and for all $(w,x) \in W \times X$. From this,

$$\begin{aligned}
\lim_{t\to\infty} \sup |e_j(t)| &= \lim_{t\to\infty} \sup |\tilde{e}_j(t) + \Psi_j(w(t),x(t))| \\
&\le \lim_{t\to\infty} \sup |\tilde{e}_j(t)| + \lim_{t\to\infty} \sup |\Psi_j(w(t),x)(t)| \\
&\le \lim_{t\to\infty} \sup |\tilde{e}_j(t)| + |\Psi_j(w,x)|_\infty \\
&\le c_1 \varepsilon^r |w|_\infty + c_2 \varepsilon \ell^j |w|_\infty
\end{aligned}$$

by which the result follows by taking an appropriate $\varepsilon^\star(\ell) \le \varepsilon_2^\star(\ell)$. □

4. CONCLUSIONS

The characterisation of the sensitivity to high-frequency measurement noise of nonlinear high-gain observers has been investigated and error bounds have been given. We showed that the asymptotic value of the j-th components of the estimation error can be bounded by a term proportional to $\varepsilon\ell^j$ where ℓ is the high-gain parameter and ε is the inverse of the noise frequency. This analysis suggests that when high-gain observers are used in practice, the high-gain parameter should be small enough with respect to the lowest frequency characterising the measurement noise, but large enough to guarantee convergence of the observer. The proposed tool relies on the approximation of a partial differential equation modelling the nonlinear steady state of the estimates. The proposed analysis can be extended also to a general class of nonlinear system in the triangular form (see Bornard and Hammouri (1991)).

The tool presented in the paper can be also successfully used to characterise the sensitivity to high-frequency noise of the"low-power" high-gain observer recently proposed in Astolfi and Marconi (2016). In that context, in particular, it is possible to show that the j-th components of the estimation error can be bounded by a term proportional to ε^k with $k > 1$ for any $j > 1$, by thus substantially improving the sensitivity at high-frequency for all the error components. Details will be presented in a journal version that is under preparation.

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