Energy Level Stabilization of Pendulum on a Cart with Restricted Cart Track Based on Elliptic Functions and Integrals

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Abstract:
This study presents an energy level stabilization algorithm for the pendulum on a cart system with restricted cart track length. The objective is to bring the pendulum to its unstable equilibrium. To do so the energy level of the pendulum is increased or decreased by accelerating the cart in the appropriate direction while keeping within imposed limit positions. To achieve, the equation of motion of the pendulum is solved by means of elliptic integrals considering the cart is moved with constant acceleration, and all the calculations are performed numerically to obtain the bounds for elapsed time and change of the energy level of the pendulum. The algorithm is tested on the system by means of numerical simulations.

1. INTRODUCTION

The stabilization of the pendulum on a cart system at the unstable equilibrium point of the pendulum is a well-known problem in the area of control theory. In general, this problem is solved by making the cart move on a linear track in such a way that it forces the pendulum to swing up. Since the system is underactuated and losses linear controllability property when the pendulum is at its horizontal configuration, many well-known nonlinear controllers can not be applied directly to this particular system. On the other hand, linear controllers are only able to stabilize the system in a small interval in the neighborhood of the unstable equilibrium.

The structure of the nonlinear controllers developed for dealing with the pendulum on a cart system can be split in two main groups. The first ones aim at asymptotically stabilizing the pendulum at its upward equilibrium. They are obtained by exploiting local properties only. The second ones aim at driving the pendulum close to its upright position. They are typically energy based methods and their objective is to drive the energy level of the pendulum to a predefined energy level which corresponds to the upward equilibrium of the pendulum.

Among the control structures to swing up the pendulum, [1] gives the general properties of the energy based approaches and compares several different type controllers that swing up the pendulum from its hanging position. A passivity based continuous controller structure presented in [2] provides to swing up the pendulum with the convergence to the unstable equilibrium. More recently, another solution is proposed in [3] for the global stabilization problem of partially linearized pendulum on a cart system which possesses a switching in the control structure. However, the motion of the cart is not analyzed in detail and the cart track length is assumed to be sufficiently long in these studies. To be more realistic, some algorithms have been developed in which the travel of the cart was also considered. A stabilization algorithm is presented in [4] that also bounds cart travel distance from above in order not to allow the pendulum to fall while it is in the upper half plane. Another study, [5], swings up the pendulum from hanging position with restricted cart track length. However, the proposed controller is not global in that study and it is defined only if the cart stands in between the limit positions. On the other hand, a globally stabilizing controller algorithm is given in [6]. In this algorithm, the motion of the cart is determined by the trajectory of the pendulum. In order to provide the energy increase or decrease, the sufficient time for the pendulum to reach to horizontal configuration or to zero velocity is shown to be bounded, and this bound is utilized to determine the desired acceleration for the cart.

An energy level stabilization algorithm for the pendulum on a cart is given in this paper. The cart is forced to move with a constant acceleration (and deceleration) to change the energy level of the pendulum. The time needed for the pendulum to reach its horizontal configuration or to zero angular velocity is calculated numerically by means of elliptic integral of first kind. This calculated time is compared to the time for the cart to reach its limit position in order to make the decision on forcing it to move. In addition to that, the change on energy level of the pendulum is also calculated by means of elliptic integral of first kind and Jacobi elliptic function 'sn' before accelerating the cart which helps to make the energy level convergent to the desired energy level.

The remainder of the paper is organized as follows. Section 2 gives the mathematical model of the pendulum on a cart system considered in this study, and before describing

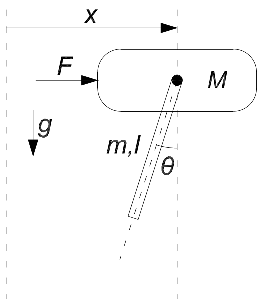

Fig. 1. Pendulum on a cart.

the aim of the paper it also presents some observations derived from mathematical model. Section 3 is devoted to explain the proposed algorithm in detail. Following, illustrative numerical simulations are presented in Section 4. Lastly, Section 5 concludes the paper, and appendices give a brief information on elliptic integral of first kind and Jacobi elliptic function 'sn', and their utilization to obtain the numerical solution to the pendulum dynamics.

2. MATHEMATICAL MODEL AND PROBLEM STATEMENT

The pendulum on a cart system considered in this study is depicted in Fig. 1. Total energy of the pendulum on a cart can be given as,

$$E_{pc} = \frac{1}{2}\left(a\dot{\theta}^2 - 2b\cos(\theta)\dot{\theta}\dot{x} + c\dot{x}^2\right) + bg(1-\cos(\theta)) \quad (1)$$

where $a = I + \frac{1}{4}ml^2$, $b = \frac{1}{2}ml$ and $c = M + m$, θ is the pendulum angle on clock-wise from the bottom and x is the cart position, and $\dot{\theta}, \dot{x}$ are angular velocity of the pendulum and velocity of the cart. I, m, l, M and g denote inertia, mass and length of the pendulum, mass of the cart and gravitational acceleration, respectively. Dynamic equations of motion can be obtained by applying Euler-Lagrange equation as

$$a\ddot{\theta} - b\cos(\theta)\ddot{x} + bg\sin(\theta) = 0 \quad (2)$$

$$-b\cos(\theta)\ddot{\theta} + c\ddot{x} + b\sin(\theta)\dot{\theta}^2 = F \quad (3)$$

where F is the force acting on the cart. Note that from (2), the $(\theta, \dot{\theta})^T$ subsystem is independent from $(x, \dot{x})^T$ subsystem by considering $\ddot{x}$ term in (2) as a virtual control input (u_0) to the $(\theta, \dot{\theta})^T$ subsystem [5] (see also [1, 4]). By setting,

$$F = \frac{b^2 g}{a}\sin(\theta)\cos(\theta) + b\sin(\theta)\dot{\theta}^2 + \left(c - \frac{b^2}{a}\cos^2(\theta)\right)u_0 \quad (4)$$

the system equations can be rewritten as,

$$\ddot{\theta} = \frac{b}{a}\cos(\theta)u_0 - \frac{bg}{a}\sin(\theta) \quad (5)$$

$$\ddot{x} = u_0 \quad (6)$$

where u_0 is the new control input. Note that the coefficient of u_0 in (4) is $\left(M + \frac{mI + \frac{1}{4}m^2l^2(1-\cos^2(\theta))}{I + \frac{1}{4}m^2l^2}\right)$ and is therefore always strictly positive. On the other hand, total energy of the pendulum can be given as,

$$E_{pen} = \frac{1}{2}a\dot{\theta}^2 + 2bg\sin^2\left(\frac{\theta}{2}\right). \quad (7)$$

Utilizing (5), the time derivative of total energy of the pendulum can be calculated as,

$$\dot{E}_{pen} = b\cos(\theta)\dot{\theta}u_0. \quad (8)$$

In order to change the energy level of the pendulum, u_0 can be selected as

$$u_0 = u \cdot \mathrm{sign}(\cos(\theta)\dot{\theta}). \quad (9)$$

For the sake of simplicity, we impose here u is a bounded constant. Then, the time derivative of total energy function takes the form of

$$\dot{E}_{pen} = bu\left|\cos(\theta)\dot{\theta}\right|. \quad (10)$$

Notice that, the maximum value for $\dot{E}_{pen}$ occurs at $\cos(\theta) = 1$ which means the transferred energy from cart to pendulum takes its maximum or minimum value around $\cos(\theta) = 1$ when the constant $u \neq 0$. Therefore, the most efficient way to increase or decrease the energy level of the pendulum is to act when the pendulum is around down vertical. Accordingly, the change of the energy level of the pendulum while applying the control input given in (9) in a time interval (t_1, t_2) can be calculated as

$$E_{pen}(t_2) - E_{pen}(t_1) = bu\left|\sin(\theta_2) - \sin(\theta_1)\right|, \quad (11)$$

unless $\cos(\theta)\dot{\theta}$ changes sign where $\theta_1 = \theta(t_1)$ and $\theta_2 = \theta(t_2)$. On the other hand, constant cart acceleration, u_0, allows us to rewrite (5) as [4]

$$\ddot{\theta} = -\frac{b\sqrt{u_0^2 + g^2}}{a}\sin\left(\theta - \tan^{-1}\left(\frac{u_0}{g}\right)\right). \quad (12)$$

Defining new coordinate

$$\bar{\theta} = \theta - \tan^{-1}\left(\frac{u_0}{g}\right), \quad (13)$$

(12) can be reconstructed as

$$\ddot{\bar{\theta}} = -\frac{b\bar{g}}{a}\sin(\bar{\theta}) \quad (14)$$

where $\bar{g}$ denotes the magnitude of the new gravity like vector. Notice that, not only magnitude but also the direction of that new vector changes depending on the sign of u_0. This representation allows us to determine the time interval elapsed between two distinct points of the pendulum by means of elliptic integral of first kind (see Appendix A and Appendix B). Comparing equations (14) and (5), one can realize that, when the cart is moved with a constant acceleration, the equilibrium points of the pendulum are also changed to

$$\theta_p = \tan^{-1}\left(\frac{u_0}{g}\right). \quad (15)$$

The angle θ_p plays an important role. Its direction with respect to down vertical configuration of the pendulum changes depending on the sign of the constant acceleration u_0. Accordingly, an instant change on the sign of constant acceleration causes a jump on equilibrium point from one side to other.

Our aim here is to stabilize the energy level of the pendulum around up vertical configuration of the pendulum by keeping the cart in the track limits. Once the energy level of the pendulum is around the desired energy level, the controller algorithm can be switched to a local linear controller to keep the pendulum at its unstable equilibrium. To make sure the cart remains between these predefined limits, the cart is accelerated forth and back. This forth and back acceleration is denoted by the term 'cycle', and it is described in two different phases namely pushing (accelerating) and braking (decelerating) phases. One needs to ensure that not only the cart stays between limit points but also the energy level of the pendulum is changed in desired direction before starting to push the cart. To do so, we first define a trajectory for the cart consisting of pushing and braking phases. This trajectory is tracked by the cart synchronized to the motion of the pendulum. In other words, if the trajectory of the pendulum allows the cart to change the energy level of the pendulum in desired direction, then the cart is forced to track this predefined trajectory.

3. PROPOSED CONTROL STRUCTURE

This section presents the stabilizing controller algorithm with brief descriptions of the computations that have to be performed for the algorithm.

3.1 Energy Pumping Algorithm

Energy pumping algorithm is proposed in order to increase the energy level of the pendulum up to the desired energy level while the cart is forced to move forth and back between assigned boundary positions. The algorithm consists of two parts, namely, pushing and braking phases which are explained in the following in details.

Let E_{pen}, ΔE and E_d denote the present pendulum energy, the possible energy increase for the upcoming cycle and the desired energy level, respectively. How ΔE can be computed is discussed in Section 3.2.

Pushing Phase. We denote by θ_0, $\dot{\theta}_0$ and x_0 the pendulum position and velocity and cart position respectively at the time the pushing phase starts. It does with the control $u_0 = u\,\mathrm{sign}(\cos(\theta_0)\dot{\theta}_0)$ if all the conditions are satisfied

(1) $E_{pen} + \Delta E < E_d$,
(2) (a) $\cos(\theta) = 1$
   or
   (b) $t_m \leqslant t_c$,
(3) $x_{lim} - \mathrm{sign}(\cos(\theta)\dot{\theta})x > 0$

where t_m denotes the time duration for the pendulum to reach from θ_0 the horizontal position or to zero velocity, t_c denotes the time for the cart to reach the middle point of its actual and limit positions, and ΔE denotes energy increase at the end of the cycle, and x_{lim} stands for the maximum allowed distance for the cart from its zero position.

Pushing phase continuous until one of the following conditions occurs

(4) $\mathrm{sign}(\cos(\theta)\dot{\theta}) \neq \mathrm{sign}(\cos(\theta_0)\dot{\theta}_0)$,
(5) $|\,\mathrm{sign}(\cos(\theta)\dot{\theta})x_{lim} + x_0| < 2\,\mathrm{sign}(\cos(\theta)\dot{\theta})x$.

Braking Phase. As soon as the pushing phase finishes, the braking phase starts. The corresponding pendulum position is denoted $\theta = \theta_{f1}$. The control is then $u_0 = -u\,\mathrm{sign}(\cos(\theta_0)\dot{\theta}_0)$ and it continuous until the cart stops. θ_{f2} is defined as the pendulum angle when the cart stops.

Remark 1. Condition 1 that has to be satisfied to start the pushing phase aims at increasing the energy level of the pendulum towards the desired energy level. Note that, unless the amount of the possible energy increase for the upcoming cycle is computed, the energy level of the pendulum may be above the desired energy level. Therefore, satisfying condition 1 ensures that the desired energy level will be converged.

Remark 2. Condition 2 that has to be satisfied to start pushing phase consists of two parts. Pushing phase is allowed to start as the pendulum passes through its down vertical configuration. This is the condition 2a. This condition deals with the situation in which $t_c < t_m$ is true for all possible configurations of the pendulum having some particular energy levels. Condition 2b is introduced to prevent the cart from reaching its half way to reach limit position before the pendulum reaches its horizontal position or zero velocity. In Section 3.3, we explain how these terms are computed.

Remark 3. Condition 3 checks if there is a sufficient distance between present and limit cart positions depending on the direction of motion. Note that, if the position of cart is out of the limit positions at the beginning, then the cart is only accelerated to one direction until it enters the allowed interval of motion.

Remark 4. Condition 4 means that either the pendulum has reached the horizontal position or the direction of its velocity changes. Condition 5 means that the cart has reached the the middle point of the interval between the limit position and x_0, position when the pushing phase started. This condition ensures that the cart will be stopped at most at the limit position.

3.2 The Change on the Energy Level of the Pendulum

When the control u is constant, the energy increase or decrease in one cycle can be computed using (11). There t_m and θ_{f1} are computed by means of elliptic integrals consistently, and, θ_{f2} can also be computed accordingly by means of Jacobi elliptic function 'sn' by taking the energy change for the pushing phase into account. Recall that the energy level of the pendulum does not change while the cart acceleration is zero.

3.3 Computation of t_c and t_m

The time duration for the cart to reach the middle (depending on direction of motion) of actual and limit positions is given with

$$t_c = \sqrt{\left|\frac{\mathrm{sign}(\dot{\theta}\cos(\theta))x_{lim} - x}{u}\right|}. \qquad (16)$$

If one is able to compute the time duration t_m for the pendulum to reach the horizontal or to the angle at which $\dot{\theta} = 0$, then this value can be compared to the value obtained by (16) and the decision on starting or not to push can be made. Notice that, for different configurations of the pendulum angle, the pendulum may approach horizontal in various ways. Therefore the evaluation of t_m depends on the pendulum angle and velocity.

3.4 Energy Removing Algorithm

We present here the energy removing algorithm only for the case where the desired energy level is equal to the upward vertical position of the pendulum. This implies that it applies only when the pendulum rotates.

Pushing Phase. We denote by θ_0, $\dot{\theta}_0$ and x_0 the pendulum position and velocity and cart position respectively at the time the pushing phase starts. It does with the control $u_0 = -u\,\text{sign}(\cos(\theta_0)\dot{\theta}_0)$ if all the conditions are satisfied

(1) $E_{pen} + \Delta E > E_d$,
(2) $t_h \leqslant t_c$,
(3) $x_{lim} + \text{sign}(\cos(\theta)\dot{\theta})x > 0$

where t_h denotes the elapsed time for the pendulum to reach from θ_0 the horizontal position.

Pushing phase continuous until one of the following conditions occurs

(4) $\text{sign}(\cos(\theta)\dot{\theta}) \neq \text{sign}(\cos(\theta_0)\dot{\theta}_0)$,
(5) $x_{lim} + \text{sign}(\cos(\theta)\dot{\theta})(2x - x_0) < 0$.

Braking Phase. As soon as the pushing phase finishes, the braking phase starts. The corresponding pendulum position is denoted $\theta = \theta_{f1}$. The control is then $u_0 = u\,\text{sign}(\cos(\theta_0)\dot{\theta}_0)$ and it continuous until the cart stops. θ_{f2} is defined as the pendulum angle when the cart stops.

3.5 Control of the Cart

The cart dynamic is not taken into account in the energy pumping and removing algorithms. It is simply assumed that the cart acceleration is constant. But ignoring this dynamic may lead to some problems in real applications because of unmodelled and uncertain dynamics, friction, noise, etc. To overcome such problems, we use the value of the constant acceleration given by the energy pumping and removing algorithms as desired values from which we can compute a desired trajectory to be tracked by the cart. Then a simple PD controller is implemented to minimize the tracking error.

The desired acceleration of the cart in one cycle

$$u_d(t) = \begin{cases} \pm u\,\text{sign}(\cos(\theta)\dot{\theta}) & \text{, if } t_0 \leqslant t < t_1 \\ \mp u\,\text{sign}(\cos(\theta)\dot{\theta}) & \text{, if } t_1 \leqslant t < t_2 \\ 0 & \text{, if } t_2 \leqslant t \end{cases} \tag{17}$$

where t_0 denotes the time at which the cycle starts, t_1 denotes the time at which pushing phase ends, and t_2 denotes the time at which the cart stops (the cycle ends). Note that, the direction of the acceleration depends on whether the energy is pumped or removed at the beginning of the cycle and switched at the end of pushing phase. Using (17), the desired position and velocity of the cart $(x_d, \dot{x}_d)$ can be determined for the cycle as

$$x_d(t) = \begin{cases} x_0 + \frac{1}{2}u_d(t-t_0)^2 & \text{, if } t_0 \leqslant t < t_1 \\ x_1 + \dot{x}_1(t-t_1) + \frac{1}{2}u_d(t-t_1)^2 & \text{, if } t_1 \leqslant t < t_2 \\ x_2 & \text{, if } t_2 \leqslant t \end{cases}, \tag{18}$$

$$\dot{x}_d(t) = \begin{cases} u_d(t-t_0) & \text{, if } t_0 \leqslant t < t_1 \\ \dot{x}_1 + u_d(t-t_1) & \text{, if } t_1 \leqslant t < t_2 \\ 0 & \text{, if } t_2 \leqslant t \end{cases} \tag{19}$$

where x_0, x_1, x_2 denote the position of the cart at the beginning of the cycle, at the end of pushing phase, at the end of the cycle, respectively, and $\dot{x}_1$ denotes the velocity of the cart at the end of pushing phase. Notice that, velocity of the cart at the beginning and at the end of the cycle is imposed to be zero.

4. NUMERICAL SIMULATIONS

This section represents the numerical simulation results that were performed to test the controller algorithms developed for the stabilization of the pendulum on a cart with restricted cart track length. In order to test both energy pumping and energy removing algorithms, two different numerical simulations were implemented. Model parameters of the pendulum on a cart system have been assigned as $m = 0.23kg$, $l = 0.6414m$, $I = 0.0078838kgm^2$ and $M = 0.7031kg$, and the gravitational acceleration has been considered to be $g = 9.81m/s^2$. The desired energy level for the pendulum has been defined as the on of the upward position of the pendulum with zero velocity, the limit positions for the cart have been assigned as $\pm 0.2m$ ($x_{lim} = 0.2m$), and the value of the desired cart acceleration has been chosen as $u = 2m/s^2$ for both simulations. Initial states have been set to $\left[\theta\ \dot{\theta}\ x\ \dot{x}\right]^T = 0$ for the first simulation and to $\left[\theta\ \dot{\theta}\ x\ \dot{x}\right]^T = [0\ 20\ 0\ 0]^T$ for the second simulation. In simulations, descending Gauss algorithm has been implemented for numerical computations of elliptic integral of first kind and Jacobi elliptic function.

Fig.2 to Fig.5 give the results for the first simulation. Notice that the pendulum is at its hanging position at the beginning with zero velocity for this simulation, and the energy level of the pendulum is very close the desired energy level after a few swings. Even the cart approaches to its limit positions, it stays in the limits. When the desired acceleration for the cart is zero, the cart is forced not to move which can be seen in Fig.4. Similarly, the results of the second simulation are depicted in Fig.6 to Fig.9. According to initial state values, it can be observed that the pendulum rotates clockwise for this simulation. The energy level of the pendulum also converges to the desired energy level, and the pendulum slows down until the desired energy level is reached while the cart stays in the predefined limit positions.

5. CONCLUSION

A globally stabilizing energy based controller algorithm for the pendulum on a cart system with restricted cart track length has been presented for which the desired energy level of the pendulum is its upright configuration. The proposed algorithm consists of two distinct parts namely energy pumping and energy removing. To achieve, the cart was considered to move with constant acceleration and it was forced to move forth and back. The computations necessary to keep the cart between predefined limit positions and to provide the convergence of the energy level of

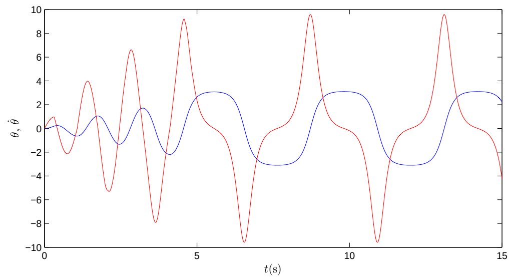

Fig. 2. Angular position(blue, darker) and velocity(red, brighter) of the pendulum.

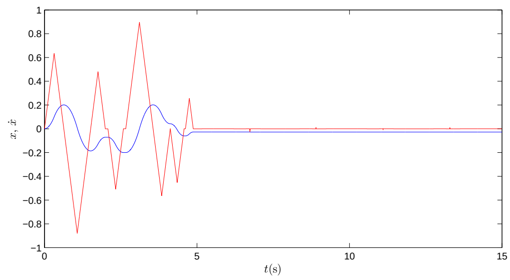

Fig. 3. Position(blue, darker) and velocity(red, brighter) of the cart.

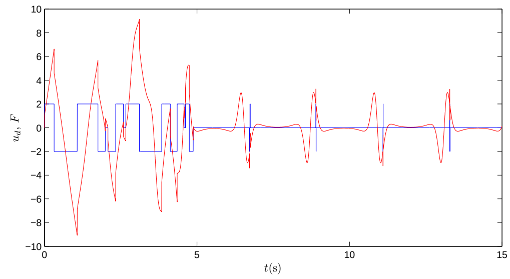

Fig. 4. Desired acceleration of the cart(blue, darker) and applied force to the cart(red, brighter).

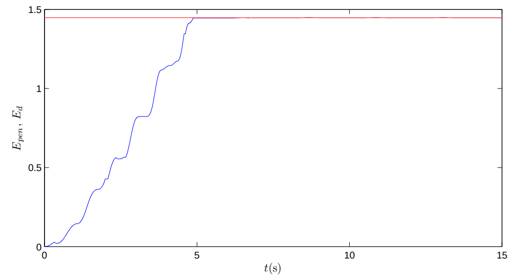

Fig. 5. The change of the energy level of the pendulum.

the pendulum to the desired energy level were performed numerically by means of elliptic integral of first kind and Jacobi elliptic function 'sn'. Numerical simulations were implemented for both of energy pumping and energy removing algorithms and results were presented.

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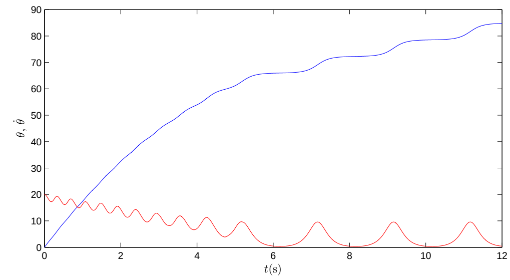

Fig. 6. Angular position(blue, darker) and velocity(red, brighter) of the pendulum.

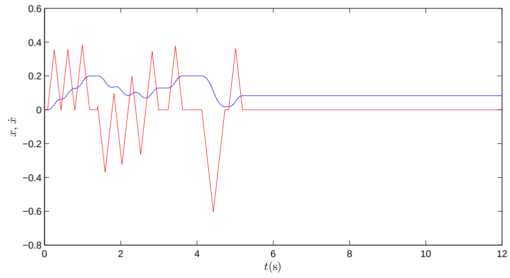

Fig. 7. Position(blue, darker) and velocity(red, brighter) of the cart.

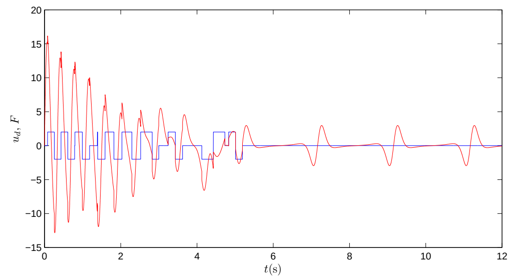

Fig. 8. Desired acceleration of the cart(blue, darker) and applied force to the cart(red, brighter).

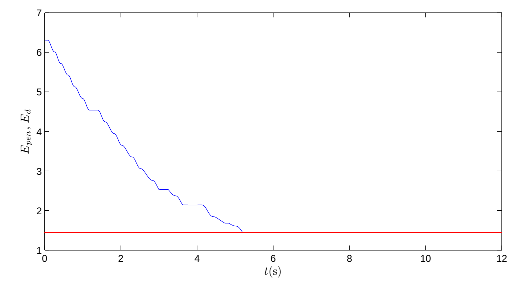

Fig. 9. The change of the energy level of the pendulum.

REFERENCES

[1] K. J. Astrom and K. Furuta, "Swinging up a pendulum by energy control", Automatica, vol. 36, no. 2, pp. 287-295, 2000.

[2] R. Lozano and I. Fantoni and D. J. Block, "Stabilization of the inverted pendulum around its homoclinic

orbit", Systems & Control Letters, vol. 47, no. 3, pp. 194-204, 2000.
[3] F. Gordillo and J. Aracil, "A new controller for the inverted pendulum on a cart", International Journal of Robust and Nonlinear Control, vol. 18, pp. 1607-1621, 2008.
[4] Q. Wei, W. P. Dayawansa and W. S. Levine, "Nonlinear controller for an inverted pendulum having restricted travel", Automatica, vol. 31, no. 6, pp. 841-850, 1995.
[5] D. Chatterjee, A. Patra and H. K. Joglekar, "Swing up and stabilization of a cart-pendulum system under restricted cart track length", Systems & Control Letters, vol. 47, no. 4, pp. 355-364, 2002.
[6] J. Zhao and M. W. Spong, "Hybrid control for global stabilization of the cart-pendulum system", Automatica, vol. 37, no. 12, pp. 1941-1951, 2001.
[7] M. Abramowitz and Irene. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, New York, 1965.
[8] J. M. Borwein and P. B. Borwein, *Pi and the AGM*, John Wiley & Sons, New York, 1987.
[9] D. F. Lawden, *Elliptic Functions and Applications*, Springer-Verlag, New York, 1989.
[10] J. V. Armitage and W. F. Eberlein, *Elliptic Functions*, Cambridge University Press, New York, 2006.

Appendix A. ELLIPTIC INTEGRAL OF FIRST KIND AND JACOBI ELLIPTIC FUNCTION 'sn'

This section gives brief definitions of the elliptic integral of first kind and Jacobi elliptic function 'sn'. Please see [7, 8, 9, 10] for further detailed information about the subject.

Elliptic integral of first kind is defined in Legendre form as

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2(\phi)}} \tag{A.1}$$

where φ denotes *amplitude* and k denotes *modulus* and they satisfy $0 \leqslant \varphi \leqslant \frac{\pi}{2}$ and $0 < k < 1$ in general. $F(\pi/2, k)$ is defined as complete elliptic integral of first kind and it is denoted by $K(k)$.

Jacobi elliptic function 'sn' is defined in terms of incomplete elliptic integral of first kind in the form

$$\mathrm{sn}(u_e, k) = \sin(\varphi) \tag{A.2}$$

where $u_e = F(\varphi, k)$.

Appendix B. RELATION BETWEEN PENDULUM ANGLE AND THE TIME

Solving (7) for dt yields

$$dt = \frac{d\theta}{\sqrt{\frac{2}{a}}\sqrt{E_{pen} - 2bg\sin^2\left(\frac{\theta}{2}\right)}}. \tag{B.1}$$

Notice that, pendulum either swings or rotates depending on its initial energy level. In order to determine the relation between pendulum angle and the elapsed time, these two different behavior of the pendulum has to be analyzed separately. Consider the rotating case first for which $E_{pen} > 2bg$. Considering the initial velocity is $\dot{\theta}_0$, then the following equation can be derived with a mathematical manipulation from (B.1)

$$\mathrm{sgn}(\dot{\theta}_0)\sqrt{\frac{2E_{pen}}{a}}dt = \frac{d\theta}{\sqrt{1 - \frac{2bg}{E_{pen}}\sin^2(\frac{\theta}{2})}}. \tag{B.2}$$

Accordingly, defining $k_r = \sqrt{\frac{2bg}{E_{pen}}}$ and substituting $\phi = \frac{\theta}{2}$, (B.2) turns out to be

$$\mathrm{sgn}(\dot{\theta}_0)\sqrt{\frac{E_{pen}}{2a}}dt = \frac{d\theta}{\sqrt{1 - k_r^2\sin^2(\phi)}}. \tag{B.3}$$

Integrating (B.3) from φ_0 to φ_1 gives

$$t_1 - t_0 = \sqrt{\frac{2a}{E_{pen}}}\left(F(\varphi_1, k_r) - F(\varphi_0, k_r)\right) \tag{B.4}$$

which is the elapsed time between the pendulum angles $\theta_0 = 2\varphi_0$ and $\theta_1 = 2\varphi_1$. Note that, the half of the rotation period of the pendulum can be computed with $\varphi_0 = 0$ and $\varphi_1 = \frac{\pi}{2}$. In the swinging case, one needs to scale the pendulum angle in order to obtain the time in terms of elliptic integral of first kind for swinging pendulum for which $E_{pen} < 2bg$. To do so, consider the equation given by (B.1). Substituting $\sin(\phi) = \sqrt{\frac{2bg}{E_{pen}}}\sin\left(\frac{\theta}{2}\right)$ gives

$$dt = \sqrt{\frac{a}{bg}}\frac{d\phi}{\sqrt{1 - k_s^2\sin^2(\phi)}} \tag{B.5}$$

with $k_s = \sqrt{\frac{E_{pen}}{2bg}}$. Integrating (B.5) from φ_0 to φ_1 yields

$$t_1 - t_0 = \sqrt{\frac{a}{bg}}\left(F(\varphi_1, k_s) - F(\varphi_0, k_s)\right) \tag{B.6}$$

which is the elapsed time between the pendulum angles θ_0 and θ_1 where $\theta_i = 2\sin^{-1}\left(\sqrt{\frac{E_{pen}}{2bg}}\sin(\varphi_i)\right)$. Note that, the quarter of the swinging period of the pendulum can be computed with $\varphi_0 = 0$ and $\varphi_1 = \frac{\pi}{2}$.

Consider the integration of (B.3) from 0 to φ ($0 \leqslant \varphi \leqslant \frac{\pi}{2}$) yielding

$$\sqrt{\frac{E_{pen}}{2a}}(t_1 - t_0) = F(\varphi, k_r). \tag{B.7}$$

If k_r and $u_r = \sqrt{\frac{E_{pen}}{2a}}(t_1 - t_0)$ are given, then the amplitude denoted by φ can be obtained as

$$\varphi = \sin^{-1}\left(\mathrm{sn}(u_r, k_r)\right). \tag{B.8}$$

Therefore, for given time interval and energy level, the angle swept out by the rotating pendulum can be expressed in terms of Jacobi elliptic function 'sn'. The same procedure can be followed for swinging pendulum to compute the swept out angle for given time interval and energy level. To do so, (B.5) is integrated from 0 to φ yielding

$$\sqrt{\frac{bg}{a}}(t_1 - t_0) = F(\varphi, k_s). \tag{B.9}$$

If k_s and $u_s = \sqrt{\frac{bg}{a}}(t_1 - t_0)$ are given, then the amplitude denoted by φ can be obtained as

$$\varphi = \sin^{-1}\left(\mathrm{sn}(u_s, k_s)\right). \tag{B.10}$$

See [7] for numerical computation algorithms of elliptic integrals and Jacobi elliptic functions.

