Asymptotic tracking of a state trajectory by output-feedback for a class of nonlinear systems

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Abstract—We consider the problem of tracking a reference trajectory with an output feedback for a class of nonlinear systems. We solve this problem by combining the techniques of dynamic scaling and homogeneity in the bi-limit.

I. INTRODUCTION

We address the following problem: Given a system \( \dot{y} = f(y, u) \) with output \( y = h(\eta) \), and a bounded reference trajectory \( (\eta_r, u_r) \), exact solution of \( \dot{y}_r = f(\eta_r, u_r) \), design an output feedback \( u = \varphi(w, y, \eta_r) \), \( w = \theta(w, y, \eta_r) \) which ensures global convergence of \( \eta \).

To illustrate our contribution we consider the system\(^1\):

\[
\begin{align*}
\dot{z} &= -z + x_2, \\
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= u + x_2 + z,
\end{align*}
\]

(1)

where \( y \in R \) is the available measurement, \( u \in R \) is the control input and \( d \) a real number in \([-1, 1)\). Consider a bounded reference trajectory \( (\eta_r, x_{1r}, x_{2r}, u_r) \), exact solution of (1), namely:

\[
\begin{align*}
\dot{z}_r &= -z_r + x_{2r}, \\
\dot{x}_{1r} &= x_{2r}, \\
\dot{x}_{2r} &= u_r + x_{2r} + z_r.
\end{align*}
\]

(2)

The problem is to find an output feedback for \( u \) such that \((z, x_1, x_2)\) converges to \((\eta_r, x_{1r}, x_{2r})\). This can be rephrased as finding \( \tilde{u} = u_r - u \), depending on \((z_r, \tilde{x}_1, x_{1r}, x_{2r}, u_r)\), rendering the origin of the error system:

\[
\begin{align*}
\dot{\tilde{z}} &= -\tilde{z} + \tilde{x}_2, \\
\dot{\tilde{x}}_1 &= \tilde{x}_2, \\
\dot{\tilde{x}}_2 &= \tilde{u} + x_{2r} + (x_{2r} + \tilde{x}_2)^{1+d} + \tilde{z},
\end{align*}
\]

globally attractive.

To solve this problem we follow a domination approach based on homogeneity. This leads to the term \( x_{1r} + d \) \( x_{2r} + \tilde{x}_2 \) \( 1+d \) \( z \) as a perturbation which can be upper-bounded as:

\[
|x_{1r} + d - (x_{2r} + \tilde{x}_2) + \tilde{z} + \tilde{z}| \leq (1 + d) |x_{2r}| + |x_{2r} + \tilde{x}_2| + |\tilde{z} + \tilde{z}| .
\]

II. MAIN RESULT OF THE PAPER

Consider a system whose dynamics are:

\[
\begin{align*}
\dot{z} &= F(z, x), \\
\dot{x}_1 &= x_2 + \delta_1(z, x), \\
\dot{x}_2 &= x_3 + \delta_2(z, x), \\
\dot{x}_n &= u + \delta_n(z, x),
\end{align*}
\]

where \( x = (x_1, \ldots, x_n) \) in \( R^n \), \( y \) is the output in \( R \), \( u \) is the input in \( R \) and \( z \) in \( R^n \) is the state of the inverse dynamics. This state can be "neglected" provided the inverse dynamics with \( \delta_i \) as output and \( x \) as input are incremental ISS (see [3]). Specifically, we make the following assumption.

\textbf{Assumption 1:} There exist a real number \( d_\infty \) in \([0, \frac{1}{n-1})\), positive real numbers \( c_\infty \), \( p \) and \( q \), non negative \( C^1 \) functions \( Z_i \), non negative continuous functions \( \Omega \), \( \gamma_i \) and \( \mu_i \), a function \( \alpha \) of class \( K_\infty \) and a continuous function \( \omega \), with strictly positive values such that, for all \( i \),
1.1 \( \mu^s_i \) is \( C^1 \), convex and satisfies \( s\mu^s_i(s) \leq q\mu^s_i(s) \), \( \mu_i(0) = 0 \).

1.2 \( \alpha(|\tilde{z}|) \omega(|z|) \leq \sum_{i=1}^n \mu_i(Z_i(z, \tilde{z})) \).

1.3 \( \frac{\partial Z_i}{\partial z}(z, \tilde{z}) F'(z, x) + \frac{\partial Z_i}{\partial \tilde{z}}(z, \tilde{z}) [F(z + \tilde{z} + x + \tilde{\chi}) - F(z, x)] \leq -Z_i(z, \tilde{z}) + \gamma_i(\tilde{\chi}) \).

1.4 \( |\delta_i(z + \tilde{z}, x + \tilde{\chi}) - \delta_i(z, x)| \leq \Omega(x, z) \sum_{j=1}^i |\tilde{z}_j| + c_\infty \sum_{j=1}^i |\tilde{z}_j|^{1-d_\infty(n-i)} + \mu_i(Z_i(z, \tilde{z})) \).

1.5 \( \mu_i((1 + v) \gamma_i(\tilde{\chi})) \leq c_\infty \sum_{j=1}^i |\tilde{z}_j| + |\tilde{z}_j|^{1-d_\infty(n-i)} \).

In Section IV, we prove:

**Theorem 1 (Main result):** Consider system (4) with Assumption 1. Then the controller given in (16) solves the tracking problem for any bounded reference trajectory \( t \mapsto (z_r(t), x_r(t), u_r(t)) \) which is a particular solution of (4).

For the system (1), the condition 1.4 of Assumption 1 follows from inequality (3) by setting \( d_\infty = d \) and picking \( \Omega(x_r, z_r) = (1 + d) |z_r|^d, Z_2(z, z_r) = |z_r|^d, \mu_2(s) = \sqrt{s}, \gamma_2(\tilde{\chi}) = |\tilde{\chi}|^2 \). Consequently, system (1) belongs to the class of systems satisfying Assumption 1 and Theorem 1 applies.

Assumption 1 is to be compared with the one in [8], where standard homogeneity and domination is also used. In that contribution, there is no \( z \) dynamics and the objective is only practical tracking. This allows the authors to work with an assumption weaker than 1.4 since it is needed only for \( (x, z, \tilde{\chi}, \tilde{\chi}) = (0, 0, \tilde{x}, \tilde{z}) \). (See footnote 4). Here we get exact tracking but under the more restrictive assumption that the reference trajectory is an exact solution of the system to be controlled. [4] and [11] are two other contributions where such a more restrictive assumptions is not needed, but on the other hand they do not allow polynomial growth in the \( \delta_i \).

Another important point is that we do not need to know in advance the whole reference trajectory or even a bound on it to design the controller. This is to be opposed for instance to the controller proposed in [8]. Compare also with [11].

**III. HOMOGENEOUS TOOLS FOR A CHAIN OF INTEGRATOR**

Throughout this section we consider a chain of integrator, with state \( \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \) described by:

\[ \dot{\mathbf{x}} = \mathbf{Sx} + Bu, \quad y = x_1, \]  

where \( B = (0, \ldots, 1)^T \) and \( \mathbf{S} \) denotes the left shift matrix of order \( n \), i.e.

\[ \mathbf{Sx} = (x_2, \ldots, x_n, 0)^T. \]

We deal now with homogeneity in the bi-limit. This notion and its properties are studied in [2]. We give in the Appendix a brief summary.

Selecting arbitrary degrees \( d_0 \leq d_\infty \) in \( (-1, \frac{1}{n-1}) \), homogeneity in the bi-limit is obtained for system (5) provided the weights \( r_0 = (r_{0,1}, \ldots, r_{0,n}) \) and \( r_\infty = (r_{\infty,1}, \ldots, r_{\infty,n}) \) are:

\[ r_{0,i} = 1 - d_0(n - i), \quad r_{\infty,i} = 1 - d_\infty(n - i). \]  

(6)

**A. Homogeneous in the bi-limit observer**

In [1], [2], we have proposed an observer for system (5) given by:

\[ \hat{\mathbf{x}} = \mathbf{S}\hat{\mathbf{x}} + Bu + K(\hat{x}_1 - x_1), \]  

where \( \hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_n) \in \mathbb{R}^n \), and \( K \) is a homogeneous in the bi-limit vector field with weights \( r_0 \) and \( r_\infty \), and degrees \( d_0 \) and \( d_\infty \). Setting:

\[ E = (e_1, \ldots, e_n)^T = \hat{\mathbf{x}} - \mathbf{x} \]

yields the error system:

\[ \dot{E} = SE + K(e_1). \]  

(8)

The design of \( K \) is done recursively in such a way that there exists a homogeneous in the bi-limit Lyapunov function \( W \) of degree \( d_W \) satisfying, for some real number \( c_1 \),

\[ \frac{\partial W}{\partial E}(SE + K(e_1)) \leq -c_1 \left( W(E)^{d_W+d_0} + W(E)^{d_W+d_\infty} \right), \]  

(9)

To combine this tool with dynamic scaling we need to establish a specific property on the error Lyapunov function \( W \). This property is a homogeneous in the bi-limit version of the one given in [14, equation (16)] or in [9, Lemma A1]. Namely, given the diagonal matrix

\[ D = \text{diag}(d_1, \ldots, d_n), \]

with \( d_i > 0 \), the function \( K \) has to be selected such that the associated error Lyapunov function \( W \) satisfies (9), and also:

\[ \frac{\partial W}{\partial E}D E \geq c_2 W(E), \]  

(10)

for some positive real number \( c_2 \).

Such a property can be obtained by modifying2 the recursive procedure given in [1], [2] as claimed in the following statement the proof of which is omitted.

**Theorem 2:** Let \( d_W \) be a positive real number satisfying \( d_W \geq 2 \max_{1 \leq j \leq n} r_{0,j} + d_\infty \) and \( D = \text{diag}(d_1, \ldots, d_n) \) with \( d_j > 0 \). There exists a homogeneous in the bi-limit vector field \( K : \mathbb{R} \to \mathbb{R}^n \), with associated triples \((r_0, d_0, K_0)\) and \((r_\infty, d_\infty, K_\infty)\) and a positive definite, proper and \( C^1 \) function \( W : \mathbb{R}^n \to \mathbb{R}_+ \), homogeneous in the bi-limit with associated triples \((r_0, d_W, W_0)\) and \((r_\infty, d_W, W_\infty)\), such that:

1) The functions \( W_0 \) and \( W_\infty \) are positive definite and proper and the functions \( \frac{\partial W}{\partial E} \) are homogeneous in the bi-limit.

---

2We multiply \( W_{i+1} \) by a sufficiently large number before using it in the definition of \( W_i \).
2) There exist two positive real numbers $c_1$ and $c_2$ such that (9) and (10) are satisfied.

B. Homogeneous in the bi-limit state feedback

In the same spirit as the homogeneous in the bi-limit observer design introduced in the previous section, we modify the recursive state feedback design introduced in [1] to make it compatible with dynamic scaling. The result is expressed in the following statement.

**Theorem 3**: Let $d_V$ be a positive real number satisfying $d_V > 2 \max 1 \leq j \leq n \eta_{0,j}$, and $D = (d_1, \ldots, d_n)$, with $d_j > 0$. There exist a homogeneous in the bi-limit function $\phi : \mathbb{R}^n \to \mathbb{R}$, with associated triples $(r_0, 1 + d_0, \phi_0)$ and $(r_\infty, 1 + d_\infty, \phi_\infty)$ and a positive definite, proper and $C^1$ function $V : \mathbb{R}^n \to \mathbb{R}_+$ homogeneous in the bi-limit with associated triples $(r_0, d_V, V_0)$ and $(r_\infty, d_V, V_\infty)$, such that:

1) The functions $V_0$ and $V_\infty$ are positive definite and proper and the functions $\frac{\partial V}{\partial x}$ are homogeneous in the bi-limit.

2) There exists $c_3 > 0$ such that, for all $x \in \mathbb{R}^n$,
$$ \frac{\partial V}{\partial x}(x) (Sx + B \phi(x)) \leq -c_3 \left( V(x)^{d_V+d_\infty} + V(x)^{d_V+d_0} \right). $$

3) There exists $c_4 > 0$ such that, for all $x \in \mathbb{R}^n$,
$$ \frac{\partial V}{\partial x}(x) D x \geq c_4 V(x). $$

IV. PROOF OF THEOREM 1

A. Rephrasing the problem as a stabilization problem

Setting
$$ \hat{u} = u - u_r, \quad \hat{x} = x - x_r, \quad \tilde{z} = z - z_r, $$
we obtain:
$$ \begin{cases} 
\dot{\tilde{z}} & = F(zr + \tilde{z}, x_r + \hat{x}) - F(z_r, x_r), \\
\dot{\tilde{x}}_1 & = \tilde{x}_2 + \delta_1(z_r + \tilde{z}, x_r + \hat{x}) - \delta_1(z_r, x_r), \\
& \vdots \\
\dot{\tilde{x}}_n & = \hat{u} + \delta_n(z_r + \tilde{z}, x_r + \hat{x}) - \delta_1(z_r, x_r).
\end{cases} $$
(13)

The objective is now to find $\hat{u}$ depending on the output $\tilde{x}_1 = y - x_r, 1$ and on the reference trajectory $(z_r, x_r)$ such that the solution $\tilde{x} = 0$ is globally attractive.

B. Output feedback using homogeneous in the bi-limit tools and dynamic scaling

The output feedback is obtained from the functions $K$ and $\phi$ given by Theorems 2 and 3 and by introducing an extra dynamically updated gain $L$.

The first step consists in selecting the parameter $D$ in Theorem 2 and 3. Following [14], let:
$$ D = \text{diag}(b, 1 + b, \ldots, n - 1 + b)^T, $$
where $b$ is a positive real number satisfying
$$ \frac{1 - d_\infty(n - i - 1)}{1 - d_\infty(n - j - 1)} < \frac{i + b}{j - 1 + b} < \frac{i}{j - 1}, $$
for all $1 \leq j \leq i \leq n$ and with $d_\infty$ as given in Assumption 1.

Selecting $d_0 = 0$ and $d_V = d_V = d_V$ sufficiently large, we apply Theorem 2 and 3 to construct the homogeneous in the bi-limit vector field $K$, the state feedback $\phi$ and the Lyapunov functions $W$ and $V$ such that (9), (10), (11) and (12) hold.

Following [1], the output feedback is given by:
$$ u = u_r + L^{n+b} \phi(\tilde{x}^{-1}), $$
and $L$ satisfying:
$$ \dot{L} = -a_1 a_2 L + L \max \{0, a_1 (a_2 L + a_3 \Omega(z_r, x_r)) \}, $$
where $a_1$, $a_2$, $a_3$ are positive real numbers to be defined with $a_2 = a_2 - a_2 > 0$.

This update law of the gain $L$ is an elaboration from the original one introduced in [14, (24)] and modified in [5, (3.12)] or [10, (134)]. Note that the driving term depends only on the reference trajectory $(x_r, z_r)$. Since this trajectory is bounded, $L$ is bounded along any closed-loop solution. Also we see that, if initialized to a value larger than $a_2$, $L$ remains larger than $a_2$ along any closed-loop solution. Moreover the presence of the term $-a_1 a_2 L$ allows to recover the main property of [14, (24)], i.e. $L$ “follows” its driving term. Specifically the Dini derivative of $\left| L - \frac{a_1 a_2 + a_3 \Omega(z_r, x_r)}{a_1} \right|$ satisfies
$$ D^+ \left| L - \frac{a_1 a_2 + a_3 \Omega(z_r, x_r)}{a_1} \right| \leq \frac{a_3}{a_1} \left| \Omega(z_r, x_r) \right|,$$
$$ \quad -a_1 \min \{a_2, a_2\} L - \frac{a_1 a_2 + a_3 \Omega(z_r, x_r)}{a_1}.$$

Hence along the solution of the closed loop system, we have
$$ \limsup_{t \to +\infty} \left| L(t) - \frac{a_1 a_2 + a_3 \Omega(z_r(t), x_r(t))}{a_1} \right| \leq \frac{a_3}{a_1} \min \{a_2, a_2\} \limsup_{t \to +\infty} \left| \Omega(z_r(t), x_r(t)) \right|. $$

Properties of the closed loop system:

Let
$$ E = (e_1, \ldots, e_n)^T, \quad \hat{X} = (\hat{x}_1, \ldots, \hat{x}_n)^T $$
and $\tau$ be scaled quantities defined as:

3We multiply $V_\infty$ by a sufficiently large number before using it in the definition of $V_{\infty+1}$.

4 In [8], $\hat{x}$ is defined as $\hat{x} = (x_1 - y_r, x_2, \ldots, x_n)$. This leads to the presence of $y_r$ as a disturbance in the $\hat{x}$ dynamics explaining why practical tracking is obtained and why a weaker assumption can be invoked.

5 This choice is always possible since, for $1 \leq j \leq i \leq n$, we have:
$$ \frac{i + b}{j - 1 + b} < \frac{i}{j - 1} \quad \forall b > 0, $$
and
$$ 1 \leq \frac{1 - d_\infty(n - i - 1)}{1 - d_\infty(n - j - 1)} < \frac{i}{j - 1} \quad \forall d_\infty \in [0, \frac{1}{n - 1}]. $$
\[
\begin{align*}
\frac{dE}{d\tau} &= SE + K(e_1) - L^{-1} \frac{dL}{d\tau} DE - \mathcal{D}(L), \\
\frac{d\hat{X}}{d\tau} &= S\hat{X} + B\phi(\hat{X}) + K(e_1) - L^{-1} \frac{dL}{d\tau} D\hat{X} - \mathcal{D}(L), \\
\text{Chain of integrator part} &\quad \text{Dynamic Scaling} & \quad \text{Nonlinearities} = \text{Disturbances}
\end{align*}
\]  
(19)

\[
T_{Cl} \leq -c_5 \left( U(\hat{X}, E) + U(\hat{X}, E) \frac{d_{\infty} + d_{\infty}}{d_{\infty}} \right)
\]  
(22)

Note that \( \ell \) and \( c_5 \) have been fixed.

**Bound on the term** \( T_{DS} \). From Claim A.1, the function \((\hat{X}, E) \mapsto \frac{\partial V}{\partial X}(\hat{X}) D\hat{X} + \ell \frac{\partial W}{\partial E}(E) D E\) is homogeneous in the bi-limit with associated weights \((r_0, r_0)\) and \((r_{\infty}, r_{\infty})\) and degrees \(d_U\) and \(d_U\). Hence Claim A.3, as well as equations (10) and (12) yield positive real numbers \(c_6\) and \(c_7\) such that:

\[
c_6 \geq \frac{\partial V}{\partial X}(\hat{X}) D\hat{X} + \ell \frac{\partial W}{\partial E}(E) D E \geq c_7.
\]

Finally, using the expression of \( \dot{L} \) in (17), this gives:

\[
T_{DS} \leq -c_7 a_3 \frac{\Omega(\hat{X}, E)}{L} U(\hat{X}, E) + c_5 a_1 U(\hat{X}, E).
\]  
(23)

**Bound on the term** \( T_{Dist} \). Let \( x_j = L^{b+j-1}(x_j - e_j) \),

By Assumption 1.4 and equation (6), we have, for all \( i \),

\[
|\mathcal{D}_i(L)| = L^{-i-b} |\delta_i(z_r + \tilde{z}, x_r + \tilde{x}) - \delta_i(z_r, x_r)| \\
\leq \Omega(z_r, x_r) \sum_{j=1}^i L^{j-i-1} |\dot{x}_j - e_j| + c_\infty L^{-i-b} \sum_{j=1}^i [L^{j+i-1}(x_j - e_j)]^{r_{\infty} + d_{\infty}} + L^{-i-b} \mu_i(Z_i).
\]

Inequalities (15) imply the existence of a real number \( \epsilon > 0 \) such that

\[
L^{-\epsilon} \geq L^{(b+j-1) r_{\infty} + d_{\infty}} \quad \forall L \geq 1.
\]

The condition \( L \geq 1 \) holds along all closed loop solutions if \( a_2 \geq 1 \). Consequently, for all \((\hat{X}, E) \in \mathbb{R}^{2n}\) and \( L \geq a_2 \geq 1 \),

\[
|\mathcal{D}_i(L)| \leq \frac{\Omega(z_r, x_r)}{L} \sum_{j=1}^i |\dot{x}_j - e_j| + c_\infty a_2^{-\epsilon} \sum_{j=1}^i (x_j - e_j)^{r_{\infty} + d_{\infty}} + L^{-i-b} \mu_i(Z_i).
\]

On the other hand, the function \( \left|\frac{\partial W}{\partial E}(E)\right| |\dot{x}_j - e_j| \) (respectively \( \left|\frac{\partial W}{\partial E}(E)\right| |\dot{x}_j - e_j|^{r_{\infty} + d_{\infty}} \)) is homogeneous in the bi-limit with weights \((r_0, r_0)\) and \((r_{\infty}, r_{\infty})\) and degrees \(d_U\) and \(d_U\) respectively \(d_U - 1 + r_{\infty} + d_{\infty} \geq d_U\) and \(d_{\infty} + d_U\). Hence, Claim A.3 yields positive real numbers \(c_8\) and \(c_9\), such that:

\[
\sum_{i=1}^L |\dot{x}_j - e_j| + c_8 a_2^{-\epsilon} \sum_{j=1}^i (x_j - e_j)^{r_{\infty} + d_{\infty}} + L^{-i-b} \mu_i(Z_i) \leq -c_9 \left( U(\hat{X}, E) + U(\hat{X}, E) \frac{d_{\infty} + d_{\infty}}{d_{\infty}} \right)
\]  
(24)
\[ \left| \frac{T_{dist}}{\ell} \right| \leq c_8 \frac{\Omega(z_r, x_r)}{L} U(\hat{X}, E) 
+ c_\infty a_2^{-\epsilon} c_9 \left( U(\hat{X}, E) + U(\hat{X}, E) \frac{d_{i, r} + d_{i, \infty}}{\kappa} \right) 
+ \sum_{i=1}^{n} \left| \frac{\partial W}{\partial e_i}(E) \right| L^{-i-b} \mu_i(Z_i) . \]

Hence, by (22) and (23), selecting \( a_1 \) sufficiently small and \( a_2 \) and \( a_3 \) sufficiently large guarantees the existence of a positive real number \( c_{10} \) such that (21) becomes:
\[
dU(\hat{X}, E) 
\leq -2 c_{10} \left( U(\hat{X}, E) + U(\hat{X}, E) \frac{d_{i, r} + d_{i, \infty}}{\kappa} \right) 
+ \sum_{i=1}^{n} \left| \frac{\partial W}{\partial e_i}(E) \right| L^{-i-b} \mu_i(Z_i) .
\]

**Small-gain arguments.** Let \( \zeta_i \) and \( \theta \) be the homogeneous in the bi-limit functions defined as
\[
\zeta_i(s) = |s|^{d_{i, r} + |s|^{d_{i, \infty} + d_{i, \infty}}} , \quad \theta(s) = |s|^{d_{i, r} + |s|^{d_{i, \infty} + d_{i, \infty}}} .
\]

We use Claim A.2 with the functions
\[
-c_{10} \theta(U(\hat{X}, E)) + \sum_{i=1}^{n} \left| \frac{\partial W}{\partial e_i}(E) \right| s_i \quad \text{and} \quad \sum_{i=1}^{n} \zeta_i(s_i),
\]
which are homogeneous in the bi-limit with weights \( (r_{\infty}, r_{\infty}) \) and \( (0, r_{\infty}) \) for \( (\hat{X}, E) \), and \( (1, r_{\infty} + d_{\infty}) \) for \( s_i \) and degrees \( d_{U} \) and \( d_{\infty} + d_{U} \). This gives a positive real number \( c_{11} \) satisfying:
\[
-c_{10} \theta(U(\hat{X}, E)) + \sum_{i=1}^{n} \left| \frac{\partial W}{\partial e_i}(E) \right| s_i \leq c_{11} \sum_{i=1}^{n} \zeta_i(s_i) .
\]

Thus, inequality (25) (in time \( t \)) implies:
\[
\dot{U}(\hat{X}, E) \leq -c_{10} L \theta(U(\hat{X}, E)) 
+ c_{11} L \sum_{i=1}^{n} \zeta_i(L^{-i-b} \mu_i(Z_i(z_r, \hat{z}))) .
\]

Since, along solutions, \( L \) is lower bounded by \( a_2 \), this implies the existence of a class \( \mathcal{KL} \) function \( \beta_u \) such that, for all \( t \) in the domain of existence of the closed loop solution and for all \( s \leq t \)
\[
U(t) \leq \beta_u(U(s), t - s) 
+ \sup_{s \leq t} \sum_{i=1}^{n} \theta^{-1} \left( \frac{2 c_{11}}{c_{10}} (L^{(r_{-i-b})} \mu_i(Z_i(z_r, \hat{z}))) \right) ,
\]
where to simplify the notation, we have defined
\[
\hat{U}(t) = U(\hat{X}(t), E(t)) \quad \text{and} \quad \hat{Z}_i(t) = Z_i(z_r(t), \hat{z}(t))
\]
and we have used the property \( \theta(a) + \theta(b) \leq \theta(a + b) \).

Now, with \( p \) as in Assumption 1, let:
\[
\hat{Y}_i \leq -p(i + b) \mu_i(Z_i(z_r, \hat{z})).
\]

We get:
\[
\dot{\hat{Y}}_i \leq -p(i + b) \hat{U} \hat{Y}_i 
+ L^{-p(i+b)} \mu_i''(Z_i(z_r, \hat{z})) \{-Z_i(z_r, \hat{z}) + \gamma_i(\hat{x}) \}. \]

Since \( \hat{L} + a_1 a_2 L \) is non-negative, using property 1.1 of \( \mu_i \) and considering the two cases \((1 + v) \gamma_i(\hat{x}) \leq Z_i(z_r, \hat{z}) \) and \( Z_i(z_r, \hat{z}) \leq (1 + v) \gamma_i(\hat{x}) \), we obtain:
\[
\hat{Y}_i \leq \left[ \frac{v}{1+v} - p(i + b) a_1 a_2 \right] Y_i 
+ q L^{-p(i+b)} \mu_i((1 + v) \gamma_i(\hat{x}))^p . \]

We remark that the functions \( \zeta_i^{-1} \left( \frac{c_{10}}{2 c_{11}} \theta(U(\hat{X}, E)) \right) \)
and \( |x_j - e_j| + |x_j - e_j| \) are homogeneous in the bi-limit with weights \((r_0, r_0), (r_\infty, r_\infty)\) and degree 1 and \( r_{\infty}, i + d_{\infty} \). Hence, using Claim A.3, we obtain the existence of a positive number \( c_{12} \) such that:
\[
c_{\infty} \sum_{i=1}^{n} \zeta_i^{-1} \left( \frac{c_{10}}{2 c_{11}} \theta(U(\hat{X}, E)) \right) \leq c_{12} . \]

Moreover, since \( L \geq a_2 \geq 1 \),
\[
L^{-i-b} \sum_{j=1}^{n} |\hat{x}_j - \hat{e}_j| + |\hat{x}_j - \hat{e}_j| \leq a_2^{-\epsilon} \sum_{j=1}^{n} |x_j - e_j| + |x_j - e_j| . \]

By Assumption 1.5, it follows that:
\[
\dot{Y}_i \leq \left[ \frac{v}{1+v} - p(i + b) a_1 a_2 \right] Y_i 
+ q \left[ a_2^{-\epsilon} c_{12} \zeta_i^{-1} \left( \frac{c_{10}}{2 c_{11}} \theta(U(\hat{X}, E)) \right) \right]^p . \]

Hence by selecting \( a_1 \) and \( a_2 \) to satisfy:
\[
a_1 a_2 < \frac{v}{p(1+v)(n+b)},
\]
we obtain the existence of a class \( \mathcal{KL} \) function \( \beta_z \) and a real number \( c_{13} \) such that, for all \( i \), all \( t \) in the domain of existence of the closed loop solution and all \( s \leq t \)
\[
L(t)^{-i-b} \mu_i(Z_i(t)) \leq \beta_z(L(s)^{-i-b} \mu_i(Z_i(s)), t - s) 
+ \sup_{r \in [s,t]} a_2^{-\epsilon} c_{13} \zeta_i^{-1} \left( \frac{c_{10}}{2 c_{11}} \theta(U(r)) \right) .
\]

Since, by choosing \( a_2 \) large enough the small gain condition [6, condition (14)] is satisfied (see also [7]), we conclude there exists a function \( \beta_u \) of class \( \mathcal{KL} \) such that, on the domain of existence of the closed loop solution, and for all \( i \),
\[
U(t) \leq \beta_u(U(0) + \sum_{i=1}^{n} L(0)^{-i-b} \mu_i(Z_i(0)), t) .
\]

By Assumptions 1.2 and 1.3, there exists another function \( \beta_z \) of class \( \mathcal{KL} \) which depends on the bounds \( \sup_t L(t) \) and \( \sup_t |z_r(t)| \) such that, on the domain of existence of the closed loop solution,
\[
\alpha(|\hat{z}(t)|) \leq \beta_z(U(0) + \sum_{i=1}^{n} \mu_i(Z_i(0)), t) .
\]

This implies that the domain of existence is \([0, +\infty)\), and the global attractiveness of \( \hat{x} = E = 0 \) and \( \hat{z} = 0 \).

**V. Conclusion**

We have solved a state trajectory tracking problem for minimum phase nonlinear systems which admit globally a strict normal form and in such a way that the nonlinearities
satisfy power growth. This has been achieved by exploiting the tools of domination, homogeneity in the bi-limit and dynamic scaling gain.

REFERENCES

APPENDIX
ON HOMOGENEITY IN THE BI-LIMIT

For details on the notion of homogeneity in the bi-limit refer to [2]. We only give the definition and state the main properties used in this paper.

Given a vector $r = (r_1, \ldots, r_n)$ in $(\mathbb{R}_+ / \{0\})^n$, we define the dilation of a vector $x$ in $\mathbb{R}^n$ as

$$\lambda^r x = (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n)^T .$$

Definition 1 (Homogeneity in the 0-limit):

- A continuous function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is said homogeneous in the 0-limit with associated triple $(r_0, d_0, \phi_0)$, where $r_0$ in $(\mathbb{R}_+ / \{0\})^n$ is the weight, $d_0$ in $\mathbb{R}_+$ the degree and $\phi_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ the approximating function, respectively, if $\phi_0$ is continuous and not identically zero and, for each compact set $C$ in $\mathbb{R}^n \setminus \{0\}$ and each $\varepsilon > 0$, there exists $\lambda^*$ such that we have :

$$\max_{x \in C} \left\{ \frac{\phi(\lambda^{r_0} \circ x)}{\lambda^{d_0}} - \phi_0(x) \right\} \leq \varepsilon \quad \forall \lambda \in (0, \lambda^*].$$

- A vector field $f = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ is said homogeneous in the 0-limit with associated triple $(r_0, d_0, f_0)$, where $f_0$ = $\sum_{i=1}^n f_{0,i} \frac{\partial}{\partial x_i}$, if, for each $i$ in $\{1, \ldots, n\}$, the function $f_i$ is homogeneous in the 0-limit with associated triple $(r_0, d_0 + r_{0,i}, f_{0,i})$.

Definition 2 (Homogeneity in the $\infty$-limit):

- A continuous function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is said homogeneous in the $\infty$-limit with associated triple $(r_\infty, d_\infty, \phi_\infty)$ where $r_\infty$ in $(\mathbb{R}_+ / \{0\})^n$ is the weight, $d_\infty$ in $\mathbb{R}_+$ the degree and $\phi_\infty : \mathbb{R}^n \rightarrow \mathbb{R}$ the approximating function, respectively, if $\phi_\infty$ is continuous and not identically zero and, for each compact set $C$ in $\mathbb{R}^n \setminus \{0\}$ and each $\varepsilon > 0$, there exists $\lambda^*$ such that we have :

$$\max_{x \in C} \left\{ \frac{\phi(\lambda^{r_\infty} \circ x)}{\lambda^{d_\infty}} - \phi_\infty(x) \right\} \leq \varepsilon \quad \forall \lambda \in [\lambda^*, +\infty).$$

- A vector field $f = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ is said homogeneous in the $\infty$-limit with associated triple $(r_\infty, d_\infty, f_\infty)$, with $f_\infty = \sum_{i=1}^n f_{\infty,i} \frac{\partial}{\partial x_i}$, if, for each $i$ in $\{1, \ldots, n\}$, the function $f_i$ is homogeneous in the $\infty$-limit with associated triple $(r_\infty, d_\infty + r_{\infty,i}, f_{\infty,i})$.

Definition 3 (Homogeneity in the bi-limit):

A continuous function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ (or a vector field $f$) is said homogeneous in the bi-limit if it is homogeneous in the 0-limit and homogeneous in the $\infty$-limit.

Claim : Let $\eta$ and $\gamma$ be two continuous homogeneous in the bi-limit functions with weights $r_0, r_\infty$, degrees $d_{\eta,0}, d_{\eta,\infty}$ and $d_{\gamma,0}, d_{\gamma,\infty}$, and continuous approximating functions $\eta_0, \eta_\infty, \gamma_0, \gamma_\infty$.

A.1. The function $x \mapsto \eta(x)\gamma(x)$ is homogeneous in the bi-limit with associated triples $(r_0, d_{\eta,0} + d_{\gamma,0}, \eta_0 \gamma_0)$ and $(r_\infty, d_{\eta,\infty} + d_{\gamma,\infty}, \eta_\infty \gamma_\infty)$.

A.2. If the degrees satisfy $d_{\eta,0} \geq d_{\gamma,0}$ and $d_{\eta,\infty} \leq d_{\gamma,\infty}$, and $\gamma(x) \geq 0$, and for $x \neq 0$

$$\gamma(x) = 0 \quad \Rightarrow \quad \eta(x) < 0 ,$$

$$\gamma_0(x) = 0 \quad \Rightarrow \quad \eta_0(x) < 0 ,$$

$$\gamma_\infty(x) = 0 \quad \Rightarrow \quad \eta_\infty(x) < 0 ,$$

then there exists a real number $k^*$ such that, for all $k \geq k^*$, and for all non zero $x$ in $\mathbb{R}^n$:

$$\eta(x) < k \gamma(x) , \eta_0(x) < k \gamma_0(x) , \eta_\infty(x) < k \gamma_\infty(x) .$$

A.3. If the degrees satisfy $d_{\eta,0} \geq d_{\gamma,0}$ and $d_{\eta,\infty} \leq d_{\gamma,\infty}$ and the functions $\gamma, \gamma_0$ and $\gamma_\infty$ are positive definite then there exists a positive real number $\varepsilon$ satisfying $\eta(x) \leq c \gamma(x)$ for all $x$ in $\mathbb{R}^n$. 

\footnote{In the case of a vector field the degree $d_0$ can be negative as long as $d_0 + r_{0,i} \geq 0$, for all $1 \leq i \leq n$.}