Nonlinear Output Regulation by Post-processing Internal Model for Multi-Input Multi-Output Systems

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Abstract: The paper deals with the problem of output regulation for the class of multi-input multi-output square nonlinear systems satisfying a minimum-phase assumption and a "positivity" condition on the high-frequency gain matrix. By following a design paradigm proposed in [12] for single-input single-output nonlinear systems, it is shown how an internal model-based controller can be obtained, whose dimension depends on the number of regulated outputs and on the dimension of the exosystem. Thanks to a scalability property of the regulator structure highlighted in [10], it is shown how a "pre-processing" internal model can be shifted from input to output, yielding in this way a "post-processing" internal model. This makes it possible to run a high-gain asymptotic analysis that bypasses the need of finding a normal form, as it would normally be the case.

1. INTRODUCTION

The problem of output regulation for nonlinear systems, beginning with the work [7], has been addressed by several authors and a rather satisfactory corpus of results has been developed, the majority of which address the problem in question for single-input single-output systems (see for instance [12], [5], [9], [14] and references therein). For the class of systems in question, (robust) regulation is typically achieved by means of a controller consisting of *an internal model that provides a control input*, to the purpose of forcing the existence of a "steady-state (invariant) manifold" on which the regulated variable vanishes, complemented by a stabilizer that makes the manifold in question attractive for the cascade of two such subsystems.

The extension of the theory of output regulation to *multi-input multi-output* nonlinear systems, in spite of a number of encouraging contributions ([6]), is still at a very preliminary state of development. There are several reasons why the extension of the theory of output regulation to such systems is a difficult and challenging research task. These can be (in good part) understood by looking at how the corresponding design problem has been successfully handled for multi-variable linear systems. In fact, for linear systems having m inputs and p outputs (and necessarily $m \geq p$, because otherwise robust regulation is not possible), the problem is solved (see [1], [4], [3]) by means of a controller embedding an internal model (consisting of p copies of the largest cyclic component of the exosystem) directly fed by the regulated output, cascaded with stabilizer which, driven by the state of the internal model as well as by any other variable available for measurement, produces the appropriate control input.

Now, this control structure cannot be easily "copied" to a multi-variable nonlinear setting, for a number of reasons.

One reason is that, while if $m = p$ the computation of a "generator" of the required steady-state input is in principle possible under reasonably weak assumptions (by means of a suitable enhanced version of the "zero-dynamics algorithm", see [8, pages 293-311]), if $m > p$ it is not clear at all how to handle the inherent redundancy in a robust fashion if one insists in following the paradigm that has proven to be successful in the case of single-input single-output systems. In fact, if the system has $m > p$ controls, only p of which are needed for regulation purposes, but all of which might be necessary for stabilization, it is not immediately clear how to identify a "robust selection" of the inputs that are to be driven by the internal model.

Second, it is not clear yet how to handle the case in which the system has q extra outputs, that are not expected to be regulated but might be necessary for stabilization (via possibly dynamic feedback). In fact, these extra outputs may not vanish in steady-state and hence they have to be somewhat "filtered" out, a problem that un-necessarily complicates the design.

A third reason is that, regardless of how the internal model is designed and where it is embedded, the resulting (augmented) system is a multi-variable nonlinear system that has to be *robustly* stabilized to a desired invariant manifold. The theory of robust stabilization of multi-variable nonlinear systems by (dynamic) output feedback is far from being complete and only special (though relevant) cases can be handled. Robust stabilization via state-feedback is in general possible for multi-variable systems that are strictly minimum phase, as shown in [11], but this is not sufficient to handle the problem of output regulation, where only outputs are available (and robust stability is sought). If $m = p$, if the system possesses a well-defined relative degree and a normal form, if the system is minimum-phase and if the so-called "high-frequency gain matrix" satisfies a suitable "positivity condition", robust stabilization can be achieved (as show in the sequel), but the stabilization procedure is rendered more complicated by the presence of the internal model (in the sequel, we will show how this problem can be efficiently handled).

Thus, in summary, the design of robust regulators for multi-variable nonlinear systems is still a largely open domain of research, that deserves appropriate attention. The purpose of the present paper is to offer a contribution in this direction, by showing how the design paradigm suggested in [12], for a single-input single-output system, to handle the case of a *nonlinear exosystem* in a very general setting, can be extended to the special class of multi-variable systems having $m = p$, a well-defined relative, a "high-frequency gain matrix" that satisfies a suitable "positivity condition" and an asymptotically stable zero dynamics.

The contribution of the paper is a novel procedure that makes it possible to overcome the need of finding, as it was done in [12], a normal form when the controlled plant is driven by the internal model. Using a design procedure recently suggested in [10]) and a "scalability" property of the (nonlinear) internal model designed in [12], it is shown how the internal model can be shifted from control input to controlled output, in which case the problem of determining the normal form trivially disappears. In this way, the paper proposes a design procedure for solving the problem of output regulation for a relevant special class of multi-variable nonlinear systems, in the presence of a rather general class of *nonlinear* exosystems.

2. PROBLEM FORMULATION

The paper deals with nonlinear multivariable square systems of the form

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f(w,z,\xi) \\ \dot{\xi}_{i,j} &= \xi_{i,j+1} \qquad j = 1,\dots r_i - 1 \\ \dot{\xi}_{i,r_i} &= q_i(w,z,\xi) + B_i(w,z,\xi)u \qquad i = 1,\dots,m \end{aligned} \tag{1}$$

in which $w \in \mathbb{R}^{n_w}$, $z \in \mathbb{R}^n$, $r_i \geq 1$, $i = 1,\dots,m$, the vector $\xi \in \mathbb{R}^{r_1+\dots+r_m}$ is defined as

$$\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} \quad \text{with} \quad \xi_i = \begin{pmatrix} \xi_{i,1} \\ \vdots \\ \xi_{i,r_i} \end{pmatrix},$$

for $i = 1,\dots,m$, and $u \in \mathbb{R}^m$. The vectors u and

$$e = \text{col}(\xi_{1,1},\, \xi_{2,1}, \cdots, \xi_{m,1})$$

are respectively the control input and the regulated error, while the variable w models exogenous inputs that might represent reference/disturbance signals to be tracked/rejected or uncertain parameters. All the functions on the right-hand side of (1) are smooth. As customary in output regulation, the exogenous input $w(t)$ is solution of an autonomous system $\dot{w} = s(w)$, referred to as the exosystem, evolving on a compact set $W \subset \mathbb{R}^{n_w}$ that is assumed to be invariant. The initial conditions $(z(0), \xi(0))$ of system (1) are assumed to range in a fixed arbitrary compact set $Z \times \Xi \subset \mathbb{R}^n \times \mathbb{R}^{r_1+\dots+r_m}$.

For this class of systems we are interested to solve the problem of semiglobal output regulation, that is to find an error feedback controller of the form

$$\begin{aligned} \dot{\eta} &= \varphi(\eta, e) \qquad \eta \in \mathbb{R}^\nu \\ u &= \vartheta(\eta, e) \end{aligned} \tag{2}$$

and a compact set $C \subset \mathbb{R}^\nu$, such that for any

$$(w(0), z(0), \xi(0), \eta(0)) \in W \times Z \times \Xi \times C$$

the resulting trajectories of the closed-loop system (1)-(2) are bounded and $\lim_{t\to\infty} e(t) = 0$ uniformly in the initial conditions.

The previous problem will be solved under a certain number of assumptions. First, we assume that system (1) has a well-defined vector relative degree $\{r_1, \cdots, r_m\}$ and a "high-frequency gain matrix" (see [8])

$$B(w,z,\xi) = \begin{pmatrix} B_1(w,z,\xi) \\ \vdots \\ B_m(w,z,\xi) \end{pmatrix} \tag{3}$$

that satisfies the following "positivity" assumption.

Assumption 1. There exists a nonsingular matrix $M \in \mathbb{R}^{m\times m}$ such that the following inequality holds

$$B(w,z,\xi)\, M + M^T\, B(w,z,\xi)^T \geq I \tag{4}$$

for all $(w,z,\xi) \in W \times \mathbb{R}^n \times \mathbb{R}^{r_1+\dots+r_m}$.

Furthermore, we assume that the system is (strongly) minimum-phase relative to the output e and input u. The minimum-phaseness condition is specified as follows.

Assumption 2. There exists a smooth function $\pi : W \to \mathbb{R}^n$ such that the system

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f(w,z,\xi) \end{aligned} \tag{5}$$

with state (w,z) and input ξ is input-to-state stable relative to the compact set

$$\mathcal{A} = \{(w,z) \in W \times \mathbb{R}^n \ : \ z = \pi(w)\}$$

with a locally linear gain function, that is there exist a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and, for all $d_\xi > 0$, there exists $\gamma > 0$ such that for all $(w(0), z(0)) \in W \times Z$ and all bounded $\xi(t)$ satisfying $\|\xi(\cdot)\|_\infty \leq d_\xi$, the resulting trajectory $(w(t), z(t))$ of (5) fulfills [1]

$$\|(w(t), z(t))\|_{\mathcal{A}} \leq \max\{\beta(\|(w(0), z(0))\|_{\mathcal{A}}, t)\,,\ \gamma\,\|\xi(\cdot)\|_\infty\}$$

for all $t \geq 0$.

Note that this implies the set $\mathcal{A}$ is invariant for (5) if $\xi(t) \equiv 0$.

In what follows, we address the problem of semiglobal output regulation in the simpler case of unitary vector relative degree, namely $r_i = 1$ for $i = 1,\dots,m$. The reason why this can be done without loss of generality follows

[1] Here and in the following we use the notation $\|x\|_{\mathcal{C}} = \min_{y\in\mathcal{C}} \|x - y\|$ to denote the distance of $x \in \mathbb{R}^n$ from a compact subset $\mathcal{C}$ of $\mathbb{R}^n$.

from classical results about output feedback stabilization which, for the sake of completeness, are briefly summarized here. Set

$$\zeta_i = \mathrm{col}(\xi_{i,1},\, \xi_{i,2}, \cdots ,\, \xi_{i,r_i-1})$$

and consider the change of variable

$$\xi_{i,r_i} \;\mapsto\; \theta_i = \xi_{i,r_i} - A_i\zeta_i \qquad i = 1, \dots m$$

where $A_i = \begin{pmatrix} a_{i,1} & \dots & a_{i,r_i-1} \end{pmatrix}$, with the coefficients $a_{i,j}$ chosen in such a way that the roots of the polynomials $\lambda^{r_i-1} + a_{i,r_i-1}\lambda^{r_i-2} + \ldots + a_{i,2}\lambda + a_{i,1} = 0$ have negative real parts, $i = 1, \dots, m$. By letting $\zeta = \mathrm{col}(\zeta_1, \dots, \zeta_m)$ and $\theta = \mathrm{col}(\theta_1, \dots, \theta_m)$, system (1) in the new coordinates reads as

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f(w, z, \xi(\zeta, \theta)) \\ \dot{\zeta}_i &= H_i\zeta_i + N_i\theta_i \\ \dot{\theta}_i &= q_i(w, z, \xi(\zeta, \theta)) - A_i(H_i\zeta_i + N_i\theta_i) + B_i(w, z, \xi(\zeta, \theta))u \\ &\qquad\qquad i = 1, \dots, m \end{aligned} \tag{6}$$

where H_i are Hurwitz matrices and

$$\xi(\zeta, \theta) = Q_1\zeta + Q_2\theta$$

with

$$Q_1 = \mathrm{blkdiag}(\begin{pmatrix} I_{r_1-1} \\ A_1 \end{pmatrix}, \dots, \begin{pmatrix} I_{r_m-1} \\ A_m \end{pmatrix})$$

$$Q_2 = \mathrm{blkdiag}(\begin{pmatrix} 0_{(r_1-1)\times 1} \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} 0_{(r_m-1)\times 1} \\ 1 \end{pmatrix})$$

By Assumption 1, system (6) has relative degree $r = \{1, \dots, 1\}$ with respect to the input u and output θ. Furthermore, by Assumption 2 and by the fact that H_i are Hurwitz, standard properties of a cascade of two ISS systems lead to the conclusion that system

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f(w, z, \xi(\zeta, \theta)) \\ \dot{\zeta}_i &= H_i\zeta_i + N_i\theta_i \quad i = 1, \dots, m\,, \end{aligned} \tag{7}$$

regarded as a system with state (w, z, ζ) and input θ, is ISS relative to the set $\mathcal{A} \times \{0\}$ with locally linear asymptotic gain. As consequence, if the system (1), with relative degree $\{r_1, \dots, r_m\}$ with respect to the input u and output e, satisfies the minimum-phase Assumption 2, also system (6), with relative degree $\{1, \dots, 1\}$ with respect to the input u and output θ, satisfies a similar assumption, with (w, z) replaced by (w, z, ζ), with ξ replaced by θ and system (5) given by (7). Now let $E \subset \mathbb{R}^{r_1+\dots+r_m-m}$ and $\Theta \subset \mathbb{R}^m$ be compact subsets such that $\xi(0) \in \Xi \Rightarrow \zeta(0) \in E$ and $\theta(0) \in \Theta$, and suppose that it is possible to design a controller of the form

$$\begin{aligned} \dot{\eta}' &= \varphi'(\eta', \theta) \qquad \eta \in \mathbb{R}^{\nu'} \\ u &= \vartheta'(\eta', \theta) \end{aligned} \tag{8}$$

able to stabilize the set $\mathcal{A} \times \{0\} \times \{0\}$ for the closed loop system with a domain of attraction containing $W \times Z \times E \times \Theta$ and to secure that $\theta(t)$ converges to zero asymptotically uniformly in the initial conditions. By the definition of θ and by the fact that the H_i are Hurwitz matrices, it follows that also the regulation error e converges to zero uniformly in the initial conditions. Furthermore, classical results about high-gain observers (see [2]) can be used to show that an "error feedback" controller of the form (2) can be obtained from a "partial state feedback" controller of the form (8) with a domain of attraction that still contains the compact sets of initial conditions. By above arguments we thus conclude that there is no loss of generality in considering the problem at hand under the simplified assumption of unitary vector relative degree. For this reason we concentrate on a system of the form

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f(w, z, e) \\ \dot{e} &= q(w, z, e) + B(w, z, e)u \end{aligned}$$

with regulated output $e \in \mathbb{R}^m$, fulfilling "positivity" Assumption 1, written with ξ replaced by e, and the (strong) minimum-phase Assumption 2.

3. THE INTERNAL MODEL

Let $(F, G) \in \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times 1}$, $d > 0$, be a controllable pair and let $\mathbf{F} \in \mathbb{R}^{md\times md}$ and $\mathbf{G} \in \mathbb{R}^{md\times md}$ be defined as

$$\begin{aligned} \mathbf{F} &= \mathrm{blkdiag}\begin{pmatrix} F & F & \cdots & F \end{pmatrix} \\ \mathbf{G} &= \mathrm{blkdiag}\begin{pmatrix} G & G & \cdots & G \end{pmatrix}. \end{aligned}$$

Moreover, let $\Psi : W \to \mathbb{R}^m$ be the smooth function defined as

$$\Psi(w) = B(w, \pi(w), 0)^{-1} q(w, \pi(w), 0)\,. \tag{9}$$

Note that $-\Psi(w)$, $w \in W$, represents a desired steady state behavior for u.

The following proposition, instrumental to present the regulator structure, follows by applying "component-wise" the main result of [12].

Lemma 1. Let $d \geq 2n_w + 2$. There exist an $\ell > 0$ and a subset $S \subset \mathbb{C}$ of zero Lebesgue measure such that if the eigenvalues of F are in $\{\lambda \in \mathbb{C} \;:\; \mathrm{Re}\lambda \leq -\ell\} \setminus S$, then there exist a differentiable function $\sigma_0 : W \to \mathbb{R}^{md}$ and a continuous bounded function $\gamma_0 : \mathbb{R}^{md} \to \mathbb{R}^m$ such that

$$\begin{aligned} \frac{\partial\sigma_0}{\partial w} s(w) &= \mathbf{F}\sigma_0(w) + \mathbf{G}M^{-1}\Psi(w) \\ M^{-1}\Psi(w) &= \gamma_0(\sigma_0(w)) \end{aligned} \tag{10}$$

for all $w \in W$.

Note that the function γ_0 solution of (10) is just guaranteed to be continuous and bounded. In this paper we require stronger properties for γ_0 specified in what follows.

Assumption 3. The function γ_0 in (10) is at least C^2, is bounded, and $\nabla\gamma_0$ is bounded with bounded derivatives.

Remark. Boundedness of $\gamma_0(\eta)$, of its gradient $\nabla\gamma_0(\eta)$ and of the derivative of the latter for all $\eta \in \mathbb{R}^{md}$ can be assumed without loss of generality. As a matter of fact, note that the only condition required to $\gamma_0(\eta)$ is that the second equation of (10) is fulfilled for all η in the compact set $\{\eta \in \mathbb{R}^{md} \ : \ \eta = \sigma_0(w),\, w \in W\}$. Standard extension results can be then used to extend $\gamma_0(\eta)$ outside the compact set by obtaining functions with the required boundedness properties. $\triangleleft$

We are now in the position of presenting the regulator and the main result of the paper, as detailed in the next proposition.

Proposition 1. Suppose Assumptions 1 and 2 hold. Let $\mathbf{F}$ and $\mathbf{G}$ be chosen as above and suppose that the function $\gamma_0(\cdot)$ in the second equation of (10) is such that Assumption 3 holds. Let C be an arbitrary compact set of $\mathbb{R}^{md}$. Then there exists a $\kappa^\star > 0$ such that for all $\kappa \geq \kappa^\star$ the controller

$$\begin{aligned} \dot\eta &= \mathbf{F}\eta + \mathbf{G}(\gamma_0(\eta) + v) \\ u &= -M(\gamma_0(\eta) + v) \\ v &= \kappa e \end{aligned} \tag{11}$$

solves the problem of semiglobal output regulation.

Remark. Practical implementation of the controller (11) is clearly affected by the design of the function $\gamma_0(\cdot)$ fulfilling (10). Lemma 1 just guarantees existence of the function without providing explicit expressions. Explicit expressions of the function $\gamma_0(\cdot)$ fulfilling (10) have been presented in [13]. In that paper also approximate expressions have been proposed that are suitable for practical implementation of the controller. $\triangleleft$

The controller (11) has exactly the same structure as the controller considered in [12]. However, as explained in the introduction, the arguments used in [12] to prove that this controller solves the problem of output regulation cannot be trivially extended to the present, *multivariable*, setting. In fact, the arguments used in [12] required a preliminary transformation whose purpose was to bring the (augmented) system

$$\begin{aligned} \dot w &= s(w) \\ \dot z &= f(w,z,e) \\ \dot e &= q(w,z,e) - B(w,z,e)M(\gamma_0(\eta) + v) \\ \dot\eta &= \mathbf{F}\eta + \mathbf{G}(\gamma_0(\eta) + v)\,, \end{aligned}$$

viewed as a system with input v and output e, in *normal form.* In this way, exploiting the asymptotic properties of the resulting (augmented) zero dynamics, it was possible to show that convergence to the desired steady-state behavior is achieved if the gain parameter κ is sufficiently large. If $m > 1$ the transformation suggested in [12] to bring the system in question in normal form is not immediately applicable, unless $B(w,z,e)$ is a constant matrix. Thus, in what follows, we propose a somewhat different argument. This argument is based on the observation that the controller (11), which as such is seen as a *pre-processor* generating the required steady-state input driven by a high-gain feedback from the regulated variable e, can be equivalently seen as a *post-processor* described by equations of the form

$$\begin{aligned} \dot{\tilde\eta} &= \mathbf{F}\tilde\eta + \mathbf{G}(\gamma_k(\tilde\eta) + e) \\ u &= -\kappa M(\gamma_k(\tilde\eta) + e) \end{aligned} \tag{12}$$

if $\tilde\eta$ and γ_k are defined as

$$\tilde\eta = \frac{1}{\kappa}\eta \qquad \gamma_\kappa(\tilde\eta) = \frac{1}{\kappa}\gamma_0(\kappa\tilde\eta)\,.$$

Remark. The two modes of control are compared in Fig. 1. Note that while swapping the internal model with the (scalar) gain parameter κ is trivially possible in case the internal model is a linear system, this is no longer an obvious option if the latter is nonlinear, as it is in the current setting. The "scalability" property used in the previous transformation, and exploited in the proof below, is yet another relevant feature of the "canonical" internal model introduced in [12]. Such a scalability property, in turn, strongly relies on the fact that the stabilizer $v = \kappa e$ is a linear function. As it will be clear from the proof of Proposition 1, the fact that the stabilizer can be taken linear is a consequence of the linearity of the asymptotic gain in the minimum-phase Assumption 2. $\triangleleft$

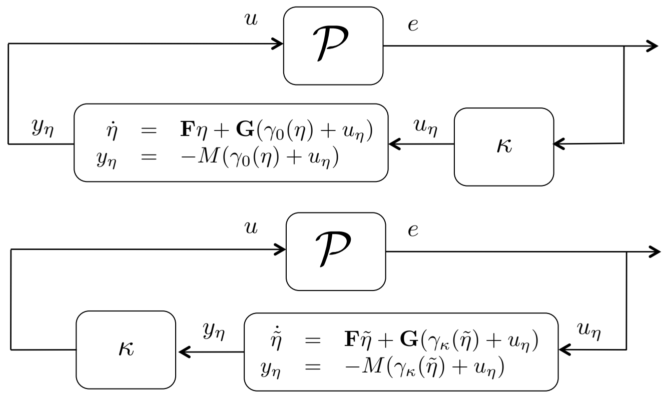

Fig. 1. Pre- and Post-processing internal models.

4. PROOF OF THE MAIN RESULT

The proof of the result relies upon high-gain arguments detailed in this section. We apply a first change of variables of the form

$$e \mapsto \bar e := e + \gamma_\kappa(\tilde\eta)$$

that transforms the closed loop system as

$$\begin{aligned} \dot w &= s(w) \\ \dot z &= f(z,w,\bar e - \gamma_\kappa(\tilde\eta)) \\ \dot{\tilde\eta} &= \mathbf{F}\tilde\eta + \mathbf{G}\bar e \\ \dot{\bar e} &= q(w,z,\bar e - \gamma_\kappa(\tilde\eta)) + B(w,z,\bar e - \gamma_\kappa(\tilde\eta))u \\ &\qquad + \nabla\gamma_\kappa(\tilde\eta)\,(\mathbf{F}\tilde\eta + \mathbf{G}\bar e) \\ u &= -\kappa M \bar e \end{aligned} \tag{13}$$

Furthermore, by bearing in mind that $-\Psi(w)$, $w \in W$, represents a desired steady state behavior for u and that $u = -\kappa M\bar e$, we further change the variable $\bar e$ as

$$\bar e \mapsto \varsigma := \bar e - \frac{1}{\kappa}M^{-1}\Psi(w)$$

putting system (13) into the form

$$
\begin{aligned}
\dot{w} &= s(w)\\
\dot{z} &= f(w,z,\varsigma+\frac{1}{\kappa}M^{-1}\Psi(w)-\gamma_\kappa(\tilde{\eta}))\\
\dot{\tilde{\eta}} &= \mathbf{F}\tilde{\eta}+\mathbf{G}(\varsigma+\frac{1}{\kappa}M^{-1}\Psi(w))\\
\dot{\varsigma} &= q(w,z,\varsigma+\frac{1}{\kappa}M^{-1}\Psi(w)-\gamma_\kappa(\tilde{\eta}))+B(\cdot)u\\
&\quad +\nabla\gamma_\kappa(\tilde{\eta})\,(\mathbf{F}\tilde{\eta}+\mathbf{G}(\varsigma+\frac{1}{\kappa}M^{-1}\Psi(w)))-\frac{1}{\kappa}M^{-1}\dot{\Psi}(w)\\
u &= -\kappa M(\varsigma+\frac{1}{\kappa}M^{-1}\Psi(w)))
\end{aligned}
\tag{14}
$$

where $B(\cdot)=B(w,z,\varsigma+\frac{1}{\kappa}M^{-1}\Psi(w)-\gamma_\kappa(\tilde{\eta}))$. Let σ_κ be defined as

$$\sigma_\kappa(\cdot)=\frac{1}{\kappa}\sigma_0(\cdot)\,.$$

By using (10) and the definition of $\gamma_\kappa(\cdot)$, it turns out that

$$
\begin{aligned}
\frac{\partial\sigma_\kappa}{\partial w}s(w) &= \mathbf{F}\sigma_\kappa(w)+\frac{1}{\kappa}\mathbf{G}M^{-1}\Psi(w)\\
\frac{1}{\kappa}M^{-1}\Psi(w) &= \gamma_\kappa(\sigma_\kappa(w))
\end{aligned}
\tag{15}
$$

for all $w\in W$. From this we observe that the $\dot{\varsigma}$ dynamics in (14) can be elaborated as

$$
\begin{aligned}
\dot{\varsigma} &= q(w,z,\varsigma+\gamma_\kappa(\sigma_\kappa(w))-\gamma_\kappa(\tilde{\eta}))\\
&\quad -B(w,z,\varsigma+\gamma_\kappa(\sigma_\kappa(w))-\gamma_\kappa(\tilde{\eta}))\,\kappa M(\varsigma+\gamma_\kappa(\sigma_\kappa(w)))\\
&\quad +\nabla\gamma_\kappa(\tilde{\eta})\,(\mathbf{F}\tilde{\eta}+\mathbf{G}(\varsigma+\gamma_\kappa(\sigma_\kappa(w))))\\
&\quad -\nabla\gamma_\kappa(\sigma_\kappa(w))\,(\mathbf{F}\sigma_\kappa(w)+\mathbf{G}\gamma_\kappa(\sigma_\kappa(w)))
\end{aligned}
\tag{16}
$$

It turns out that the right-hand side of (16) is identically zero if $\varsigma=0$ and $\tilde{\eta}=\sigma_\kappa(w)$ for all $w\in W$. Furthermore, $\varsigma=0$ and $\tilde{\eta}=\sigma_\kappa(w)$ imply that $e=0$. This fact suggests a final change of variables of the form

$$\tilde{\eta}\mapsto x:=\tilde{\eta}-\sigma_\kappa(w)$$

that transforms system (14) into

$$
\begin{aligned}
\dot{w} &= s(w)\\
\dot{z} &= f(w,z,\,\rho_\kappa(w,x,\varsigma))\\
\dot{x} &= \mathbf{F}x+\mathbf{G}\varsigma\\
\dot{\varsigma} &= -\kappa\bar{B}_\kappa(w,z,x,\varsigma)M\varsigma+\delta_{0\kappa}(w,z,x,\varsigma)+\delta_{1\kappa}(w,x,\varsigma)
\end{aligned}
\tag{17}
$$

in which

$$
\begin{aligned}
\rho_\kappa(w,x,\varsigma) &= \varsigma+\gamma_\kappa(\sigma_\kappa(w))-\gamma_\kappa(x+\sigma_\kappa(w)))\\
\bar{B}_\kappa(w,z,x,\varsigma) &= B(w,z,\varsigma+\gamma_\kappa(\sigma_\kappa(w))-\gamma_\kappa(x+\sigma_\kappa(w)))\\
\delta_{0\kappa}(w,z,x,\varsigma) &= q(w,z,\varsigma+\gamma_\kappa(\sigma_\kappa(w))-\gamma_\kappa(x+\sigma_\kappa(w)))\\
&\quad -\bar{B}_\kappa(w,z,x,\varsigma)\Psi(w)\\
\delta_{1\kappa}(w,x,\varsigma) &= (\nabla\gamma_\kappa(x+\sigma_\kappa(w))-\nabla\gamma_\kappa(\sigma_\kappa(w)))\cdot\\
&\quad\cdot(\mathbf{F}\sigma_\kappa(w)+\mathbf{G}\gamma_\kappa(\sigma_\kappa(w)))\\
&\quad +\nabla\gamma_\kappa(x+\sigma_\kappa(w))\,(\mathbf{F}x+\mathbf{G}\varsigma)
\end{aligned}
$$

Note that $\delta_{0\kappa}(w,z,0,0)=0$ and $\delta_{1\kappa}(w,0,0)=0$ for all $(w,z)\in\mathcal{A}$ and for all κ.

We start analyzing the (w,z)-dynamics in system (17), regarded as a system with state (w,z) and inputs (x,ς). By definition of $\gamma_\kappa(\cdot)$ and $\sigma_\kappa(\cdot)$, and by the fact that γ_0 is locally Lipschitz and bounded, it follows that there exists a positive L_1, independent of κ, such that

$$\|\gamma_\kappa(x+\sigma_\kappa(w))-\gamma_\kappa(\sigma_\kappa(w))\|=L_1\|x\|\quad\forall\,w\in W\,,\ x\in\mathbb{R}^{md}\,. \tag{18}$$

By this, the definition of $\rho_\kappa(\cdot)$ and by Assumption 2, it follows that the system in question is Input-to-State Stable relative to $\mathcal{A}$ with respect to the inputs (x,ς) with linear asymptotic gains independent of κ. This fact and the fact that F (and thus $\mathbf{F}$) is a Hurwitz matrix, immediately imply, by standard cascade arguments, that also the system given by the first three equations in (17), regarded as a system with state (w,z,x) and input ς, is ISS with linear asymptotic gain independent of κ. In particular, there exist a class-$\mathcal{KL}$ function $\beta'(\cdot,\cdot)$ and, for any compact set $X\subset\mathbb{R}^{md}$ and any $d_\varsigma>0$, a positive γ', such that for any $(\mathbf{z}(0),x(0))\in\mathbf{Z}\times X$, any bounded $\varsigma(t)$ satisfying $\|\varsigma(\cdot)\|_\infty\le d_\varsigma$, and any $\kappa\ge1$, the resulting trajectory $(\mathbf{z}(t),x(t))$ fulfills

$$\|(\mathbf{z}(t),x(t))\|_{\mathcal{A}\times\{0\}}\le\max\{\beta'(\|(\mathbf{z}(0),x(0))\|_{\mathcal{A}\times\{0\}},t)\,,\ \gamma'\,\|\varsigma(\cdot)\|_\infty\}$$

We now shift the attention on the ς dynamics of (17) by finding bounds on the terms $\delta_{0\kappa}(w,z,x,\varsigma)$ and $\delta_{1\kappa}(w,x,\varsigma)$. By adding and subtracting the terms $q(w,\pi(w),0)$ and $B(w,\pi(w),0)\Psi(w)$ to the function $\delta_{0\kappa}(\cdot)$, the latter reads as (bear in mind (9))

$$\delta_{0\kappa}(w,z,x,\varsigma)=\Delta q(w,z,x,\varsigma)+\Delta B(w,z,x,\varsigma)$$

where

$$
\begin{aligned}
&\Delta q(w,z,x,\varsigma)=\\
&q(w,z,\varsigma+\gamma_\kappa(\sigma_\kappa(w))-\gamma_\kappa(x+\sigma_\kappa(w)))-q(w,\pi(w),0)\\
&\Delta B(w,z,x,\varsigma)=(B(w,z,\varsigma+\gamma_\kappa(\sigma_\kappa(w))-\\
&\qquad\gamma_\kappa(x+\sigma_\kappa(w))-B(w,\pi(w),0))\Psi(w)
\end{aligned}
$$

By using the fact that $q(\cdot)$ and $B(\cdot)$ are smooth and (18), it turns out that there exists two positive numbers c_1 and c_2, independent of κ, such that

$$\|\delta_{0\kappa}(w,z,x,\varsigma)\|\le c_1\|(w,z,x)\|_{\mathcal{A}\times\{0\}}+c_2\|\varsigma\|\,.$$

As far as $\delta_{1\kappa}(w,x,\varsigma)$ is concerned, we observe that, by the definition of $\gamma_\kappa(\cdot)$, $\sigma_\kappa(\cdot)$, by Assumption 3, and by the fact that W is compact, there exist three positive numbers L_i, $i=2,3,4$, such that [2]

$$
\begin{aligned}
\|\nabla\gamma_\kappa(x+\sigma_\kappa(w))\| &\le L_2\\
\|\nabla\gamma_\kappa(x+\sigma_\kappa(w))-\nabla\gamma_\kappa(\sigma_\kappa(w))\| &\le L_3\kappa\|x\|\\
\|\mathbf{F}\sigma_\kappa(w)+\mathbf{G}\gamma_\kappa(\sigma_\kappa(w))\| &\le \frac{1}{\kappa}L_4.
\end{aligned}
$$

for all $w\in W$, $x\in\mathbb{R}^{md}$ and $\kappa\ge1$. As a consequence, there exist two positive numbers c_3, c_4 such that

$$\|\delta_{1\kappa}(w,x,\varsigma)\|\le c_3\|x\|+c_4\|\varsigma\|\,.$$

These relations immediately lead to conclude that the last dynamics in (17), regarded as a system with state ς and inputs (w,z,x), can be rendered ISS with an arbitrarily small linear gain, by taking κ sufficiently large. As a matter of fact, consider the ISS-Lyapunov function $V=\varsigma^T\varsigma$ whose derivative, by bearing in mind Assumption 1, can be bounded as

[2] Note that $\nabla\gamma_\kappa(\tilde{\eta})=\frac{1}{\kappa}\left.\frac{\partial\gamma_0}{\partial\tilde{\eta}_0}\right|_{\tilde{\eta}_0=\kappa\tilde{\eta}}\frac{\partial(\kappa\tilde{\eta})}{\partial\tilde{\eta}}=\nabla\gamma_0(\kappa\tilde{\eta})$.

$$
\begin{aligned}
\dot{V} &= 2\varsigma^T \left(-\kappa \bar{B}_\kappa(w,z,x,\varsigma) M \varsigma + \delta_{0\kappa}(\cdot) + \delta_{1\kappa}(\cdot)\right) \\
&\leq -\kappa \|\varsigma\|^2 + 2\|\varsigma\| \|\delta_{0\kappa}(\cdot)\| + 2\|\varsigma\| \|\delta_{1\kappa}(\cdot)\| \\
&\leq -\|\varsigma\| \left((\kappa - \bar{c})\|\varsigma\| - 2c_1 \|(w,z,x)\|_{\mathcal{A}\times\{0\}} - 2c_3\|x\|\right) \\
&\leq -\|\varsigma\| \left((\kappa - \bar{c})\|\varsigma\| - 2(c_1+c_3) \|(w,z,x)\|_{\mathcal{A}\times\{0\}}\right)
\end{aligned}
$$

where $\delta_{0\kappa}(\cdot) = \delta_{0\kappa}(w,z,x,\varsigma)$, $\delta_{1\kappa}(\cdot) = \delta_{1\kappa}(w,z,x,\varsigma)$ and $\bar{c} = 2(c_2 + c_4)$. From this, elementary ISS-Lyapunov arguments can be used to show that for any $\varepsilon > 0$ there exists a $\kappa^\star > 0$ such that for all $\kappa \geq \kappa^\star$ the ς subsystem is ISS (without restrictions on the initial state and on the inputs) with linear asymptotic gain that can be bounded by ε. From the previous fact, small gain arguments for ISS systems with restrictions ([15]) can be used to show that for any compact set $\bar{E} \subset \mathbb{R}^m$ and any compact set $X \subset \mathbb{R}^{md}$ there exists a $\kappa^\star > 0$ such that for all $\kappa \geq \kappa^\star$ the set $\mathcal{A} \times \{0\} \times \{0\}$ is asymptotically (locally exponentially) stable for system (17) with a domain of attraction containing $\mathbf{Z} \times X \times \bar{E}$. In order to complete the proof of Proposition 1 we have to verify how compact sets of initial conditions for the original variables $(\tilde{\eta}, e)$ map into compact sets of initial conditions for the transformed variables (x, ς), by checking that "peaking" phenomena in the parameter κ are prevented. In this respect, we observe that, by the definition of $\bar{e}$, x, ς, σ_κ and γ_κ, for any compact sets $E \subset \mathbb{R}^m$ and $C \subset \mathbb{R}^{md}$ there exist compact sets $\bar{E} \subset \mathbb{R}^m$ and $X \subset \mathbb{R}^{md}$ such that for all $(e(0), \tilde{\eta}(0)) \in E \times C$ then $(\varsigma(0), x(0)) \in \bar{E} \times X$ for all $\kappa \geq 1$ and for all $w \in W$.

5. CONCLUSIONS

The problem of output regulation for multi-input multi-output square nonlinear systems of the form (1) has been dealt with. Under a "positivity" condition on the high-frequency matrix (Assumption 1) and a strong minimum-phase assumption (Assumption 2), it has been shown that a regulator of the form (11) solves the problem with a κ sufficiently large provided that the function $\gamma_0(\cdot)$, whose main properties are highlighted in Lemma 1, fulfills the technical conditions detailed in Assumption 3. It has been shown how the proposed control structure can be *equivalently* interpreted in terms of "pre-processing" internal model, namely as a cascade of a stabilizer feeding an internal model that directly acts on the control input (see the scheme on the top of Figure 1), or "post-processing" internal model, in which the internal model and the stabilizer are swapped (see the scheme at the bottom of Figure 1). In other words, the operators from e to u are the same in the two perspectives but their realizations are different. The post-processing realization, in turn, has been shown to be the right one in order to succeed in an high-gain asymptotic analysis without transforming the system in normal form, the latter being not easy to obtain in case the high frequency matrix is state dependent.

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