

Tools for observers based on coordinate augmentation

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Abstract—Designing a non linear observer often requires to immerse the system dynamics into a space of greater dimension. In order to avoid the difficult left inversion of the immersion, coordinate augmentation has been proposed in [9]. However, difficulties in this augmentation via continuous Jacobian completion, losses of observability, or the question of completeness of solutions may arise in practice. This paper illustrates with the help of a toy-system how to overcome them by using tools and tricks such as extending the image of a diffeomorphism or adding fictitious outputs.

Index Terms—observers for non linear systems, coordinate augmentation, high gain observers, non linear Luenberger observers, Jacobian completion, diffeomorphism extension

I. INTRODUCTION

A. Context

In many applications, estimating the state of a dynamical system is crucial either to build a controller or simply to obtain real time information on the system. A lot of efforts have thus been made in the scientific community to find universal methods for the construction of observers. Although very satisfactory solutions are known for linear systems ([1]), nonlinear observer designs still suffer from a significant lack of generality.

For nonlinear systems, "general purpose" observers with guaranteed "non local" (e.g. not linked to a linear approximation) convergence as high gain observers ([2], [3], [4], [5] etc), nonlinear Luenberger observers ([6], [7], [8]) or others do not demand any particular structure and only require some basic observability properties. But then, in general, their observer state is living in a space different from the one of the system space, often with larger dimension, and the state estimate is obtained typically by solving on-line a nonlinear equation, which may be very complicated, not to mention possible singularities.

Tools, such as coordinate augmentation [9], have been proposed to simplify the implementation of these observers as well as take care of extra constraints and/or extend their domain of validity. In this paper we introduce another tool based on diffeomorphism extension. We motivate and illustrate the interest of these tools with the help of a toy system made of a linear oscillator with unknown frequency. This system has received a lot of attention due to its link

with phase lock loop and adaptive notch filters. Our objective here is not to compare with the many corresponding observers which have been proposed in the literature. We work with it because of its simple structure exhibiting however observability singularities. It allows us to show how some difficulties encountered in implementing an observer can be rounded.

B. Problem statement

For the system

$$\dot{x} = f(x) \quad , \quad y = h(x) \quad (1)$$

with x in \mathbb{R}^n and y in \mathbb{R} , we assume we are given :

- a C^1 function $\tau^* : \mathbb{R}^n \rightarrow \mathbb{R}^m$ whose restriction to the closure $\text{cl}(\mathcal{S})$ of an open subset \mathcal{S} of \mathbb{R}^n is an injective immersion,
- and a converging observer of the type:

$$\dot{\hat{\xi}} = \varphi(\hat{\xi}, y) \quad , \quad \hat{x} = \tau(\hat{\xi}) \quad , \quad (2)$$

where the functions φ and τ are continuous, the latter being a left inverse of τ^* .

Convergence being assumed, we concentrate our attention on the implementation aspect. A difficulty is to compute $\tau(\hat{\xi})$ and [9] proposes to round this difficulty by extending τ^* into a diffeomorphism $\tau_e^* : \mathcal{S} \times \mathcal{S}_w \rightarrow \mathbb{R}^m$, thus adding $m - n$ dimensions to the state through a new variable w . Indeed, under some basic assumptions given in [9, Prop1], we obtain the convergence of the observer:

$$\begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix} = \left(\frac{\partial \tau_e^*}{\partial (\hat{x}, \hat{w})}(\hat{x}, \hat{w}) \right)^{-1} \varphi(\tau_e^*(\hat{x}, \hat{w}), y) \quad . \quad (3)$$

The left-inversion of an injective function is thus replaced by a simple matrix inversion. However, at least two questions arise :

- how to manage the extension of an injective immersion τ^* on \mathcal{S} into a diffeomorphism τ_e^* on $\mathcal{S} \times \mathcal{S}_w$?
- what is the domain of existence of the solutions to observer (3) ?

[9, Prop 2] states that the first can be achieved through continuous completion of the Jacobian of τ^* , in so far as the function defined on $\mathcal{S} \times \mathcal{S}_w$ by

$$\tau_e^*(x, w) = \tau^*(x) + \psi(x) w \quad (4)$$

where $\psi(x)$ is a $m * (m - n)$ matrix verifying

$$\det \left(\frac{\partial \tau^*}{\partial x}(x) \quad \psi(x) \right) \neq 0 \quad \forall x \in \text{cl}(\mathcal{S}) \quad (5)$$

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meets the right conditions when w is small enough. Unfortunately, there does not always exist a universal solution to the Jacobian completion ([13]).

As for the second issue, although we may know the system trajectories remain in $\mathcal{S} \times \mathcal{S}_w$ where the Jacobian is invertible, we have no guarantee the ones of the observer do. Therefore, we must find means to ensure the estimate do not leave this set in order to obtain convergence and completeness of solutions. To achieve this goal, a possible approach is the extension of the image of the diffeomorphism τ_e^* . Sufficient conditions for such a design are listed in Theorem 2.

When these conditions are not satisfied, a possible solution is to modify the given immersion τ^* and redesign the observer. For doing so, we propose the trick of adding fictitious outputs since it may enable to remove observability singularities, thus extending the domain of existence of the solutions.

In Section II, we present our theoretical tools. We illustrate their use with the example of the oscillator in Section III when the given observer (2) is a high gain observer and a nonlinear Luenberger observer. We redo this in Section IV with a redesigned τ^* in order to address a possible loss of observability.

II. TOOLS PRESENTATION

Before entering our illustration, we give in this section a brief presentation of the Jacobian completion tool we shall use. We also state the diffeomorphism extension result.

A. Wasevski theorem and Jacobian completion

In our case, the question of Jacobian completion can be formalized as :

Given $m \times n$ functions φ_{ij} which are continuous in the n -dimensional variable x in $\text{cl}(\mathcal{S})$, look for $m \times (m - n)$ functions ψ_{kl} which are continuous on $\text{cl}(\mathcal{S})$ and such that the following matrix is invertible for all x in $\text{cl}(\mathcal{S})$:

$$P(x) = \begin{pmatrix} \varphi(x) & \psi(x) \end{pmatrix} . \quad (6)$$

We have :

Theorem 1 ([12, Theorems 1 and 3] and [13, page 127]): On any subset of $\text{cl}(\mathcal{S})$ which, equipped with the subspace topology of \mathbb{R}^n , is a contractible space, there exist continuous functions ψ_{kl} making invertible the matrix P in (6).

Actually, in some very particular cases, listed in [13], [9], the ψ_{kl} can be expressed in terms of the φ_{ij} via universal formulae. Examples will be given in the following.

B. Diffeomorphism extension

As mentioned in the introduction, a second issue may arise if we take a closer look at the time-domain of definition of maximal solutions of system (3). The estimated system state is obtained by applying the inverse of τ_e^* to the observer-state. Therefore, if we want the estimation to remain in $\mathcal{S} \times \mathcal{S}_w$, where injectivity and Jacobian invertibility are ensured, the observer-state must live in $\tau_e^*(\mathcal{S} \times \mathcal{S}_w)$. Unfortunately,

if $\tau_e^*(\mathcal{S} \times \mathcal{S}_w)$ is a strict subset of \mathbb{R}^m , the trajectories of the convergent observer in the image space may leave $\tau_e^*(\mathcal{S} \times \mathcal{S}_w)$, and the Jacobian may lose its invertibility, thus preventing the convergence of (3). A solution in this case, is to extend the image of the diffeomorphism τ_e^* to make it cover \mathbb{R}^m .

Let us introduce the following property:

An open subset E of \mathbb{R}^m is said to verify condition \mathbb{B} if there exist a C^1 function $\kappa : \mathbb{R}^m \rightarrow \mathbb{R}$, a bounded C^1 vector field χ , and a closed set K_0 contained in E such that:

- 1) $E = \{x \in \mathbb{R}^m, \kappa(x) < 0\}$
- 2) K_0 is globally attractive for χ
- 3) we have the following transversality assumption:

$$\frac{\partial \kappa}{\partial x}(x)\chi(x) < 0 \quad \forall x \in \mathbb{R}^m : \kappa(x) = 0.$$

The possibility of an image extension is given in the following theorem which is proved in [10] :

Theorem 2: Let $\psi : \mathcal{D} \subset \mathbb{R}^m \rightarrow \psi(\mathcal{D}) \subset \mathbb{R}^m$ be a diffeomorphism. If $\psi(\mathcal{D})$ verifies condition \mathbb{B} or \mathcal{D} is C^2 -diffeomorphic to \mathbb{R}^m and ψ is C^2 , then for any compact set K in \mathcal{D} there exists a diffeomorphism $\psi_e : \mathcal{D} \rightarrow \mathbb{R}^m$ satisfying :

- $\psi_e(\mathcal{D}) = \mathbb{R}^m$
- $\psi_e(x) = \psi(x) \quad \forall x \in K.$

Note that for \mathcal{D} to be diffeomorphic to \mathbb{R}^m , it must be contractible.

If $\mathcal{S} \times \mathcal{S}_w$ verifies the assumptions of Theorem 2, it is possible to extend τ_e^* into $\overline{\tau_e^*} : \mathcal{S} \times \mathcal{S}_w \rightarrow \mathbb{R}^m$, and the maximal solutions of

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{w}} \end{bmatrix} = \left(\frac{\partial \overline{\tau_e^*}}{\partial (\hat{x}, \hat{w})}(\hat{x}, \hat{w}) \right)^{-1} \varphi(\overline{\tau_e^*}(\hat{x}, \hat{w}), y) . \quad (7)$$

are defined for all $t \geq 0$ if the same holds for (1)-(2). However, for the time being we have no complete explicit algorithm to find an expression for such a diffeomorphism. Fortunately, as shown in the illustration, it is possible (in some specific cases) to construct a global diffeomorphism without relying on the above theorem. See Section IV-A.

III. ILLUSTRATION

We enter now the core of this paper with an illustration of the technicalities summarized above. Our intent is to show step by step what could be a systematic construction of observers. We consider the oscillator with unknown frequency, which combines both simplicity in terms of computations, and richness in terms of underlying observability issues :

$$\begin{cases} \dot{x}_1 = x_2 , \\ \dot{x}_2 = -x_1 x_3 , \\ \dot{x}_3 = 0 , \\ y_1 = x_1 \end{cases} \quad (8)$$

When x_3 is non negative, the solution going through $x = (x_1, x_2, x_3)$ at time 0 is :

$$\begin{aligned} X_1(x, t) &= x_1 \cos(\sqrt{x_3}t) + \frac{x_2}{\sqrt{x_3}} \sin(\sqrt{x_3}t) , \\ X_2(x, t) &= -x_1 \sqrt{x_3} \sin(\sqrt{x_3}t) + x_2 \cos(\sqrt{x_3}t) , \\ X_3(x, t) &= x_3 . \end{aligned}$$

So, from the only knowledge of the system dynamics and the function $t \mapsto y(t) = X_1(x, t)$, we cannot get any information on x_3 if we have $x_1^2 + x_2^2 = 0$. This motivates us for restricting our attention to the set

$$\mathcal{S} = \{x \in \mathbb{R}^3 : x_3 \geq 0, x_3 x_1^2 + x_2^2 \neq 0\}.$$

It is invariant and, for any s and any strictly positive real number σ , the function

$$x \in \mathcal{S} \mapsto (t \in [s, s + \sigma] \mapsto y(t) = X_1(x, t) \in \mathbb{R}),$$

which associates to x the path of the measured X_1 component of the solution, observed on the time interval $[s, s + \sigma]$, is injective. Namely, the system is instantaneously observable on \mathcal{S} . Actually it is strongly differentially observable of order 4 on this set. Indeed, it can be checked that the function :

$$x \mapsto H_4(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ L_f^2 h(x) \\ L_f^3 h(x) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ -x_1 x_3 \\ -x_2 x_3 \end{pmatrix}$$

is an injective immersion on \mathcal{S} .

We conclude from this analysis that an observer can be obtained by invoking the high gain observer or the nonlinear Luenberger observer design techniques.

To illustrate the coordinate augmentation method and its possible difficulties, we will consecutively build both, and, in each case, we will endeavor to highlight the different ideas or "tricks", which the reader could find useful in other applications.

A. 4th order high gain observer

Inspired by the previous observability analysis, we consider the following function with values in \mathbb{R}^4 :

$$\tau^*(x) = (x_1, x_2, -x_1 x_3, -x_2 x_3), \quad (9)$$

which is an injective immersion on \mathcal{S} , and build the high gain observer :

$$\dot{\hat{\xi}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \hat{\xi} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \text{sat}(\hat{x}_1 \hat{x}_3^2) \end{pmatrix} + \begin{pmatrix} \ell k_1 \\ \ell^2 k_2 \\ \ell^3 k_3 \\ \ell^4 k_4 \end{pmatrix} [y - \hat{\xi}_1] \quad (10)$$

where sat is an appropriate saturation function, ℓ is to be chosen large enough and the k_i are so that the following polynomial is Hurwitz :

$$\lambda^4 + k_1 \lambda^3 + k_2 \lambda^2 + k_3 \lambda + k_4.$$

Due to the difference of dimensions between the x -system and the ξ -system, we can proceed to a coordinate augmentation of the x -system, through Jacobian completion as in [9]. For example, we consider the C^1 extension $\tau_e^* : \mathbb{R}^4 \rightarrow \mathbb{R}^4$:

$$\tau_e^*(x, w) = (x_1, x_2, -x_1 x_3 + x_2 w, -x_2 x_3 - x_1 w)$$

which is full rank on $\mathcal{S} \times \mathbb{R}$ and whose inverse function is C^1 on $\mathbb{R}^4 \setminus (\{(0, 0)\} \times \mathbb{R}^2)$ and defined by :

$$\tau_e(\xi) = \left(\xi_1, \xi_2, -\frac{\xi_1 \xi_3 + \xi_2 \xi_4}{\xi_1^2 + \xi_2^2}, \frac{\xi_2 \xi_3 - \xi_1 \xi_4}{\xi_1^2 + \xi_2^2} \right).$$

Note that we have Lipschitzness on the compact image by τ_e^* of :

$$\left\{ (x, w) \in \mathbb{R}^4 : x_1^2 + x_2^2 \in \left[\frac{1}{r}, r\right], x_3 \in [0, r], w \in [-r, r] \right\},$$

$r > 0$. It follows that a possible implementation of the observer (10) is :

$$\begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \\ \dot{\hat{w}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\hat{x}_3 & \hat{w} & -\hat{x}_1 & \hat{x}_2 \\ -\hat{w} & -\hat{x}_3 & -\hat{x}_2 & -\hat{x}_1 \end{pmatrix}^{-1} \times \left[\begin{pmatrix} \hat{x}_2 \\ -\hat{x}_1 \hat{x}_3 + \hat{x}_2 \hat{w} \\ -\hat{x}_2 \hat{x}_3 - \hat{x}_1 \hat{w} \\ \text{sat}(\hat{x}_1 \hat{x}_3^2) \end{pmatrix} + \begin{pmatrix} \ell k_1 \\ \ell^2 k_2 \\ \ell^3 k_3 \\ \ell^4 k_4 \end{pmatrix} [y - \hat{x}_1] \right].$$

Notice that here, the triangular structure of the Jacobian makes it invertible for any w in \mathbb{R} , and the added state has therefore no impact on the convergence. However, although we know the real state remains in \mathcal{S} , we have no guarantee that the estimated state will. In particular, (\hat{x}_1, \hat{x}_2) may go through the observability singularity $(0, 0)$, where the Jacobian is not invertible. We will see in Section IV how to avoid this problem.

B. 4th order Luenberger observer

Instead of a high gain observer we can choose a nonlinear Luenberger observer whose dynamics are simply :

$$\dot{\xi}_i = -\lambda_i \xi_i + y,$$

where λ_i are complex numbers with strictly positive real parts and i is in $\{1, \dots, m\}$. If one can find an injective function τ^* such that $\xi = \tau^*(x)$ is invariant, i-e such that for all i in $\{1, \dots, m\}$, τ_i^* satisfies the following partial differential equation :

$$\frac{\partial \tau_i^*}{\partial x}(x) f(x) = -\lambda_i \tau_i^*(x) + y,$$

then ξ converges exponentially to $\tau^*(x)$, and one can estimate x from ξ by left inversion of τ^* (see [8] for example).

Coming back to the oscillator, straight-forward computation gives:

$$\tau_i^*(x) = \frac{\lambda_i x_1 - x_2}{\lambda_i^2 + x_3}. \quad (11)$$

It is shown in [11] that the injectivity on \mathcal{S} is achieved for $m \geq 4$ with λ_i 's in \mathbb{R} . Notice that in this case, the singularity at $x_1 = x_2 = 0$ remains, and other singularities appear for x_3 . We obtain the following Jacobian :

$$\frac{\partial \tau^*}{\partial x}(x) = \begin{pmatrix} \frac{\lambda_1}{\lambda_1^2 + x_3} & -\frac{1}{\lambda_1^2 + x_3} & -\frac{\tau_1^*(x)}{\lambda_1^2 + x_3} \\ \frac{\lambda_2}{\lambda_2^2 + x_3} & -\frac{1}{\lambda_2^2 + x_3} & -\frac{\tau_2^*(x)}{\lambda_2^2 + x_3} \\ \frac{\lambda_3}{\lambda_3^2 + x_3} & -\frac{1}{\lambda_3^2 + x_3} & -\frac{\tau_3^*(x)}{\lambda_3^2 + x_3} \\ \frac{\lambda_4}{\lambda_4^2 + x_3} & -\frac{1}{\lambda_4^2 + x_3} & -\frac{\tau_4^*(x)}{\lambda_4^2 + x_3} \end{pmatrix}$$

Once again, the difference of state dimension between the system and the observer makes it difficult to left invert τ^* .

Therefore, we proceed with a Jacobian completion and we add a new state component w . The completion is quite easy in this case because there is only one dimension to add : we just add a column $\psi(x)$ consisting of the corresponding minors (see [9]).

For all $r > 0$ there exists $\varepsilon(r) > 0$ such that the function $\tau_e^*(x, w) = \tau^*(x) + \psi(x)w$ (see (4)) is a diffeomorphism on the open bounded set

$$C = \left\{ (x, w) \in \mathbb{R}^4 : x_1^2 + x_2^2 \in \left(\frac{1}{r}, r\right), \right. \\ \left. x_3 \in (0, r), w \in (-\varepsilon(r), \varepsilon(r)) \right\}. \quad (12)$$

It leads to the observer :

$$\begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \\ \dot{\hat{w}} \end{pmatrix} = \left(\frac{\partial \tau_e^*}{\partial x}(\hat{x}, \hat{w}) \right)^{-1} (-\lambda \tau_e^*(\hat{x}, \hat{w}) + By)$$

where $B = [1, 1, 1, 1]^T$ and $\lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. We do not give an explicit expression of the extended Jacobian because of its complexity.

As before, the observability singularity $\hat{x}_1 = \hat{x}_2 = 0$ remains. We will see in Section IV how to avoid this issue. However, in this case, two other problems appear : due to the absence of triangular structure, w must be small for the Jacobian to be invertible (whereas it was invertible for all w in the high gain observer case) ; besides, \hat{x}_3 must stay away from $\{-\lambda_i^2\}$. Therefore, to obtain the convergence of this observer, we would have to ensure (\hat{x}, \hat{w}) remains in C defined in (12). This is precisely the kind of situation where the diffeomorphism extension of Theorem 2 is useful. Unfortunately, the set C where τ_e^* is a diffeomorphism is not contractible and the theorem does not apply. We round this problem by modifying the immersion τ^* , from which τ_e^* is obtained.

IV. REMOVAL OF AN OBSERVABILITY SINGULARITY THROUGH ADDITION OF A FICTITIOUS OUTPUT

In this section, we want to remove the observability singularity for $x_1 = x_2 = 0$, both for the high gain observer and the Luenberger observer.

A. 6th order high gain observer

The problem we have when $x_1 = x_2 = 0$ is that we lose observability of x_3 . When we know the system solutions stay away from this singularity, precisely they remain in the set

$$\mathcal{S}_r = \left\{ x \in \mathbb{R}^3 : x_1^2 + x_2^2 \in \left[\frac{1}{r}, r\right], x_3 \in [0, r] \right\}, \quad (13)$$

for some known r , we can round this problem by adding the fictitious output:

$$y_2 = \psi(x_1, x_2) x_3$$

where

$$\psi(x_1, x_2) = \max \left\{ 0, \frac{1}{r} - (x_1^2 + x_2^2) \right\}^2.$$

The interest of y_2 is to give us access to x_3 directly.

And, for all system solutions remaining in the set \mathcal{S}_r , the corresponding value of y_2 is known to be 0.

This motivates us for replacing the former function τ^* given in (9) by the following one based on $(y_1, \dot{y}_1, \ddot{y}_1, y_2)$, i-e

$$\tau^*(x) = (x_1, x_2, -x_1 x_3, -x_2 x_3, \psi(x_1, x_2) x_3). \quad (14)$$

τ^* is C^1 on \mathbb{R}^3 and its Jacobian is :

$$\frac{\partial \tau^*}{\partial x}(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_3 & 0 & -x_1 \\ 0 & -x_3 & -x_2 \\ \frac{\partial \psi}{\partial x_1} x_3 & \frac{\partial \psi}{\partial x_2} x_3 & \psi \end{pmatrix},$$

which has full rank 3 on \mathbb{R}^3 , since $\psi(x_1, x_2) \neq 0$ when $x_1 = x_2 = 0$. It follows that the singularity has disappeared and τ^* is an injective immersion on \mathbb{R}^3 .

We can replace the former high gain observer of dimension 4 given in (10) by the following one of dimension 5 corresponding to the new τ^* in (14) :

$$\begin{cases} \dot{\xi}_1 = \xi_2 - \ell k_1 (\xi_1 - y_1) \\ \dot{\xi}_2 = \xi_3 - \ell^2 k_2 (\xi_1 - y_1) \\ \dot{\xi}_3 = \xi_4 - \ell^3 k_3 (\xi_1 - y_1) \\ \dot{\xi}_4 = \text{sat}(\xi_1 \hat{x}_3^2) - \ell^4 k_4 (\xi_1 - y_1) \\ \dot{\xi}_5 = -a \xi_5, \end{cases} \quad (15)$$

where sat are appropriate saturation functions for x in \mathcal{S}_r and a is a strictly positive number.

In order to obtain a diffeomorphism, we augment the coordinates according to the method presented in [9]. However, the above Jacobian does not fit into any cases where a universal formula exists. Therefore, for mere practicality in the Jacobian completion, we add a third output

$$y_3 = 0$$

and thus a state ξ_6 in the observer with the dynamic

$$\dot{\xi}_6 = -b \xi_6. \quad (16)$$

The immersion becomes

$$\tau^*(x) = (x_1, x_2, -x_1 x_3, -x_2 x_3, \psi(x_1, x_2) x_3, 0).$$

which is still injective and C^1 on \mathbb{R}^3 and its Jacobian

$$\frac{\partial \tau^*}{\partial x}(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_3 & 0 & -x_1 \\ 0 & -x_3 & -x_2 \\ \frac{\partial \psi}{\partial x_1} x_3 & \frac{\partial \psi}{\partial x_2} x_3 & \psi \\ 0 & 0 & 0 \end{pmatrix},$$

is still of rank 3 on \mathbb{R}^3 . Completing this Jacobian, we obtain the extension $\tau_e^* : \mathbb{R}^6 \rightarrow \mathbb{R}^6$:

$$\tau_e^*(x, w) = \left(x_1, x_2, \right. \\ \left. [-x_1 x_3 + x_2 w_1 - \psi(x_1, x_2) w_2], \right. \\ \left. [-x_2 x_3 - x_1 w_1 - \psi(x_1, x_2) w_3], \right. \\ \left. [\psi(x_1, x_2) x_3 - x_1 w_2 - x_2 w_3], \right. \\ \left. [\psi(x_1, x_2) w_1 + x_2 w_2 - x_1 w_3] \right). \quad (17)$$

It is C^1 and its Jacobian is :

$$\frac{\partial \tau_e^*}{\partial x}(x, w) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -x_3 - \frac{\partial \psi}{\partial x_1} w_2 & w_1 - \frac{\partial \psi}{\partial x_2} w_2 & -x_1 & x_2 & -\psi & 0 \\ -w_1 - \frac{\partial \psi}{\partial x_1} w_3 & -x_3 - \frac{\partial \psi}{\partial x_2} w_3 & -x_2 & -x_1 & 0 & -\psi \\ \frac{\partial \psi}{\partial x_1} x_3 - w_2 & \frac{\partial \psi}{\partial x_2} x_3 - w_3 & \psi & 0 & -x_1 & -x_2 \\ \frac{\partial \psi}{\partial x_1} w_1 - w_3 & \frac{\partial \psi}{\partial x_2} w_1 + w_2 & 0 & \psi & x_2 & -x_1 \end{pmatrix}$$

It follows that τ_e^* is full rank on \mathbb{R}^6 . Its inverse function is defined on \mathbb{R}^6 , C^1 and given by :

$$\tau_e(\xi) = \left(\xi_1, \xi_2, \frac{\xi_1 \xi_3 + \xi_2 \xi_4 - \psi(\xi_1, \xi_2) \xi_5}{\xi_1^2 + \xi_2^2 + \psi(\xi_1, \xi_2)^2}, \frac{\xi_2 \xi_3 - \xi_1 \xi_4 + \psi(\xi_1, \xi_2) \xi_6}{\xi_1^2 + \xi_2^2 + \psi(\xi_1, \xi_2)^2}, \frac{\xi_2 \xi_6 - \xi_1 \xi_5 - \psi(\xi_1, \xi_2) \xi_3}{\xi_1^2 + \xi_2^2 + \psi(\xi_1, \xi_2)^2}, \frac{\xi_2 \xi_5 + \xi_1 \xi_6 + \psi(\xi_1, \xi_2) \xi_4}{\xi_1^2 + \xi_2^2 + \psi(\xi_1, \xi_2)^2} \right).$$

It follows that a possible implementation of the observer (15) augmented with (16) is :

$$\begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \\ \dot{\hat{w}}_1 \\ \dot{\hat{w}}_2 \\ \dot{\hat{w}}_3 \end{pmatrix} = \left(\frac{\partial \tau_e^*}{\partial x}(\hat{x}, \hat{w}) \right)^{-1} \times \begin{pmatrix} \hat{\xi}_2 + \ell k_1 (y - \hat{x}_1) \\ \hat{\xi}_3 + \ell^2 k_2 (y - \hat{x}_1) \\ \hat{\xi}_4 + \ell^3 k_3 (y - \hat{x}_1) \\ \text{sat}(\hat{x}_1 \hat{x}_3^2) + \ell^4 k_4 (y - \hat{x}_1) \\ -a \hat{\xi}_5 \\ -b \hat{\xi}_6 \end{pmatrix}$$

where $\hat{\xi} = \tau_e^*(\hat{x}, \hat{w})$ with τ_e^* given in (17).

Again, the block triangular structure enables to obtain a Jacobian whose invertibility does not depend on the added states (w_1, w_2, w_3) . This differs from the Luenberger observer to be presented next. Besides, as announced, the observability singularity has disappeared and this observer converges even when initialized at $\hat{x}_1 = \hat{x}_2 = 0$. Therefore, by exploiting the knowledge that the system solutions remain in \mathcal{S}_r , we have managed to ensure convergence whatever the initial conditions and the observer trajectories are.

B. 6th order Luenberger observer

Similarly, let us now see how to remove the observability singularity in the Luenberger observer. Again, we assume the system solutions remain in \mathcal{S}_r given in (13) and add a fictitious output y_2 which is different from zero when (x_1, x_2) is close to the origin:

$$y_2 = \psi(x_1, x_2, x_3) x_3$$

where

$$\psi(x_1, x_2, x_3) = \max \left\{ 0, \frac{1}{r} - (x_1^2 x_3 + x_2^2) \right\}^2.$$

Notice that this time, the output is taken as a function of the invariant $x_1^2 x_3 + x_2^2$. Then, with letting

$$\tau_5^*(x) = \frac{1}{\mu} \psi(x_1, x_2, x_3)^2 x_3,$$

we obtain :

$$\overline{\tau_5^*(x)} = -\mu \tau_5^*(x) + y_2.$$

This motivates for selecting the function τ^* as

$$\tau^*(x_1, x_2, x_3) = (\tau_1^*(x), \tau_2^*(x), \tau_3^*(x), \tau_4^*(x), \tau_5^*(x))$$

where $\tau_i^*(x)$ for $i = 1 \dots 4$ are given in (11). It is injective on

$$\mathcal{S}_e = \mathbb{R}^2 \times \mathbb{R}_+.$$

The observability singularity has disappeared and we obtain the following Jacobian :

$$\frac{\partial \tau^*}{\partial x}(x) = \begin{pmatrix} \frac{\lambda_1}{\lambda_1^2 + x_3} & -\frac{1}{\lambda_1^2 + x_3} & -\frac{\tau_1^*(x)}{\lambda_1^2 + x_3} \\ \frac{\lambda_2}{\lambda_2^2 + x_3} & -\frac{1}{\lambda_2^2 + x_3} & -\frac{\tau_2^*(x)}{\lambda_2^2 + x_3} \\ \frac{\lambda_3}{\lambda_3^2 + x_3} & -\frac{1}{\lambda_3^2 + x_3} & -\frac{\tau_3^*(x)}{\lambda_3^2 + x_3} \\ \frac{\lambda_4}{\lambda_4^2 + x_3} & -\frac{1}{\lambda_4^2 + x_3} & -\frac{\tau_4^*(x)}{\lambda_4^2 + x_3} \\ A(x_1, x_2, x_3) & B(x_1, x_2, x_3) & C(x_1, x_2, x_3) \end{pmatrix}$$

Once again, we need to achieve an extension of dimension through Jacobian completion. Since, as above we are not in any of the specific cases given in [9], we add a null output

$$y_3 = 0,$$

and the dynamics

$$\dot{\xi}_6 = -\gamma \xi_6$$

to the observer. Then it can be checked that the completion is possible after a series of invertible transformations. Indeed, we can first simplify the matrix by multiplying the first four lines by $\lambda_i^2 + x_3$. Then, left-multiplying the obtained matrix by the block matrix

$$\begin{pmatrix} \mathcal{D}^{-1}(\lambda_i) & 0_{4 \times 2} \\ 0_{2 \times 4} & I_{2 \times 2} \end{pmatrix}$$

where $\mathcal{D}^{-1}(\lambda_i)$ is the inverse of a Vandermonde matrix associated to the λ_i , we manage to change the matrix into

$$\begin{pmatrix} 1 & 0 & m_1 \\ 0 & 1 & m_2 \\ 0 & 0 & m_3 \\ 0 & 0 & m_4 \\ A(x_1, x_2, x_3) & B(x_1, x_2, x_3) & C(x_1, x_2, x_3) \\ 0 & 0 & 0 \end{pmatrix},$$

which we can complete using permutations (case $(n = 1, m$ even), see [9]) and the formula:

$$\det(M) = \det(Z - XY)$$

when M is in the block form

$$M = \begin{pmatrix} I_2 & X \\ Y & Z \end{pmatrix}.$$

Reversing the transformations, we thus manage to extend the Jacobian of τ^* into a matrix of dimension 6 whose determinant is non-zero on \mathcal{S}_e . Applying the method of [9], we add three state components to the system state. We can get an expression for a function ψ , such that, for any open bounded subset of \mathcal{S}_e containing \mathcal{S}_r , there exists an open ball B_ε , centered at the origin in \mathbb{R}^3 , such that τ_e^* , given in (4), is a diffeomorphism on

$$C = \mathcal{S}_e \times B_\varepsilon. \quad (18)$$

See [9, Prop 2]. With this, we get the observer :

$$\begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \\ \dot{\hat{w}}_1 \\ \dot{\hat{w}}_2 \\ \dot{\hat{w}}_3 \end{pmatrix} = \left(\frac{\partial \tau_e^*}{\partial x}(\hat{x}, \hat{w}) \right)^{-1} (-\lambda \tau_e^*(\hat{x}, \hat{w}) + B_1 y_1) \quad (19)$$

where :

$$B_1 = [1, 1, 1, 1, 0, 0]^T, \quad \lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \mu, \gamma)$$

and γ is the eigenvalue chosen for ξ_6 . We do not give the Jacobian of the extended function due to its complexity, the computations themselves can be done with symbolic computations.

We have removed the singularity at $(\hat{x}_1, \hat{x}_2) = 0$ from the observer, but we still have to ensure that \hat{x}_3 remains positive, or at least greater than $-\min\{\lambda_i^2\}$. Besides, unlike the high gain observer, the invertibility of the extended Jacobian is only guaranteed for w in B_ε . More precisely, the estimate (\hat{x}, \hat{w}) must remain in C defined in (18). If C is a set verifying the conditions of Theorem 2, we can find an extension $\overline{\tau}_e^*$ whose image is \mathbb{R}^6 and which matches τ_e^* almost everywhere (in the sense of the theorem). Replacing τ_e^* by $\overline{\tau}_e^*$ in (19) would give a convergent observer whose solutions are defined for all $t \geq 0$.

V. CONCLUSION

This paper follows [9] which provided a new method for implementing, without immersion inversion, observers involving a dynamic extension. Nevertheless, a few difficulties can arise and we have presented via an illustration some tools and techniques to possibly round them. In particular, we have to keep the observer state in the image of a diffeomorphism to avoid Jacobian singularities. To do so, a fix is to extend its image to the whole space. We gave sufficient conditions for this in Theorem 2. But for the time being, we do not have a systematic construction to obtain an explicit expression for this extension.

The interested reader will find more details both theoretical and practical in [10].

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