LOCAL ANALYSIS OF A ONE STEP AHEAD ADAPTIVE CONTROLLER

L. Praly ** and D. Rhode
Coordinated Science Laboratory
University of Illinois
1101 W. Springfield Avenue
Urbana, Illinois 61801

Abstract

A local analysis of the one step ahead adaptive controller of [1] is presented. We study the behavior of the closed loop adaptive system, in presence of periodic inputs, around a trajectory defined as the solution of a nominal system. We establish that if this solution is unique, it is uniformly asymptotically stable and we can tolerate a wide class of small perturbations from this nominal system.

1. Introduction

Consider the following adaptive controller [1].

\[ \dot{\gamma}(k) \equiv (u(k), \ldots, u(k-n-d+1), y(k), \ldots, y(k-n+1)) \]

\[ \dot{\gamma}(k) = \dot{\gamma}(k-1) + y(k)z(k-d)(y(k) - \tilde{z}(k-d)) \]

\[ \dot{\gamma}(k) = \dot{\gamma}(k-d) \]

where \( \gamma(k) \) is a scalar gain, \( y_m(k+d) \) is a reference output known \( d \) steps in advance and \( u(k), y(k) \) are the input and output of a plant which can be described by:

\[ A(q^{-1})y(k) = q^{-1}B(q^{-1})u(k) + w(k) \]

\[ A(q^{-1}) \text{ and } B(q^{-1}) \text{ are polynomials of unknown degree.} \]

\( \{u(k)\} \) is a sequence which may depend upon \( \gamma(k) \) and \( y(k) \). The nominal plant is given when \( w(k) = 0 \).

Goodwin, Ramadge and Caines have shown in [2] if \( n, m \) are not less than the degree of \( A(q^{-1}), B(q^{-1}) \), respectively, \( d \) is the plant delay, the plant is minimum phase then the adaptive controller ensures global stability and the tracking error tends to zero for the nominal plant or the nominal test reference output. However, this type of statement is only known for \( \gamma(k) \) small. This corresponds to the case of slow adaptation.

In this paper we will study a problem similar to [4]. However, instead of assuming that there exists a constant parameter vector \( \gamma^* \) which gives for the nominal plant a tracking transfer function equal to 1 for any frequency, we now assume that this holds only at the frequencies contained in \( y_m(k) \). This is not possible in general, but is possible for some particular sequences \( \{y_m(k)\} \) called test reference outputs (see definition in Section 2).

In our problem formulation, the nominal case is the data of \( A(q^{-1}), B(q^{-1}) \) and a test reference output \( \{y_m(k)\} \). The constant parameter vector \( \gamma^* \) and the corresponding signals \( u^*(k), y^*(k) \) are called the tuned solution of the adaptive system. We establish that under the assumption of uniqueness of the tuned solution, this solution is exponentially stable. It follows that the stability is robust to a wide class of small perturbations of the nominal plant or the nominal test reference output.

Since we will analyze only the local behavior of the system around the tuned solution, the projection and the normalization \( \gamma(k) = s(k)\gamma \) introduced in [3,4], respectively, are not active. We will concentrate our attention on slow adaptation, i.e., with \( s \) small:

\[ \gamma(k) = \frac{1}{s(k)^2} \]

In [5] global stability is shown if there exists a constant parameter vector \( \gamma^* \) which gives for the nominal plant a tracking transfer function, (which may be noncausal), having the following properties:

i) The Nyquist plot of its causal part lies entirely within a circle of radius 1 and center 1. This is a conicity condition.

ii) The noncausal part is sufficiently small.

It is important to note that the preceding results made no assumptions concerning the reference output. But, as an extension of the work of Anderson and Johnson [6], Kosut and Anderson [7], Astron [8], Krause [9], Anderson, Bitmead, Johnson and Kosut [10], Riedle and Kokotovic [11], Kokotovic, Riedle and Praly [12], ... on the stability of error models, it has recently been established in [13] that if this signal is taken into account, then the classical conicity condition can be relaxed to a signal dependent conicity condition. This implies that in our case we can expect the following sufficient condition: The transfer function evaluated only at frequencies contained in \( y_m(k) \) lies in a circle of radius 1 and center 1. However, this type of statement is only known for \( \gamma(k) \) small. This corresponds to the case of slow adaptation.

In our problem formulation, the nominal case is the data of \( A(q^{-1}), B(q^{-1}) \) and a test reference output \( \{y_m(k)\} \). The constant parameter vector \( \gamma^* \) and the corresponding signals \( u^*(k), y^*(k) \) are called the tuned solution of the adaptive system. We establish that under the assumption of uniqueness of the tuned solution, this solution is exponentially stable. It follows that the stability is robust to a wide class of small perturbations of the nominal plant or the nominal test reference output.

In the following a more useful representation of the plant (1.3) will be the nominal state space representation defined as (cancellation of poles and zeros at 0):

\[ \gamma(k) = \frac{1}{s(k)^2} \]

2. Problem Statement

In the following a more useful representation of the plant (1.3) will be the nominal state space representation defined as (cancellation of poles and zeros at 0):
\[ X(k+1) = FX(k) + G_1u(k) + G_2w(k) \]
\[ y(k) = h'X(k) \]
\[
(2.1)
\]
with
\[
X(k) = [y(k),\ldots,y(k-n*+1),u(k-l),\ldots,u(k-m*)]'
\]
and \(n^*\) and \(m^*\) are large enough for \(X(k)\) to contain all the elements of \((k-d)\). It follows that a matrix \(H\) exists satisfying
\[
\dot{z}(k-d) = HX(k).
\]
We also introduce the following notation:
\[
\dot{z}(k)' = (\dot{z}(k) - \dot{z}(k-1))',
\]
\[
\dot{z}(k) = \dot{z}(k-1) - \dot{z}(k-1) - \tau HX(k)X(k)'(h - \dot{z}(k-d) - \dot{z}(k-d)).
\]
(2.2)
We are interested in the existence of a bounded solution and in its stability. For the existence problem, let us make the following:

**Definition:** A test reference output is an \(N\)-periodic sequence \(\gamma^*(k)\) for which there exists an \(N\)-periodic sequence \(\lambda(k)\) and constants \(t^*\) with \(t^* \neq 0\) satisfying:
\[
G_1 \dot{\lambda}' = \dot{J}(G_1 \dot{z}) X(k) + G_1 \dot{\lambda}(k-d) + G_2 w(k)
\]
\[
\dot{z}(k) = \dot{z}(k-1) - \tau HX(k)X(k)'(h - \dot{z}(k-d) - \dot{z}(k-d)).
\]
(2.2)
We are interested in the existence of a bounded solution and in its stability. For the existence problem, let us make the following:

**Definition:** A test reference output is an \(N\)-periodic sequence \(\gamma^*(k)\) for which there exists an \(N\)-periodic sequence \(\lambda(k)\) and constant \(t^*\) with \(t^* \neq 0\) satisfying:
\[
G_1 \dot{\lambda}' = \dot{J}(G_1 \dot{z}) X(k) + G_1 \dot{\lambda}(k-d) + G_2 w(k)
\]
\[
\dot{z}(k) = \dot{z}(k-1) - \tau HX(k)X(k)'(h - \dot{z}(k-d) - \dot{z}(k-d)).
\]
(2.2)
We are interested in the existence of a bounded solution and in its stability. For the existence problem, let us make the following:

**Definition:** A test reference output is an \(N\)-periodic sequence \(\gamma^*(k)\) for which there exists an \(N\)-periodic sequence \(\lambda(k)\) and constant \(t^*\) with \(t^* \neq 0\) satisfying:
\[
G_1 \dot{\lambda}' = \dot{J}(G_1 \dot{z}) X(k) + G_1 \dot{\lambda}(k-d) + G_2 w(k)
\]
\[
\dot{z}(k) = \dot{z}(k-1) - \tau HX(k)X(k)'(h - \dot{z}(k-d) - \dot{z}(k-d)).
\]
(2.2)
We are interested in the existence of a bounded solution and in its stability. For the existence problem, let us make the following:

**Definition:** A test reference output is an \(N\)-periodic sequence \(\gamma^*(k)\) for which there exists an \(N\)-periodic sequence \(\lambda(k)\) and constant \(t^*\) with \(t^* \neq 0\) satisfying:
\[
G_1 \dot{\lambda}' = \dot{J}(G_1 \dot{z}) X(k) + G_1 \dot{\lambda}(k-d) + G_2 w(k)
\]
\[
\dot{z}(k) = \dot{z}(k-1) - \tau HX(k)X(k)'(h - \dot{z}(k-d) - \dot{z}(k-d)).
\]
(2.2)
We are interested in the existence of a bounded solution and in its stability. For the existence problem, let us make the following:

**Definition:** A test reference output is an \(N\)-periodic sequence \(\gamma^*(k)\) for which there exists an \(N\)-periodic sequence \(\lambda(k)\) and constant \(t^*\) with \(t^* \neq 0\) satisfying:
\[
G_1 \dot{\lambda}' = \dot{J}(G_1 \dot{z}) X(k) + G_1 \dot{\lambda}(k-d) + G_2 w(k)
\]
\[
\dot{z}(k) = \dot{z}(k-1) - \tau HX(k)X(k)'(h - \dot{z}(k-d) - \dot{z}(k-d)).
\]
(2.2)
We are interested in the existence of a bounded solution and in its stability. For the existence problem, let us make the following:

**Definition:** A test reference output is an \(N\)-periodic sequence \(\gamma^*(k)\) for which there exists an \(N\)-periodic sequence \(\lambda(k)\) and constant \(t^*\) with \(t^* \neq 0\) satisfying:
\[
G_1 \dot{\lambda}' = \dot{J}(G_1 \dot{z}) X(k) + G_1 \dot{\lambda}(k-d) + G_2 w(k)
\]
\[
\dot{z}(k) = \dot{z}(k-1) - \tau HX(k)X(k)'(h - \dot{z}(k-d) - \dot{z}(k-d)).
\]
(2.2)
We are interested in the existence of a bounded solution and in its stability. For the existence problem, let us make the following:

**Definition:** A test reference output is an \(N\)-periodic sequence \(\gamma^*(k)\) for which there exists an \(N\)-periodic sequence \(\lambda(k)\) and constant \(t^*\) with \(t^* \neq 0\) satisfying:
\[
G_1 \dot{\lambda}' = \dot{J}(G_1 \dot{z}) X(k) + G_1 \dot{\lambda}(k-d) + G_2 w(k)
\]
\[
\dot{z}(k) = \dot{z}(k-1) - \tau HX(k)X(k)'(h - \dot{z}(k-d) - \dot{z}(k-d)).
\]
(2.2)
We are interested in the existence of a bounded solution and in its stability. For the existence problem, let us make the following:

**Definition:** A test reference output is an \(N\)-periodic sequence \(\gamma^*(k)\) for which there exists an \(N\)-periodic sequence \(\lambda(k)\) and constant \(t^*\) with \(t^* \neq 0\) satisfying:
\[
G_1 \dot{\lambda}' = \dot{J}(G_1 \dot{z}) X(k) + G_1 \dot{\lambda}(k-d) + G_2 w(k)
\]
\[
\dot{z}(k) = \dot{z}(k-1) - \tau HX(k)X(k)'(h - \dot{z}(k-d) - \dot{z}(k-d)).
\]
(2.2)
We are interested in the existence of a bounded solution and in its stability. For the existence problem, let us make the following:

**Definition:** A test reference output is an \(N\)-periodic sequence \(\gamma^*(k)\) for which there exists an \(N\)-periodic sequence \(\lambda(k)\) and constant \(t^*\) with \(t^* \neq 0\) satisfying:
\[
G_1 \dot{\lambda}' = \dot{J}(G_1 \dot{z}) X(k) + G_1 \dot{\lambda}(k-d) + G_2 w(k)
\]
\[
\dot{z}(k) = \dot{z}(k-1) - \tau HX(k)X(k)'(h - \dot{z}(k-d) - \dot{z}(k-d)).
\]
(2.2)
Since $N(k,\epsilon)$ is $N$-periodic, there exists an $N$-
periodic Lyapunov transformation which maps $N(k,\epsilon)$ into
a time invariant matrix whose eigenvalues are the
characteristic values of the system. In particular for
$\epsilon = 0$, this transformation is given by

$$
\begin{bmatrix}
I & -L(k) \\
0 & I
\end{bmatrix}
$$

where $L(k)$ is the unique $N$-periodic solution of:

$$
G_k \hat{X} + \epsilon \Delta_1(k,\epsilon) = \left(F - \frac{1}{\epsilon} R \right) L(k) + \frac{1}{2} \epsilon \Delta_2(k,\epsilon)
$$

Using the same transformation for $\epsilon \neq 0$ gives:

$$
\hat{N}(k,\epsilon) = \begin{bmatrix}
I & -L(k+1) \\
0 & I
\end{bmatrix} \hat{N}(k,\epsilon) \\
\begin{bmatrix}
I & -L(k) \\
0 & I
\end{bmatrix}
$$

where $\Delta_1(k,\epsilon)$'s are bounded $N$-periodic matrices and
$\psi(k)$ is defined by:

$$
\psi(k) = - (h - \psi \delta) L(k) + \epsilon \psi(k-d)^{-1}
$$

In appendix, we prove:

**Property 1:** Under Assumption Al, we have:

$$
\psi(k) = \phi(k-d)^{-1}
$$

To complete our proof, let us take the product from
$k = 0$ to $N-1$ of $\hat{N}(k,\epsilon)$. One can see by induction that:

$$
\hat{N}(k,\epsilon) = \begin{bmatrix}
I & A_{22} & A_{21} \\
0 & I & \epsilon A_{11}
\end{bmatrix}
$$

with

$$
A_{22} = (F - \frac{1}{\epsilon} R) N - I + \epsilon \Delta_1(\epsilon)
$$

$$
A_{21} = \Delta_2(\epsilon) , A_{12} = \Delta_3(\epsilon)
$$

$$
A_{11} = \sum_{k=0}^{N-1} \epsilon \psi(k-d)^{-1} \psi(k-d)^{-1} + \epsilon \Delta_4(\epsilon)
$$

where the $\Delta_i(\epsilon)$'s are bounded.

The result follows by noticing that, as a consequence of Lemma 1 of [15], this product is similar to:

$$
\begin{bmatrix}
I + A_{22} + \epsilon^2 L A_{12} \\
\epsilon A_{12} & I + \epsilon A_{11} + \epsilon^2 A_{12} L
\end{bmatrix}
$$

where $L$ is a matrix whose existence and boundedness are guaranteed for $\epsilon$ small enough satisfying (see (11) of [15]):

$$
\|A_{22}\| \leq \frac{1}{2} \epsilon \left(\|A_{11}\| + \|A_{12}\| \|A_{22}\|\right)^{-1}
$$

where invertibility of $A_{22}$, $A_{11}$ is given by assumption.

This lemma leads us to introduce a new assumption.

**Assumption A2:** The test reference output $\psi(k)$ is such that the corresponding tuned solution satisfies:

$$
\sum_{k=0}^{N-1} \psi(k) \phi(k)^{-1} > a \quad \text{for } a > 0
$$

This is the classical persistent spannig assumption. In the appendix we show that this condition is equivalent to the uniqueness of the tuned solution and generically to the fact that the test reference output is persistently exciting of order at least equal to the number of parameters.

**Theorem 1:** Under Assumptions Al, A2, there exists $\epsilon_0$ such that for all $\epsilon, 0 < \epsilon < \epsilon_0$, the tuned solution is an exponentially stable solution of the nominal system $(\nu(k) = \omega(k) = 0)$. In particular, there exists a constant $C$ such that:

$$
\sum_{k=0}^{N-1} M(j,\epsilon) \leq C(1 - \epsilon_0)^{N-1}
$$

**Proof:** This is a direct consequence of Lemma 1. In particular, for $\epsilon$ sufficiently small, the spectral radius of

$$
\sum_{k=0}^{N-1} M(k,\epsilon) \leq C_0(1 - \epsilon \min_{k} \phi(k) \psi(k)^{-1} + \epsilon 0(1))
$$

where the second inequality is obtained from Assumption A3 and the property of $O(1)$, for $\epsilon$ small enough. Then using the $N$-periodicity we obtain:

$$
\sum_{k=0}^{N-1} M(j,\epsilon) \leq C_0(1 - \epsilon_0)^N
$$

where $C_0$ is defined by:

$$
\sum_{k=0}^{N-1} M(j,\epsilon) \leq C_0(1 - \epsilon_0)^N
$$

This theorem extends the result of Anderson and Johnson [6] which was established for the ideal case: any reference output is a test reference output.

It is remarkable that for this one step ahead adaptive controller, the number of frequencies and not their location is important. Both the tuning condition (Assumption Al) and the persistent spanning condition (Assumption A2) do not generically constrain the location of the frequencies.

However, Assumption Al imposes also that the eigenvalues of $\left(F - \frac{1}{\epsilon} R \right)$ lie in the unit circle, which is related to the frequency location via $\phi(k)$. It has been demonstrated by Astrom in ([8],[14]) that violation of this stability condition can by itself create instability.

**4. Total Stability of the Tuned Solution**

Since the tuned solution is an exponentially stable solution of the nominal system, it has also stability properties with respect to all kinds of permanent perturbations (see 11.8 of [16]). In particular, we have:

**Theorem 2:** Under Assumptions Al, A2 and for all $\epsilon, 0 < \epsilon < \epsilon_0$ there exist $N_0(\epsilon), N_0(\epsilon), \delta_0(\epsilon)$ such that if the initial conditions satisfy:

$$
\|x(0) - x^{(0)}\| + \|\hat{x}(0) - \hat{x}\| \leq N_0(\epsilon)
$$

and if we have the following noise to signal ratio bounds:

$$
|\nu(k)| \leq \nu(\epsilon) a
$$

$$
|\omega(k)| \leq \omega(\epsilon) \left(\sum_{i=0}^{k-1} (|x(i)| + \omega)\right)
$$

then the solution $x(k), \theta(k)$ to the actual system (2.2)
satisfies:
\[ |x(k) - x^*(k)| + |z(k) - z^*| \leq N(r) \]
for some value \( N(r) \).

This theorem states that if the nominal plant and its adaptive controller are such that there exists a test reference output which is persistently exciting of order equal to the number of adapted parameters then generically the actual system has a bounded solution provided that the persistent perturbations \( v(k) \) and \( w(k) \) satisfy noise to signal ratio limitations. In particular, the disturbance on the reference output is limited by the degree of persistent spanning of \( z^*(k) \).

In [7] Kosut and Anderson were aiming at such a result for a continuous time scheme, but their analysis was incomplete. Here we have obtained more complete results but in a more restricted case, because we studied the local behavior around a particular solution of the nominal adaptive scheme.

Proof: Let \( Y(k+1) \) be the complete state of system (2.5). By the variation of constants formula, we have
\[
Y(k+1) = \sum_{i=0}^{k} N(i) Y(i) + \sum_{i=0}^{k} N(i)
\]

where \( N(k) \) is the complete state of system (2.5) with the induction assumption: we want to show
\[
N(i) \leq N(i+1) \leq N(i+2).
\]

We define \( N(k) = |Y(k)| \).

With Theorem 1, we obtain using abbreviated notations
\[
N(k+1) \leq |C| |y(k)| + \sum_{i=0}^{k} |G| |v(i)| + |w(i)|
\]

With \( k = 0 \), the proof proceeds by induction. We want to show
\[
N(0) \leq N(k) \leq N(k+1) \leq N(k+2).
\]

Since
\[
|y(k)| \leq \sup_{k} |y^*(k)| + |y(j)|
\]

we have with the induction assumption:
\[
|w(k)| \leq \sup_{k} |w^*(k)| + |w(k)|
\]

with
\[
N(k) = \sup_{k} |y^*(k)| + |w(k)|
\]

Bound on \( R_{k} \) \( R_{k} \): Assuming that \( z(k) \leq \frac{\|z\|}{2} \) and using the Schwarz and triangle inequalities, it is easy to derive from the expressions for \( R_{k} \) and \( R_{k} \) the following inequalities:
\[
|R_{k}(1)| \leq |y^*(k)| P_{2}(N(i)) + \sum_{i=1}^{k} |y(i)| P_{4}(N(i))
\]

With the induction assumption; it follows that, given a bound \( N \), there exists a constant \( K \), such that if \( i \geq \frac{N}{2}/2 \):
\[
|R_{k}(1)| \leq K \|v\| N + N^2 \|y\| P_{4}(N) + N^2 \|y\| P_{4}(N) \leq N^2 \|y\| P_{4}(N)
\]

Substituting these bounds into Equation (4.1) gives:
\[
N(k+1) \leq \frac{C}{C} (C + \|v\| + K \|v\| + N + K \|v\| N) \leq N(k+1)
\]

Therefore, \( N(k+1) \) will be smaller than \( N(k) \).

This inequality has a solution in \( N \) satisfying \( 0 < N \leq 0 \) iff:
\[
C|G| |v| + \|y\| \leq 0
\]

This gives the bounds \( z^*(k), v^*(k), N(k) \). We see that these bounds tend to zero as \( r \) tends to zero.

Appendix: Properties of the Tuned Solution

Here we assume that the sequence \( y^*(k) \) can be written as:
\[
y^*(k) = y_{m}^* + z^*_k \quad , \quad w_{l}
\]

with
\[
x_{m} = x_{m-1} + \frac{1}{N} \quad , \quad z = \exp(j2\pi \frac{1}{N}) \quad , \quad y_{m}^* = \text{conjugate of } y_{m}^*
\]

where \( y_{m}^* \) is the complex conjugate of \( y_{m}^* \).

In the following we denote by \( E \) the set
\[
E = : 1; y_{m}^* \neq 0
\]

Its cardinal is \( n_{E} \).

The vector \( V(z^{-1}) \) is defined as:
\[
V(z^{-1}) = (A(z^{-1}), ..., A(z^{-1}), z^{-1}, B(z^{-1}), ..., z^{-1}) \quad , \quad \text{where } A(z^{-1}) = z^{-1} \text{, } B(z^{-1}) = z^{-1}
\]

Property A1: If
\[
B(z^{-1}) \neq 0 , \quad w_{l} \in E
\]

Then the tuning condition is equivalent to the following linear system:
\[
z_{k}^{-1} B(z^{-1}) = V(z^{-1}) \quad \text{, } \quad w_{l} \in E
\]

Corollary A1: Under the same assumption the tuning condition is satisfied iff:
\[
B(z_{1}^{-1}) \quad \text{, } \quad E \text{-Range } (V(z_{1}^{-1}))
\]

This condition is only expressed in terms of the nominal plant. It shows that the tuning condition is satisfied generically if:
\[
x_{E} \leq m + d + n
\]

This means that \( y^*(k) \) contains no more than \( m + d + n \) frequencies.

Proof: The tuning condition can be written in:
\[
y^*(k) = z^* \quad , \quad w_{l}
\]

and the control is obtained from
\[
y_{m}^*(k+d) = z^* \quad , \quad w_{l}
\]

It follows readily that:
\[
y^*(k) = y_{m}^* \quad , \quad w_{l}
\]

1865
Now we notice that the characteristic polynomial of the tuned closed loop system is:

\[ F(z^{-1}) = V(z^{-1}) \cdot e^* \]

Let \( T_y(z^{-1}) \) be the transfer function between \( y^*(k+d) \) and \( y^*(k) \). From the state space representation, we have:

\[ T_y(z^{-1}) = h'(zI - (F - G_1 \frac{e^*}{\bar{e}^*} J))^{-1} G_1 \frac{1}{\bar{e}^*} \]

And from the polynomial representation, we have:

\[ T_y(z^{-1}) = \frac{z^{-1}B(z^{-1})}{V(z^2)} \]

The identity of the sequences \( \{y^*(k)\} \) and \( \{\gamma^*_m(k)\} \) yields:

\[ \sum_{i \in E} y^*_m(z_i^{-1})y_i^{-1} = 0 \quad \forall k \]

But since the sequences \( \{z_i^{-1}\} \) are linearly independent, this relation shows that, with the expression of \( T_y(z^{-1}) \),

\[ z_i^{-1}B(z_i^{-1}) = V(z_i^{-1}) \cdot e^* \]

And since \( B(z_i^{-1}) \) is not zero. We have obtained the following necessary condition:

\[ z_i^{-1}B(z_i^{-1}) = V(z_i^{-1}) \cdot e^* \quad \forall i \in E \]

Clearly, this is also a sufficient condition.

Corollary A2: Under Assumption A1, we have:

\[ \{u^*(k-d)\} = T_y(q^{-1})\{u^*(k)\} \quad \{y^*(k-d)\} = T_y(q^{-1})\{y^*(k)\} \]

Proof: The second relation has already been established. For the first one, we notice that \( u^*(k) \) is also \( N \)-periodic and, therefore:

\[ u^*(k) = \sum_{i \in E} u_i^* \quad \forall k \]

The conclusion follows from the identity:

\[ T_y(z^{-1}) = z_i^{-d} \quad \forall i \in E \]

Property A2: If the tuning condition is satisfied and \( B(z_i^{-1}) \neq 0 \quad \forall i \in E \).

The uniqueness of \( e^* \) and persistent spanning (Assumption A2) are equivalent properties.

Proof: Let \( T_u(z^{-1}) \) be the transfer function between \( y^*_m(k+d) \) and \( u^*(k) \), we have:

\[ T_u(z^{-1}) = \frac{A(z^{-1})}{P(z^{-1})} \]

Let \( T_i(z^{-1}) \) be the transfer function between \( y^*_m(k+d) \) and \( y^*_i(k) \). From the state space representation, we have:

\[ T_i(z^{-1}) = z_i^{-d} h_i(zI - (F - G_1 \frac{e^*}{\bar{e}^*} J))^{-1} G_1 \frac{1}{\bar{e}^*} \]

But also comparing the definitions of \( e^*(k) \) and \( V(z^{-1}) \) and using the expression of \( T_u(z^{-1}), T_i(z^{-1}) \), we have:

\[ T_i(z^{-1}) = \frac{V(z^{-1})}{P(z^{-1})} \]

This expression allows us to rewrite the persistent spanning condition in the frequency domain as:

\[ \sum_{k=0}^{N-1} \omega^*(k)e^*(k) = \sum_{i \in E} \left| y_i^* \right|^2 \frac{V(z_i^{-1})V(z_i^{-1})}{P(z_i^{-1})} > a_1 \quad \forall i \in E \]

And we know that both \( y^*_m \) and \( P(z^{-1}) \) are not equal to zero for \( i \in E \). Therefore, the fact that this matrix is positive definite is equivalent to the fact that the matrix \( V(z_i^{-1}) \in E \) is full rank. But with Property A1 we know that this is equivalent to the uniqueness of \( e^* \).

Let us now prove Property 1.

Lemma: For any \( e^* \), we have the following identity:

\[ e^*H(zI - (F - G_1 \frac{e^*}{\bar{e}^*} J))^{-1} G_1 \frac{1}{\bar{e}^*} = z_i^{-d} \]

Proof: This transfer function is just \( z_i^{-d}e^*T_y(z_i^{-1}) \), where \( T_y(z_i^{-1}) \) is defined in the proof of Property A2.

Since we have:

\[ T_y(z_i^{-1}) = \frac{V(z_i^{-1})}{\bar{e}^*} \]

The conclusion follows readily.

Now to prove Property 1, we have to show that the transfer function:

\[ H(z^{-1}) = z_i^{-d}H(zI - (F - G_1 \frac{e^*}{\bar{e}^*} J))^{-1} G_1 \frac{1}{\bar{e}^*} \]

with \( \{\gamma^*_m(k)\} \) as input, has \( \{\gamma^*_m(k-d)\} \) as output. From this lemma, we can rewrite \( H(z_i^{-1}) \) simply as:

\[ H(z_i^{-1}) = h'(zI - (F - G_1 \frac{e^*}{\bar{e}^*} J))^{-1} G_1 \frac{1}{\bar{e}^*} \]

But this is just \( T_y(z_i^{-1}) \). Therefore, the conclusion follows from the definition of \( e^*(k) \) and Corollary A2.

References


