Dynamics and solutions to some control problems for water-tank systems

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Abstract

We consider a tank containing a fluid. The tank is subjected to directly controlled translations and rotations. The fluid motion is described by linearized wave equations under shallow water approximations. For irrotational flows, a new variational formulation of Saint-Venant equations is proposed. This provides a simple method to establish the equations when the tank is moving. Several control configurations are studied: one and two horizontal dimensions; tank geometries (straight and non-straight bottom, rectangular and circular shapes), tank motions (horizontal translations with and without rotations). For each configuration we prove that the linear approximation is steady-state controllable and provide a simple and flatness-based algorithm for computing the steering open-loop control. These algorithms rely on operational calculus. They lead to second order equations in space variables whose fundamental solutions define delay operators corresponding to convolutions with compact support kernels. For each configurations several controllability open-problems are proposed and motivated.

Keywords:. Wave equations, boundary control, flatness, controllability, motion planning, delay operators.

Introduction

The following study is derived from an industrial problem for which tanks filled with liquid are to be moved to different steady-state workbenches as fast as possible. For such start and stop motions, the fluid mass has a significant contribution in the dynamics of the whole system. Several recent publications deal with this question, see, for example, [1], [2], [3], [4], [5]. This paper is a first attempt to base the control design on wave equations describing the fluid surface dynamics .

We concentrate on finding open-loop tank trajectories such that if the liquid is initially at rest then it returns to rest when the tank stops. This is a typical motion planning problem: finding open-loop control steering in finite time from one steady-state to another one. For finite dimensional systems, flatness based methods [6], [7] are very efficient to solve this problem. In [8], [9], [10], [11], [12], [13], infinite dimensional extensions are proposed for several systems described by partial differential equations with boundary control. We employ such a "flatness based" methodology, working on physical models of the system and we establish several controllability results: positive results consider exact steady-state controllability in finite time T, i.e., proving that there exists a control $[0,T] \ni t \mapsto u(t)$ steering the system from any steady-state to any other one, on the other hand negative results describe the lack of approximate controllability.

The first contribution of the paper consists of models. The major modelling difficulty lies in the fact that the fluid surface is unknown. A "rigorous" modelling involving Euler or Navier-Stoke equations with free surface boundaries is out of reach. Thus we restrict our study to classical modelling based on shallow water approximation [14]. Even for such restrictive modelling, the motion equations are not so simple to derive when the tank is moving. Thus in a first step, we propose a variational formulation of the Saint-Venant equations for irrotational flows and fixed tank: their solutions are extremal of the action under the constraint formed by the mass conservation equation. Then, when the tank is moving, we derive a similar formulation by adding the contribution of the tank motion in the kinetic and potential energy and proceed as before to get the dynamics.

The second contribution is relative to motion planning. The Saint-Venant equations are nonlinear hyperbolic equations. Only few results are available concerning their nonlinear stability, stabilization and controllability (see, e.g., [15], [16] and a recent result in [17]). Preliminary results [18] sketched in appendix lead us to thinking that, when used properly, linearized Saint-Venant equations can be an insighful approximation for motion planning purposes. We restrict our study to such linearized wave equations and show how to obtain open-loop control algorithms that are computationally straightforward. They are derived from formulas presented in lemmas 3, 4, 5 and 6 and are based on symbolic computations and involve operational calculus, Bessel functions, and Paley-Wiener theorem.

We would like to emphasize that although wave equations with Neuman boundary control have been intensively studied and many precise and general results are available on their controllability and stabilization (see, e.g., [19], [20], [21], [22], [23]), the classical results do not apply in a simple way to the problem presented in this paper. First reason: we have a linear wave equation controlled via Neuman boundary control but, the control is not distributed on the boundary; even for the simple tank described by system (26), the same control u, the acceleration of the tank, appears at both edges $x=-a$ and $x=a$. Second reason: the controlled wave equation is coupled with a double integrator $\ddot{D}=u$; one has to control not only the surface waves inside the tank but also the position and velocity of the tank.

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The paper is organized as follows. The dynamics of the systems under consideration are the subject of section I. Sections II and III are devoted to control problems for the one-dimensional and two-dimensional cases respectively.

More precisely, in section I we detail the variational formulation: the Saint-Venant equations for a moving tank are established. In section II, we consider one-dimensional cases: the translation case with a straight bottom is treated in details; the non-straight bottom is also addressed; combination of tank translation and rotation are investigated. Section III is devoted to two-dimensional cases: the translation of rectangular and circular tanks are solved; the combination of tank translation and rotation for an arbitrary geometry are also solved. In conclusion, we show that shallow water approximation is essential to ensure an explicit formulation of the dynamics: a more general modelling with a non-horizontal fluid velocity leads to an implicit formulation, an infinite dimensional analogue of an index one differential-algebraic system. In the appendix we prove a technical lemma devoted to symbolic analysis of the one dimensional wave equations when the speed depends on space: it can be seen as a generalization of d'Alembert formulas. We also recall in appendix some preliminary results and nonlinear simulations based on Godunov scheme for the Saint-Venant equation.

In this paper we pin point several open problems. As far as we know, the techniques presented in this paper give only partial answer, if any, in these tricky situations. We hope researchers in this area will welcome these challenges.

Some preliminary results relative to the horizontal translation of a tank in a vertical plane and with a straight bottom can be found in [18]. A preliminary version of this paper can be found in [24].

I. Variational formulations

The Saint-Venant equations describe the motion of a perfect fluid under gravity g with a free boundary (the shallow water assumption). We provide a variational formulation of these equations that, up to our knowledge, is new although not surprising (see, e.g., [25], [26], [14]). This variational formulation is interesting: it gives directly the dynamics equations when the tank is moving (translation and rotation).

A. The one-dimensional cases

A.1 One-dimensional straight bottom fixed tank

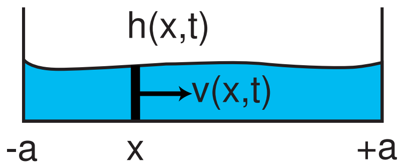

Fig. 1. The one dimensional tank.

We assume that the tank is at rest and study the motion of the fluid.

A.1.a Notations. As displayed on figure 1, the system is described by the following quantities
- a horizontal coordinate $x \in [-a, a]$ where $2a$ is the length of the tank;
- the height profile $[-a, a] \ni x \mapsto h(x, t)$ with $h(x, t) > 0$;
- the velocity profile $[-a, a] \ni x \mapsto v(x, t)$ with respect to the tank
- g is the gravity, ρ is the specific mass of the fluid.

A.1.b Physics. The mass conservation equation is

$$\frac{\partial h}{\partial t} + \frac{\partial (hv)}{\partial x} = 0. \tag{1}$$

The momentum conservation equation is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -g\frac{\partial h}{\partial x}. \tag{2}$$

The boundary condition is

$$v(-a, t) = v(a, t) = 0. \tag{3}$$

The kinetic energy T is

$$T(h, v) = \frac{\rho}{2}\int_{-a}^{a} h(x, t)v^2(x, t)\ dx. \tag{4}$$

The potential energy is

$$U(h, v) = \frac{\rho g}{2}\int_{-a}^{a} h^2(x, t)\ dx. \tag{5}$$

Under these hypothesis the following lemma holds

Lemma 1: Take a positive time $\tau > 0$. Equation (2) , i.e. the momentum conservation equation, results from the Euler-Lagrange first-order stationarity conditions deduced from

$$\delta\left(\int_0^\tau (T(h,v) - U(h,v))\ dt\right) = 0 \tag{6}$$

under the constraints formed by the mass equation (1), the boundary conditions (3) and fixed initial and final values for h and v: $h(x,0) = h_0(x)$, $v(x,0) = v_0(t)$, $h(\tau,x) = h_\tau(x)$, $v(\tau,x) = v_\tau(x)$.

Proof: Denote by $\lambda(x,t)$ the multiplier associated to the constraint (1) and by $\mathcal{L}(h,v,\lambda)$ the Lagrangian[1]

$$\mathcal{L} = \int_0^\tau (T(h,v) - U(h,v))\ dt + \int_0^\tau \int_{-a}^{a} \lambda(x,t)\left(\frac{\partial h}{\partial t} + \frac{\partial (hv)}{\partial x}\right)\ dx\ dt.$$

The condition $\delta\mathcal{L} = 0$ for any small variation δh of h such that $\delta h(x,0) = \delta h(x,\tau) = 0$ yields

$$\int_0^\tau \int_{-a}^{a} \left[\rho(v^2/2 - gh)\delta h + \lambda\left(\frac{\partial(\delta h)}{\partial t} + \frac{\partial(v\delta h)}{\partial x}\right)\right] dx\ dt = 0,$$

and then thanks to an integration by parts

$$\int_0^\tau \int_{-a}^{a} \left[\rho(v^2/2 - gh) - \frac{\partial \lambda}{\partial t} - v\frac{\partial \lambda}{\partial x}\right]\ \delta h\ dx\ dt = 0.$$

Thus

$$\frac{\partial \lambda}{\partial t} + v\frac{\partial \lambda}{\partial x} = \rho(v^2/2 - gh).$$

Similarly, variation δv of v such that $\delta v(x,0) = \delta v(x,\tau) = 0$ and $\delta v(-a,t) = \delta v(a,t) = 0$, gives

$$\frac{\partial \lambda}{\partial x} = \rho v.$$

Gathering these last two stationarity equations we get

$$\frac{\partial \lambda}{\partial t} + \frac{\rho}{2}v^2 = -g\rho h.$$

A differentiation with respect to x yields

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -g\frac{\partial h}{\partial x} \tag{7}$$

which is indeed identical to (2). ■

A.2 One-dimensional non-straight bottom moving tank

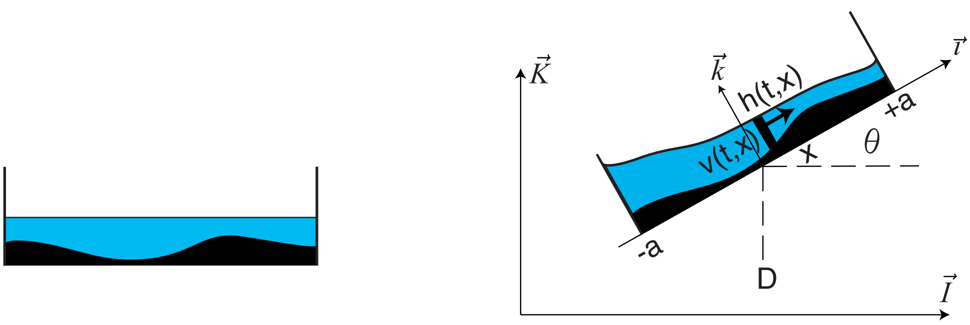

Fig. 2. The non-straight bottom tank at rest (right) and in movement (translation and rotation) (left).

We assume that the tank is moving.

[1]The Lagrangian we use is the classical Lagrangian as used in optimization: the constraints are adjoined with their Lagrange multipliers to the function (or functional) that is minimized.

A.2.a Notations. $(\vec{I}, \vec{K})$ is the fixed frame with $\vec{I}$ horizontal and $(\vec{\imath}, \vec{k})$ is the tank frame: $\vec{\imath} = \cos\theta \vec{I} + \sin\theta \vec{K}$ and $\vec{k} = -\sin\theta \vec{I} + \cos\theta \vec{K}$.

As displayed on figure 2, the tank motion is described by an horizontal position $\mathbf{R}^+ \ni t \mapsto D(t) \in \mathbf{R}$ and a rotation angle $\theta(t)$ around the horizontal axis orthogonal to the translation axis.

We still assume that the fluid can be described by $[-a, a] \times \mathbf{R}^+ \ni (x, t) \mapsto h(x, t)$ and $[-a, a] \times \mathbf{R}^+ \ni (x, t) \mapsto v(x, t)$, the velocity with respect to the tank. Notice that the space coordinate x is relative to the tank. Moreover we assume that the tank bottom is not straight but described by a smooth profile $[-a, a] \ni x \mapsto b(x) \in \mathbf{R}$. As before g is the gravity, ρ is the specific mass of the fluid.

A.2.b Physics and derivation of the model. The momentum conservation equation is derived from the variational formulation of lemma 1 with the following kinetic and potential energies (the boundary and constraint conditions remain unchanged)

$$T(h, v) = \frac{\rho}{2} \int_{-a}^{a} h \left(\dot{D}\vec{I} + v\vec{\imath} + x\dot{\theta}\vec{k} \right)^2 \, dx \tag{8}$$

$$U(h, v) = \rho g \int_{-a}^{a} \int_{b}^{b+h} \left(x\vec{\imath} + z\vec{k} \right) \cdot \vec{K} \, dz \, dx. \tag{9}$$

Denote by $\lambda(x, t)$ the multiplier associated to the mass conservation constraint and by $\mathcal{L}$ the resulting Lagrangian

$$\mathcal{L}(h, v, \lambda) = \int_0^\tau (T(h, v) - U(h, v)) \, dt + \int_0^\tau \int_{-a}^{a} \lambda(x, t) \left(\frac{\partial h}{\partial t} + \frac{\partial (hv)}{\partial x} \right) \, dx \, dt.$$

Stationary condition of $\mathcal{L}$ with respect to any small variation δv of v such that $\delta v = 0$ for $t = 0, \tau$ and $x \in \{-a, a\}$, yields

$$\frac{\partial \lambda}{\partial x} = \rho(\dot{D}\cos\theta + v). \tag{10}$$

Stationary condition of $\mathcal{L}$ with respect to any small variation δh of h such that $\delta h = 0$ for $t = 0, \tau$ yields

$$\frac{\partial \lambda}{\partial t} + v\frac{\partial \lambda}{\partial x} = \frac{\rho}{2} \left(\dot{D}\vec{I} + v\vec{\imath} + x\dot{\theta}\vec{k} \right)^2 - \rho g \left(x\vec{\imath} + (b + h)\vec{k} \right) \cdot \vec{K}. \tag{11}$$

We differentiate (11) with respect to x and substitute $\frac{\partial \lambda}{\partial x}$ by the righthand side of (10). We obtain the momentum conservation equation for v. The full dynamics are then described by the following set of equations

$$\left\{ \begin{array}{l} \dfrac{\partial h}{\partial t} + \dfrac{\partial (hv)}{\partial x} = 0 \\ \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = -\ddot{D}\cos\theta - g\sin\theta + x\dot{\theta}^2 - g\cos\theta \dfrac{\partial (b + h)}{\partial x} \\ \quad v(-a, t) = v(a, t) = 0. \end{array} \right. \tag{12}$$

Notice that these equations are indeed invariant under Galilean transformations, i.e. uniform translations, $D \mapsto D + p_1 t + p_0$ with p_1 and p_0 arbitrary constants.

Assume now that $\dot{\theta}$ is small ($\dot{\theta}^2 a \ll g$), that $h(x, t) = \bar{h}(x) + H(x, t)$, with $\bar{h}(x) = \varpi - b(x) > 0$ (where ϖ is a constant) is the steady-state height profile and with $|H| \ll \bar{h}$, and $|v| \ll \sqrt{g\bar{h}}$. Notice that we neither assume θ small nor $|\ddot{D}| \ll g$. Up to second order-terms the "linearized" dynamics read

$$\left\{ \begin{array}{l} \dfrac{\partial H}{\partial t} = -\dfrac{\partial (\bar{h}v)}{\partial x} \\ \dfrac{\partial v}{\partial t} = -\ddot{D}\cos\theta - g\sin\theta - g\cos\theta \dfrac{\partial H}{\partial x} \\ v(-a, t) = v(a, t) = 0. \end{array} \right.$$

We end up with the following model

Model 1: Elimination of v yields to a wave equation for H

$$\left\{ \begin{array}{l} \dfrac{\partial^2 H}{\partial t^2} = \dfrac{\partial}{\partial x} \left[\bar{h} \left(\ddot{D}\cos\theta + g\sin\theta + g\cos\theta \dfrac{\partial H}{\partial x} \right) \right] \\ g\cos\theta \dfrac{\partial H}{\partial x}(a, t) = g\cos\theta \dfrac{\partial H}{\partial x}(-a, t) = -\ddot{D}\cos\theta - g\sin\theta \end{array} \right. \tag{13}$$

where $[-a,a] \ni x \mapsto \bar{h}(x)$ is the steady-state height profile and $h(x,t) = \bar{h}(x) + H(x,t)$ is up-to second order terms the liquid height. The control variables are $\ddot{D}(t)$ the horizontal acceleration of the tank, and $\dot{\theta}(t)$ its angular velocity. At any given time t, $[-a,a] \ni x \mapsto (H(x,t), \frac{\partial H}{\partial t}(x,t))$, $D(t)$, $\dot{D}(t)$ and $\theta(t)$ constitute the state of the system.

These equations are a good approximation as soon as

$$\dot{\theta}^2 a \ll g, \quad |H| \ll \bar{h}, \quad |v| \ll \sqrt{g\bar{h}}.$$

B. The two-dimensional cases

B.1 Two-dimensional straight bottom fixed tank

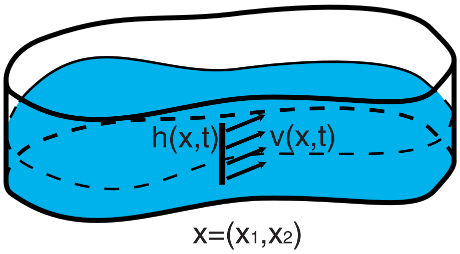

Fig. 3. The two-dimensional tank.

B.1.a Notations. As displayed on figure 3, the system is described by the following quantities:
- two horizontal coordinates $x = (x_1, x_2) \in \Omega$ where Ω is an open bounded connected domain of $\mathbf{R}^2$ with smooth boundary $\partial\Omega$;
- the height profile $\Omega \ni x \mapsto h(x,t)$ with $h(x,t) > 0$;
- the velocity profile $\Omega \ni x \mapsto \vec{v}(x,t) \in \mathbf{R}^2$.

We assume that the tank is at rest. As usual we denote by ∇ the operator

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix}.$$

The mass conservation equation is

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\vec{v}) = 0 \tag{14}$$

where $\vec{v}$ is the velocity field of coordinates (v_1, v_2).

B.1.b Physics. The momentum conservation equation is

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -g\nabla h. \tag{15}$$

The boundary condition is

$$\vec{v} \cdot \vec{n} = 0 \text{ on } \partial\Omega \tag{16}$$

where $\vec{n}$ is the normal to $\partial\Omega$.

We restrict our study to potential flow $\vec{v}$, i.e., solutions of (14,15,16) such that $\nabla \times \vec{v} = 0$. This makes sense since if the initial velocity profile is irrotational, it remains irrotational

$$\frac{\partial(\nabla \times \vec{v})}{\partial t} + \nabla \times [(\nabla \times \vec{v}) \times \vec{v}) = 0.$$

For irrotational $\vec{v}$, (15) reads also

$$\frac{\partial \vec{v}}{\partial t} + \frac{1}{2}\nabla(\vec{v}^2) = -g\nabla h.$$

The kinetic energy T is

$$T(h, \vec{v}) = \frac{\rho}{2} \int_\Omega h(x,t)\vec{v}^2(x,t)\ dx_1 dx_2. \tag{17}$$

The potential energy is

$$U(h, \vec{v}) = \frac{\rho g}{2} \int_\Omega h^2(x,t)\ dx_1 dx_2. \tag{18}$$

As for the one dimensional case, we have the following lemma

Lemma 2: Take a positive time $\tau > 0$ and consider irrotational solutions of (14,15,16) ($\nabla\times\vec{v} = 0$). Then equation (15), i.e. the momentum conservation equation, results from the Euler-Lagrange first-order stationarity conditions deduced from

$$\delta\left(\int_0^\tau (T(h,\vec{v}) - U(h,\vec{v}))\ dt = 0\right) \tag{19}$$

under the constraints formed by the mass conservation equation (14), the boundary conditions (16) and fixed initial and final values for h and v, $h(x,0) = h_0(x)$, $\vec{v}(x,0) = \vec{v}_0(t)$, $h(\tau,x) = h_\tau(x)$, $\vec{v}(\tau,x) = \vec{v}_\tau(x)$.
The proof is very similar to the one dimensional one and is left to the reader.

B.2 Two-dimensional non-straight bottom moving tank

In the following we sketch the main steps to derive from lemma 2 the dynamical equations when the tank is moving with an irrotational velocity profile $\vec{v}$.

B.2.a Notations. The fixed frame is denoted by $(\vec{I}_1,\vec{I}_2,\vec{K})$ where $\vec{K}$ is the upwards vertical unit vector. The tank frame is $(\vec{\imath}_1,\vec{\imath}_2,\vec{k})$ where the liquid height is along the $\vec{k}$ axis. The rotation of the tank is described by the instantaneous rotation vector $\vec{\omega}$ defined by

$$\dot{\vec{\imath}}_1 = \vec{\omega}\times\vec{\imath}_1,\quad \dot{\vec{\imath}}_2 = \vec{\omega}\times\vec{\imath}_2,\quad \dot{\vec{k}} = \vec{\omega}\times\vec{k}.$$

Since we are looking for irrotational flows, we will see that, necessarily, $\vec{\omega}\cdot\vec{k} = 0$: the tank cannot spin around axis $\vec{k}$.

The fluid remains described by the height profile $\Omega \ni x \mapsto h(x,t) > 0$, where Ω is as before an open bounded connected domain of $\mathbf{R}^2$ with a piecewise smooth boundary $\partial\Omega$, and the velocity profile $\vec{v}(x,t) = v_1\vec{\imath}_1 + v_2\vec{\imath}_2$ where $x = (x_1,x_2)$ are Cartesian coordinates along a plane attached to the tank and parallel to $(\vec{\imath}_1,\vec{\imath}_2)$. We assume that the tank bottom is given by the profile $\Omega \ni x \mapsto b(x)$. We denote by (D_1,D_2,Z) the coordinates in the fixed frame of the point D attached to the tank. Vertical acceleration can be included by changing gravity g into $g+\ddot{Z}$ and just considering horizontal motions for D. Without loss of generality, we assume in the sequel that $Z\equiv 0$ and

$$\dot{D} = \dot{D}_1\vec{I}_1 + \dot{D}_2\vec{I}_2,\quad \ddot{D} = \ddot{D}_1\vec{I}_1 + \ddot{D}_2\vec{I}_2.$$

Once more, g denotes the gravity and ρ is the specific mass of the fluid.

B.2.b Physics and derivation of the model. With the above notations the kinetic and potential energies are

$$T(h,\vec{v}) = \frac{\rho}{2}\int_\Omega h\left(\dot{D} + \vec{v} + \vec{\omega}\times(x_1\vec{\imath}_1 + x_2\vec{\imath}_2)\right)^2\ dx_1dx_2 \tag{20}$$

$$U(h,\vec{v}) = \rho g Z + \rho g\ \int_\Omega\int_b^{b+h}\left(x_1\vec{\imath}_1 + x_2\vec{\imath}_2 + z\vec{k}\right)\cdot\vec{K}\ dzdx_1dx_2. \tag{21}$$

Denote by $\lambda(x,t)$ the multiplier associated to the mass conservation constraint and by $\mathcal{L}$ the resulting Lagrangian

$$\mathcal{L}(h,\vec{v},\lambda) = \int_0^\tau (T(h,\vec{v}) - U(h,\vec{v}))\ dt + \int_0^\tau\int_\Omega \lambda(x,t)\left(\frac{\partial h}{\partial t} + \nabla\cdot(h\vec{v})\right)\ dx_1dx_2\ dt.$$

Stationary condition of $\mathcal{L}$ with respect to any small variation $\delta\vec{v} = \delta\vec{v}_1\vec{\imath}_1 + \delta\vec{v}_2\vec{\imath}_2$ of $\vec{v}$ such that $\delta\vec{v} = 0$ for $t = 0,\tau$ and $\delta\vec{v}\cdot\vec{n} = 0$ for $x\in\partial\Omega$, yields

$$\frac{\partial\lambda}{\partial x_\sigma} = \rho\left(\dot{D} + \vec{v} + \vec{\omega}\times(x_1\vec{\imath}_1 + x_2\vec{\imath}_2)\right)\cdot\vec{\imath}_\sigma,\qquad \sigma = 1,2. \tag{22}$$

Stationary condition of $\mathcal{L}$ with respect to any small variation δh of h such that $\delta h = 0$ for $t = 0,\tau$ yields

$$\begin{aligned}\frac{\partial\lambda}{\partial t} + \vec{v}\cdot\nabla\lambda \quad &= \quad \frac{\rho}{2}\left(\dot{D} + \vec{v} + \vec{\omega}\times(x_1\vec{\imath}_1 + x_2\vec{\imath}_2)\right)^2 \\ &\qquad -\rho g\left(x_1\vec{\imath}_1 + x_2\vec{\imath}_2 + (b+h)\vec{k}\right)\vec{K}.\end{aligned} \tag{23}$$

According to (22)

$$\vec{v}\cdot\nabla\lambda = \rho\vec{v}\cdot\left(\dot{D} + \vec{v} + \vec{\omega}\times(x_1\vec{\imath}_1 + x_2\vec{\imath}_2)\right).$$

We then apply $\frac{\partial}{\partial x_1}$ on (23) and substitute $\frac{\partial\lambda}{\partial x_1}$ by the righthand side of (22):

$$\begin{aligned}&\frac{\partial v_1}{\partial t} + \vec{v}\cdot\frac{\partial\vec{v}}{\partial x_1} = x_2\frac{d(\vec{\omega}\cdot\vec{k})}{dt} + \ldots \\ &\ldots + \frac{\partial}{\partial x_1}\left(\frac{1}{2}(\vec{\omega}\times(x_1\vec{\imath}_1 + x_2\vec{\imath}_2))^2 - (\ddot{D} + g\vec{K})\cdot(x_1\vec{\imath}_1 + x_2\vec{\imath}_2) - g(b+h)\vec{k}\cdot\vec{K}\right).\end{aligned}$$

Similarly we have

$$\frac{\partial v_2}{\partial t} + \vec{v}\cdot\frac{\partial \vec{v}}{\partial x_2} = -x_1\frac{d(\vec{\omega}\cdot\vec{k})}{dt} + \dots$$
$$\dots + \frac{\partial}{\partial x_2}\left(\frac{1}{2}(\vec{\omega}\times(x_1\vec{\imath}_1 + x_2\vec{\imath}_2))^2 - (\ddot{D}+g\vec{K})\cdot(x_1\vec{\imath}_1+x_2\vec{\imath}_2) - g(b+h)\vec{k}\cdot\vec{K}\right).$$

This provides the vectorial momentum conservation equation for

$$\frac{\partial v}{\partial t} + \frac{1}{2}\nabla\vec{v}^2 = \frac{d(\vec{\omega}\cdot\vec{k})}{dt}(x_2\vec{\imath}_1 - x_1\vec{\imath}_2) + \dots$$
$$\dots + \nabla\left(\frac{1}{2}(\vec{\omega}\times(x_1\vec{\imath}_1 + x_2\vec{\imath}_2))^2 - (\ddot{D}+g\vec{K})\cdot(x_1\vec{\imath}_1+x_2\vec{\imath}_2) - g(b+h)\vec{k}\cdot\vec{K}\right).$$

$\vec{v}$ must be kept irrotational to apply lemma 2. Thus we restrict rotations by $\vec{\omega}\cdot\vec{k}\equiv 0$.

The full dynamics is then described by the following set of equations

$$\left\{\begin{array}{l}\dfrac{\partial h}{\partial t} + \nabla\cdot(h\vec{v}) = 0\\ \dfrac{\partial \vec{v}}{\partial t} + \dfrac{1}{2}\nabla\vec{v}^2 = \dfrac{1}{2}\nabla\left((\vec{\omega}\times(x_1\vec{\imath}_1+x_2\vec{\imath}_2))^2\right) + \dots\\ \dots + \nabla\left(-(\ddot{D}+g\vec{K})\cdot(x_1\vec{\imath}_1+x_2\vec{\imath}_2) - g(b+h)\vec{k}\cdot\vec{K}\right)\\ \vec{v}\cdot\vec{n} = 0 \text{ on } \partial\Omega.\end{array}\right. \tag{24}$$

Assume now that $\vec{\omega}$ is small ($\vec{\omega}^2 a \ll g$, where a is the typical size of Ω), that $h(x,t) = \bar{h}(x) + H(x,t)$, with $\bar{h}(x) = cte - b(x) > 0$ is the steady-state height profile and with $|H| \ll \bar{h}$, and that $|\vec{v}| \ll \sqrt{g\bar{h}}$. Up to second order-terms the "linearized" dynamics read

$$\left\{\begin{array}{l}\dfrac{\partial H}{\partial t} = -\nabla\cdot(\bar{h}\vec{v})\\ \dfrac{\partial \vec{v}}{\partial t} = \nabla\left(-(\ddot{D}+g\vec{K})\cdot(x_1\vec{\imath}_1+x_2\vec{\imath}_2) - g\vec{k}\cdot\vec{K}H\right).\end{array}\right.$$

We end up with the following model

Model 2: Elimination of $\vec{v}$ yields to a wave equation for H

$$\left\{\begin{array}{l}\dfrac{\partial^2 H}{\partial t^2} = \nabla\cdot\left(\bar{h}\nabla\left[(\ddot{D}+g\vec{K})\cdot(x_1\vec{\imath}_1+x_2\vec{\imath}_2) + gH\vec{k}\cdot\vec{K}\right]\right)\\ \nabla\left[(\ddot{D}+g\vec{K})\cdot(x_1\vec{\imath}_1+x_2\vec{\imath}_2) + gH\vec{k}\cdot\vec{K}\right]\cdot\vec{n} = 0 \quad \text{on } \partial\Omega\end{array}\right. \tag{25}$$

where $\bar{h}(x)$ is the steady-state height profile and $h(x,t) = \bar{h}(x) + H(x,t)$ is up-to second order terms the liquid height.

The control variables are $\ddot{D}$ the tank acceleration and $\vec{\omega}$ its instantaneous rotation vector (remember that $\vec{\omega}\cdot\vec{k} = 0$). At any given time t, $[-a,a] \ni x \mapsto (H(x,t), \frac{\partial H}{\partial t}(x,t))$, $D(t)$, $\dot{D}$ and the three Euler angles defining the orientation of the tank constitute the state of the system.

These equations are a good approximation as soon as

$$\vec{\omega}^2 a \ll g, \quad |H| \ll \bar{h}, \quad \|\vec{v}\| \ll \sqrt{g\bar{h}}.$$

II. Several control problems and their solutions for the one-dimensional linearized model

A. Translation and straight bottom

Model 3: Assume that $[-a,a] \ni x \mapsto b(x) = 0$ and $\theta = 0$. Then $[-a,a] \ni x \mapsto \bar{h}(x)$ is constant and model 1 reads

$$\left\{\begin{array}{c}\dfrac{\partial^2 H}{\partial t^2} = \bar{h}g\dfrac{\partial^2 H}{\partial x^2}\\ \dfrac{\partial H}{\partial x}(a,t) = \dfrac{\partial H}{\partial x}(-a,t) = -\dfrac{u}{g}\\ \ddot{D} = u\end{array}\right. \tag{26}$$

with $(H, \frac{\partial H}{\partial t}, D, \dot{D})$ as state and u as control.

The controllability of the above system can be studied directly by considering the dual system and its observability (see, e.g., [19], [21], [23]). The dual system reads

$$\begin{cases} \dfrac{\partial^2 P}{\partial t^2} = \bar{h}g\dfrac{\partial^2 P}{\partial x^2} \\ \dfrac{\partial P}{\partial x}(a,t) = \dfrac{\partial P}{\partial x}(-a,t) = 0 \\ \ddot{\xi} = 0 \end{cases}$$

with output $y = P(a,t) - P(-a,t) + \xi$ and is clearly not observable (any even solution $x \mapsto P(x,t)$ with $\xi \equiv 0$ gives $y = 0$). The approximate controllability is not even valid. Nevertheless, the system is steady-state controllable. This results from the following elementary lemma.

Lemma 3 (Parametrization of the trajectories) Denote $c = \sqrt{g\bar{h}}$ the velocity of the waves. The general solution of (26) is given by

$$\begin{cases} H(x,t) = \dfrac{c}{2g}\left(\dot{y}(t-\dfrac{x}{c}) - \dot{y}(t+\dfrac{x}{c})\right) + \dfrac{1}{2}\left(F(t+\dfrac{x}{c}) + F(t-\dfrac{x}{c})\right) + k_0 t \\ D(t) = \dfrac{1}{2}\left(y(t+\dfrac{a}{c}) + y(t-\dfrac{a}{c})\right) \\ u(t) = \dfrac{1}{2}\left(\ddot{y}(t+\dfrac{a}{c}) + \ddot{y}(t-\dfrac{a}{c})\right) \end{cases} \tag{27}$$

where k_0 is an arbitrary constant, F an arbitrary $2a/c$-periodic time function and y an arbitrary time function. Moreover

$$\begin{cases} k_0 = \dfrac{c}{2a}(H(0,2a/c) - H(0,0)) \\ F(t) = H(0,t) - \dfrac{c}{2a}(H(0,2a/c) - H(0,0))t \\ y(t) = D(t) + \dfrac{1}{2\bar{h}}\left(\displaystyle\int_0^a H(x,t)\,dx - \int_{-a}^0 H(x,t)\,dx\right). \end{cases} \tag{28}$$

Proof: When H and D are given by (27), standard computations show that they satisfy (26). Let us prove in details the converse: any solution of (26) admits the form (27) with k_0, F and y defined by (28).

The general solution of

$$\frac{\partial^2 H}{\partial t^2} = \bar{h}g\frac{\partial^2 H}{\partial x^2}$$

is given by the d'Alembert's formula

$$H(x,t) = \varphi(t+\frac{x}{c}) + \psi(t-\frac{x}{c})$$

where φ and ψ are smooth functions.

The idea of the proof is to turn the boundary conditions of the model into functional equations with φ and ψ as variables, and then to solve these equations.

The boundary conditions can be expressed as

$$\begin{cases} \dot{\varphi}(t+\dfrac{a}{c}) - \dot{\psi}(t-\dfrac{a}{c}) = -\dfrac{c}{g}\ddot{D}(t) \\ \dot{\varphi}(t-\dfrac{a}{c}) - \dot{\psi}(t+\dfrac{a}{c}) = -\dfrac{c}{g}\ddot{D}(t). \end{cases} \tag{29}$$

Elimination of D yields

$$\dot{\varphi}(t+\frac{a}{c}) + \dot{\psi}(t+\frac{a}{c}) = \dot{\varphi}(t-\frac{a}{c}) + \dot{\psi}(t-\frac{a}{c}).$$

Thus $f \equiv \dot{\varphi} + \dot{\psi}$ is a periodic function with period $2\frac{a}{c}$. Since $\frac{\partial H}{\partial t}(0,t) = f(t)$. F defined in (28) is a $2a/c$ periodic function and

$$\dot{F}(t) = f(t) - \frac{c}{2a}\int_0^{2a/c} f.$$

Consider y defined in (28). Since H is solution of (26), we have

$$\ddot{y}(t) = -g\frac{\partial H}{\partial x}(0,t).$$

Simple computations show also that

$$c\frac{\partial H}{\partial x}(0,t) = \dot{\varphi}(t) - \dot{\psi}(t).$$

So we have

$$\dot{\varphi}(t) - \dot{\psi}(t) = -\frac{c}{g}\ddot{y}$$
$$\dot{\varphi}(t) + \dot{\psi}(t) = \dot{F}(t) + \frac{c}{2a}\int_0^{2a/c} f.$$

Thus

$$\dot{\varphi}(t) = \frac{1}{2}\dot{F}(t) - \frac{c}{2g}\ddot{y}(t) + \frac{c}{4a}\int_0^{2a/c} f$$
$$\dot{\psi}(t) = \frac{1}{2}\dot{F}(t) + \frac{c}{2g}\ddot{y}(t) + \frac{c}{4a}\int_0^{2a/c} f$$

that is

$$\varphi(t) = m + \frac{1}{2}F(t) - \frac{c}{2g}\dot{y}(t) + \frac{ct}{4a}\int_0^{2a/c} f$$
$$\psi(t) = n + \frac{1}{2}F(t) + \frac{c}{2g}\dot{y}(t) + \frac{ct}{4a}\int_0^{2a/c} f$$

where n and m are two constants. Yet $F(0) = H(0,0)$ and $H(0,0) = \varphi(0) + \psi(0)$. Thus $m + n = 0$ and

$$H(x,t) = \frac{1}{2}\left(F(t+\frac{x}{c}) + F(t-\frac{x}{c})\right) + \frac{c}{2g}\left(\dot{y}(t-\frac{x}{c}) - \dot{y}(t+\frac{x}{c})\right) + \frac{ct}{2a}\int_0^{2a/c} f.$$

With this relation, we compute $\int_0^a H(x,t)dx$ and $\int_{-a}^0 H(x,t)dx$ and derive D via $D(t) = y(t) - (\int_0^a H(x,t)dx - \int_{-a}^0 H(x,t)dx)/(2\bar{h})$. This gives

$$D(t) = \frac{1}{2}\left(y(t+\frac{a}{c}) + y(t-\frac{a}{c}))\right).$$

■

Remark 1 (Inspection of the controllability) The explicit parameterization (27) implies that (26) is not controllable neither exact nor approximate. To see this, take an initial state $(H_0(x), \dot{H}_0(x))$, $x \in [-a,a]$ that is zero. This means that ϕ and ψ are zeros on $[-a/c, a/c]$ since

$$2\varphi(t) = H_0(ct) + \int_0^{ct}\dot{H}_0(x)dx, \quad 2\psi(t) = H_0(ct) - \int_0^{ct}\dot{H}_0(x)dx.$$

Thus $F(t) = 0$ on $[-a/c, a/c]$. Take any final state $(H_T(x), \dot{H}_T(x))$ at time T. It will provide another $F(t)$ that will not vanish over $[T - a/c, T + a/c]$, in general. Since F is $2a/c$-periodic, there does not exist a trajectory joining such two states: F is an invariant quantity that cannot be modified by control. From an algebraic point of view, it corresponds to the torsion sub-module of the module attached to (29) (see [27] for more details). This means (26) is not controllable.

If we assume the initial state is zero then both F and k_0 vanish. We have an explicit description of all the trajectories passing though $(H_0, \dot{H}_0) = 0$. It suffices to take (27) with $k_0 = F = 0$. This provides a very simple way to steer the system from any steady-position in $D = p$ to any other steady-position in $D = q$. The system is steady-state controllable. More precisely there is a one-to-one correspondence between the trajectories starting from the steady position $D = p$ at time $t = 0$ and arriving at time $T > 2a/c$ at the steady position $D = q$, and the smooth functions $t \mapsto y(t)$ such that

$$y(t) = \begin{cases} p & \text{if } t \leq a/c \\ \text{arbitrary} & \text{if } a/c < t < T - a/c \\ q & \text{if } t \geq T - a/c \end{cases} \tag{30}$$

via the following formulas

$$\begin{cases} D(t) = \frac{1}{2}\left(y(t+\frac{a}{c}) + y(t-\frac{a}{c})\right) \\ H(x,t) = \frac{c}{2g}\left(\dot{y}(t+\frac{x}{c}) - \dot{y}(t-\frac{x}{c})\right). \end{cases} \tag{31}$$

Remark 2: The reader might believe that the problem of finding $t \mapsto D(t)$ with $D(t \leq 0) = p$ and $D(t \geq T) = q$ such that the solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial^2 H}{\partial t^2} = \bar{h} g \dfrac{\partial^2 H}{\partial x^2} \\ \dfrac{\partial H}{\partial x}(a,t) = \dfrac{\partial H}{\partial x}(-a,t) = -\dfrac{\ddot{D}}{g} \end{cases}$$

starting from zeros at $t = 0$ and arriving at zero at $t = T$ could be obtained via basic symmetry arguments and invariance with respect to $t \mapsto -t$ and $x \mapsto -x$: this is false. The fact that $\ddot{D}(t) = -\ddot{D}(T-t)$ does not ensure that H and $\dot{H}$ return to zero at time T. The proposed method does.

Remark 3 (Physical meaning of the flat output) The quantity y appearing in (27) is the position of a particular point of the system. It is the center of gravity of the two punctual masses M^+ (the mass at the front of the tank) and M^- (the mass at the rear of the tank) placed at the edges of the tank ($x = a$ and $x = -a$):

$$M^+ = \int_0^a (\bar{h} + H(x,t))dx, \quad M^- = \int_{-a}^0 (\bar{h} + H(x,t))dx, \quad y(t) = D(t) + \frac{M^+ - M^-}{2\bar{h}}$$

(remember that $M^+ + M^- = 2a\bar{h}$, by mass conservation).

We have thus proved that the first-order linear approximation of

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial (hv)}{\partial x} = 0 \\ \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = -\ddot{D}(t) - \dfrac{\partial (h)}{\partial x} \\ v(-a,t) = v(a,t) = 0 \end{cases}$$

with $\ddot{D} = u$ as control is steady-state controllable but not controllable. Coron [17] has proved very recently using first return and fixed-point methods that the above nonlinear model itself is also steady-state controllable.

Remark 4 (Relevance of the linearization approach) Nonlinear simulations, see [18], show that the motions computed via formulas (26), i.e. parameterization of the trajectories of the linearized model, approximate the trajectories of the nonlinear system.

B. Translation and non-straight bottom

Model 4: When $\theta = 0$, model 1 reads

$$\begin{cases} \dfrac{\partial^2 H}{\partial t^2} = \dfrac{\partial}{\partial x}\left[\bar{h}(x)\ddot{D}(t) + g\bar{h}(x)\dfrac{\partial H}{\partial x}\right] \\ \dfrac{\partial H}{\partial x}(a,t) = \dfrac{\partial H}{\partial x}(-a,t) = -\dfrac{u(t)}{g} \\ \ddot{D}(t) = u(t) \end{cases} \tag{32}$$

where $\bar{h}(x)$ the the steady-state height profile and where $(H, \frac{\partial H}{\partial t}, D, \dot{D})$ is the state.

We will not study the controllability of (32) in details as for (26). We will just prove that for any $\bar{h}(x)$, this system is steady-state controllable: one can steer the system from the steady-position $D = p$ to another steady-position $D = q$ in finite time.

Lemma 4 (Steady-state controllability) Take p and q two reals, and $T > 2\Delta$ where

$$\Delta = \int_{-a}^{a} \frac{dx}{\sqrt{g\bar{h}(x)}}$$

is the propagation time between the two edges. There exists a smooth control $t \mapsto D(t)$ such that $D(t) = p$ for $t \leq 0$, $D(t) = q$ for $t \geq T$ and the solution of (32) starting from $(H, \dot{H}) = 0$ at time $t = 0$ returns to 0 at time T.

Proof: It is based on symbolic computations and the technical lemma 7 given in appendix. The proof is constructive in the sense that the control D is obtained via convolutions with L^2 kernels of compact support and deduced from the function $\mathcal{B}(x, \xi)$ of lemma 7. Just for this proof, we will assume that the liquid is between $x = 0$ and $x = 2a$. In the Laplace domain we have the following second order differential system[2] :

$$\begin{cases} (g\bar{h}H')' = s^2 H - s^2 \bar{h}' D \\ gH'(0,t) + s^2 D = gH'(2a,t) + s^2 D = 0 \end{cases} \tag{33}$$

[2]We do not consider extra terms such as $D(0)$ $\dot{D}(0)$ since s is just a formal variable that represents the derivation.

where $'$ is the derivation with respect to x. The general solution of $(g\bar{h}H')' = s^2H - s^2\bar{h}'D$ reads

$$H = s^2(X + D\beta)A - s^2(Y + D\alpha)B \tag{34}$$

where X and Y are the integration constants, A and B the solutions of $(g\bar{h}A')' = s^2A$ and $(g\bar{h}B')' = s^2B$ with $A(0) = 1$, $A'(0) = 0$, $B(0) = 0$, $g\bar{h}(0)B'(0) = 1$, and

$$\alpha(x,s) = \int_0^x \bar{h}'(x)A(x)dx, \quad \beta(x,s) = \int_0^x \bar{h}'(x)B(x)dx.$$

The fact that H given by (34) is solution results from the classical Wronskian identity

$$\begin{vmatrix} A & B \\ g\bar{h}A' & g\bar{h}B' \end{vmatrix} \equiv 1.$$

Since

$$H' = s^2(X + D\beta)A' - s^2(Y + D\alpha)B'$$

the boundary conditions read

$$\begin{cases} \bar{h}(0)D = Y \\ \quad D/g = -(X + D\beta_+)A'_+ + (Y + D\alpha_+)B'_+ \end{cases} \tag{35}$$

where $A'_+(s) = A'(2a,s)$, ...Notice that $A(0,s) = 1$, $\alpha(0,s) = \beta(0,s) = 0$ and $B'(0,s) = 1/(g\bar{h}(0))$. Elimination of D yields

$$PX = QY$$

where

$$\begin{aligned} P(s) &= g\bar{h}(0)A'_+ \\ Q(s) &= -(1 + g(\beta_+A'_+ - \alpha_+B'_+)) + g\bar{h}(0)B'_+. \end{aligned}$$

Let us examine in details the structure of the operators $P(s)$ and $Q(s)$. According to lemma 7, see appendix, with $c^2(x) = g\bar{h}(x)$, A and B read

$$\begin{aligned} A(x,s) &= \sqrt{\frac{c(0)}{c(x)}}\cosh(s\sigma(x)) + \int_{-\sigma(x)}^{\sigma(x)} \mathcal{A}(x,\xi)\exp(\xi s)d\xi \\ B(x,s) &= \int_{-\sigma(x)}^{\sigma(x)} \mathcal{B}(x,\xi)\exp(\xi s)d\xi \end{aligned}$$

where $\mathcal{A}$ and $\mathcal{B}$ are L^2 functions of ξ. Since

$$g\bar{h}A' = s^2\int_0^x A(\xi,s)\ d\xi, \quad g\bar{h}B' = 1 + s^2\int_0^x B(\xi,s)\ d\xi,$$

we have

$$\begin{aligned} A'(x,s) &= s^2\int_{-\sigma(x)}^{\sigma(x)} \bar{A}(x,\xi)\exp(\xi s)d\xi \\ B'(x,s) &= \frac{1}{g\bar{h}(x)} + s^2\int_{-\sigma(x)}^{\sigma(x)} \bar{B}(x,\xi)\exp(\xi s)d\xi \end{aligned}$$

for some L^2 functions of ξ, $\bar{A}$ and $\bar{B}$. Since

$$\begin{aligned} \alpha(x,s) &= \bar{h}(x)A(x,s) - \bar{h}(0) - \int_0^x \bar{h}A' \\ \beta(x,s) &= \int_0^x \bar{h}'B \end{aligned}$$

we have similarly

$$\alpha(x,s) = \bar{h}(x)A(x,s) - \bar{h}(0) + s^2 \int_{-\sigma(x)}^{\sigma(x)} \bar{\alpha}(x,\xi)\exp(s\xi)d\xi$$

$$\beta(x,s) = \int_{-\sigma(x)}^{\sigma(x)} \bar{\beta}(x,\xi)\exp(s\xi)d\xi$$

where $\bar{\alpha}$, $\bar{\beta}$ are L^2 functions of ξ with $\bar{\alpha}(0,\xi) = \bar{\beta}(0,\xi) \equiv 0$. Thus

$$P = s^2 \int_{-\sigma(2a)}^{\sigma(2a)} \bar{P}(\xi)\exp(\xi s)d\xi$$

$$Q = g\bar{h}(2a)A_+B'_+ - 1 + s^2 \int_{-\sigma(a)}^{\sigma(a)} \bar{Q}(\xi)\exp(\xi s)d\xi$$

where $\bar{P}$ and $\bar{Q}$ are L^2 functions of ξ. Thanks to the identity $g\bar{h}(AB' - A'B) = 1$, $g\bar{h}(2a)A_+B'_+ - 1$ is equal to $gB_+A'_+$ and can be represented as

$$s^2 \int_{-\sigma(2a)}^{\sigma(2a)} \bar{f}(\xi)\exp(\xi s)d\xi$$

via some L^2 function f. Thus Q reads

$$Q = s^2 \left(\int_{-\sigma(2a)}^{\sigma(2a)} (\bar{Q}(\xi) + \bar{f}(\xi))\exp(\xi s)d\xi \right).$$

and we have the following factorization $P(s) = s^2R(s)$ and $Q(s) = s^2S(s)$ with

$$\begin{cases} R = \displaystyle\int_{-\sigma(2a)}^{\sigma(2a)} \bar{P}(\xi)\exp(\xi s)d\xi \\ S = \displaystyle\int_{-\sigma(2a)}^{\sigma(2a)} (\bar{Q}(\xi) + \bar{f}(\xi))\exp(\xi s)d\xi. \end{cases} \tag{36}$$

The operators R and S correspond to convolution with L^2 kernels whose supports are included in $[-\sigma(2a), \sigma(2a)]$. For any quantity $Z(s)$

$$X = SZ, \quad Y = RZ, \quad D = \frac{R}{\bar{h}(0)}Z$$

formally satisfies the boundary conditions (35) and $H(x,s)$ defined by (34) is a solution of (33). Yet, we have seen that for each x, the operators A, B, α and β are also convolutions with compact kernels. In the time domain, all this machinery defines, for any arbitrary smooth time function $t \mapsto Z(t)$, a solution of $(t,x) \mapsto H(x,t)$ and $t \mapsto D(t)$ of (32). Moreover $D(t)$ depends on the values of Z over the interval $[t-\Delta, t+\Delta]$ where $\Delta = \sigma(2a)$ is the propagation time between the two edges. When Z is constant for $t < 0$, D is constant for $t < -\Delta$ and $H(x,t)$ is 0 for t small enough, i.e., $t < -3\Delta$ since

$$H = s^2[SA - RB + (\beta)A - \alpha B)R/\bar{h}(0)] \;\; Z$$

and for each x, each operator, S, R, A, B, α, β is a convolution with a kernel of support included in $[-\Delta, +\Delta]$. In fact $H(x,t)$ is 0 for $t < -\Delta$. This results from Holgrem uniqueness theorem: every quantity is smooth and $H = 0$ is also solution of (32) over $[-d, -\Delta]$ with $D = cte$ and $H_{t=-d} = 0$ and $\dot{H}_{t=-d} = 0$ for any $d > 3\Delta$. For Z constant we have

$$D = \frac{4a}{\bar{h}(2a)}Z$$

since $D = gA'(2a,s)/s^2 Z$ and for $s = 0$, $A'(2a,s)/s^2$ is a well defined function of x, $\Lambda(x)$, solution of the differential equation

$$(g\bar{h}\Lambda')' = 2A(x,0) = 2,$$

the second s-derivative of (33) in $s = 0$ with 0 initial values ($\Lambda'(x) = 2x/(g\bar{h}(x))$). This relation explains the factorization by s^2 between P, Q and R, S: without it, we will not be able to steer the system from different steady-states via smooth functions Z that are constant outside $[-\Delta, \Delta]$; with P instead of R, D will always return to 0 when Z becomes constant; with R, the motion planning problem can be solved as in section II using a sigmoid function similar to (30). ■

Remark 5: For a general bottom profile, one can conjecture[3] that the minimum transition time is 2Δ, i.e., the double of the travelling time from one edge to the over one. This is to compare with the straight bottom case where the minimum transition case is just Δ. Notice that, when the bottom profile is symmetric one can prove that the minimum transition is also Δ . It suffices to define $A(x,s)$ and $B(x,s)$ such that A is symmetric and B is anti-symmetric and to adapt the above proof: with such A and B computations simplify and provide Δ as minimum transition time.

C. Translation and rotation

Assume that we have two controls D and θ and that we want to steer the system from rest to rest, i.e. from D_0 at time $t=0$ to D_T at time $t=T>0$. Take any smooth function $[0,T] \ni t \mapsto D(t)$ such that $D(0)=D_0$, $D(T)=D_T$ and $D^{(i)}(0)=D^{(i)}(T)=0$, $i=1,2$. Set $\theta = -\arctan(\ddot{D}/g)$. The solution $t \mapsto H(x,t)$ of (13) starting from 0 satisfies

$$\begin{cases} \dfrac{\partial^2 H}{\partial t^2} = \dfrac{\partial}{\partial x}\left[\bar{h}(x) g \cos\theta \dfrac{\partial H}{\partial x}\right] \\ g\cos\theta \dfrac{\partial H}{\partial x}(a,t) = g\cos\theta\dfrac{\partial H}{\partial x}(-a,t) = 0. \end{cases}$$

We can deduce from that $H(x,t)=0, \forall x \in [-a,a], \forall t \in [0,T]$. The control $\theta(t) = -\arctan(\ddot{D}(t)/g)$ steers the system from rest to rest. In practice such open-loop control will be valid if $\dot{\theta}^2 a \ll g$, i.e., for all $t \in [0,T]$,

$$|D^{(3)}(t)| \ll \frac{g^2 + (D^{(2)}(t))^2}{\sqrt{ga}}.$$

D. Open problems

D.1 Controllability of the non-straight bottom system

With the single control D ($\theta \equiv 0$), we have seen that, when the bottom is straight, the system is not controllable. Is it still true for a non-straight bottom ? This suggests the following problem: characterize in term of $\bar{h}(x)$, the controllability of

$$\begin{cases} \dfrac{\partial^2 H}{\partial t^2} = \dfrac{\partial}{\partial x}\left[\bar{h}(x)u(t) + g\bar{h}(x)\dfrac{\partial H}{\partial x}\right] \\ \dfrac{\partial H}{\partial x}(a,t) = \dfrac{\partial H}{\partial x}(-a,t) = -\dfrac{u(t)}{g} \\ \ddot{D}(t) = u(t). \end{cases}$$

Since $\frac{d^2}{dt^2}\left(\int_{-a}^{a} H(x,t)dx\right) = 0$ we assume that $\int_{-a}^{a} H \equiv 0$: this is just the global conservation of the fluid in the tank. An interesting fact is that one can prove, from the observability of the adjoint system [19], that, when $[-a,a] \ni x \mapsto h(x)$ is even ($h(x)=h(-x)$), the system is not controllable: the adjoin system

$$\frac{\partial^2 P}{\partial t^2} = \frac{\partial}{\partial x}\left(g\bar{h}(x)\frac{\partial P}{\partial x}\right),$$
$$\frac{\partial P}{\partial x}(-a,t) = \frac{\partial P}{\partial x}(a,t) = 0$$
$$\ddot{x}i = 0$$

with

$$y(t) = \xi\bar{h}(a)P(a,t) - \bar{h}(-a)P(-a,t) - \int_{-a}^{a} P(x,t)\bar{h}'(x)dx$$

as output is not observable ($y \equiv 0$ for solutions $P(x,t)$ that are even x-function and $\xi = 0$). This particular case is important in practice, but more precisely speaking, what are, if any, the necessary and sufficient conditions on $[-a,a] \ni x \mapsto h(x)$ for the system to be controllable?

D.2 Use of an extra control

We know that the straight bottom tank with the single control D is not controllable. Is-it still true with the additional control θ? This suggests the study of the controllability of the following system where the nonlinearity is due to the

[3] This conjecture has been suggested by Jean-Michel Coron.

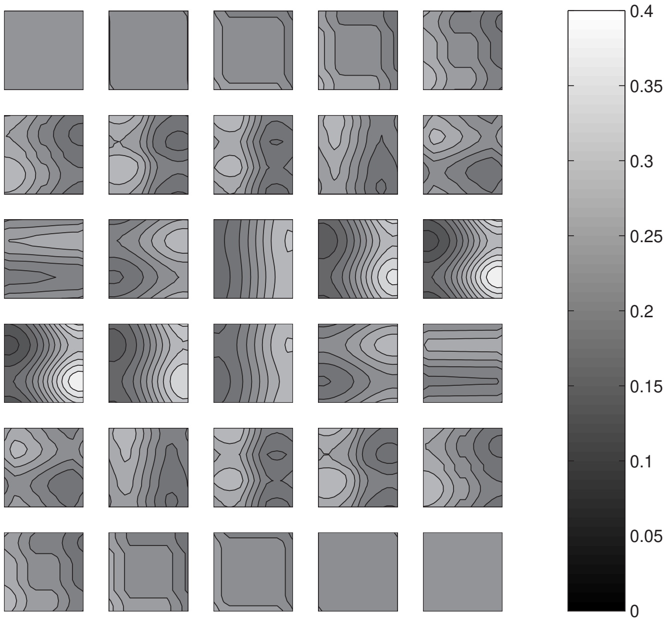

Fig. 4. Height contours sequence of the free surface of a square tank filled with fluid. Finite-time excursion from a steady point ($a_1 = a_2 = .5$, $\bar{h} = .2$, $g = 10$, $time = 2.2$)

control:

$$\begin{cases} \dfrac{\partial^2 H}{\partial t^2} = \dfrac{\partial}{\partial x}\left[\bar{h}\left(\ddot{D}\cos\theta + g\sin\theta + g\cos\theta\dfrac{\partial H}{\partial x}\right)\right] \\ g\cos\theta\dfrac{\partial H}{\partial x}(a,t) = g\cos\theta\dfrac{\partial H}{\partial x}(-a,t) = -\ddot{D}\cos\theta - g\sin\theta \end{cases}$$

with $\ddot{D} = u(t)$ and $\dot{\theta} = \omega$ as control variables (we still assume that $\int_{-a}^{a} H \equiv 0$).

III. Control of the two-dimensional linearized model: first issues

A. Translation of the rectangular tank

Model 5: When $Z \equiv 0$, $\vec{\omega} \equiv 0$ and $(\vec{\imath}_1, \vec{\imath}_2, \vec{k}) \equiv (\vec{I}_1, \vec{I}_2, \vec{K})$, model 2 becomes for a straight bottom ($\bar{h}$ constant):

$$\begin{cases} \ddot{H} = g\bar{h}\Delta H \\ g\nabla H \cdot \vec{n} = -u.\vec{n} \quad \text{on } \partial\Omega \\ \ddot{D} = u \end{cases} \tag{37}$$

Assume that Ω is the rectangle $[-a_1, a_1] \times [-a_2, a_2]$. The following lemma provides a constructive answer to the motion planing problem.

Lemma 5 (Flatness of the rectangular tank) Take two arbitrary C^3 time functions y_1 and y_2. Then D and H defined by ($c^2 = g\bar{h}$)

$$\begin{cases} D_1(t) = \dfrac{1}{2}\left(y_1(t + \dfrac{a_1}{c}) + y_1(t - \dfrac{a_1}{c})\right) \\ D_2(t) = \dfrac{1}{2}\left(y_2(t + \dfrac{a_2}{c}) + y_2(t - \dfrac{a_2}{c})\right) \\ H(x_1, x_2, t) = \dfrac{c}{2g}\left(\dot{y}_1(t + \dfrac{x_1}{c}) - \dot{y}_1(t - \dfrac{x_1}{c}) + \dot{y}_2(t + \dfrac{x_2}{c}) - \dot{y}_2(t - \dfrac{x_2}{c})\right). \end{cases} \tag{38}$$

satisfy (37) automatically.

The proof is straightforward. When y_1 and y_2 are constant, $D_1 = y_1$, $D_2 = y_2$ and $H = 0$. Steering from steady position (p_1, p_2) to steady position (q_1, q_2) can then be solved as in section II with a sigmoid function for y_1 and y_2 similar to (30).

In fact, equations (38) can be seen as the superposition of solutions of two one-dimensional wave equations whose boundary conditions are decoupled, see [12]. We represent on figure 4 successive contours of the free surface of a rectangular tank filled with fluid steered from two different steady points, using bump functions in equations (38).

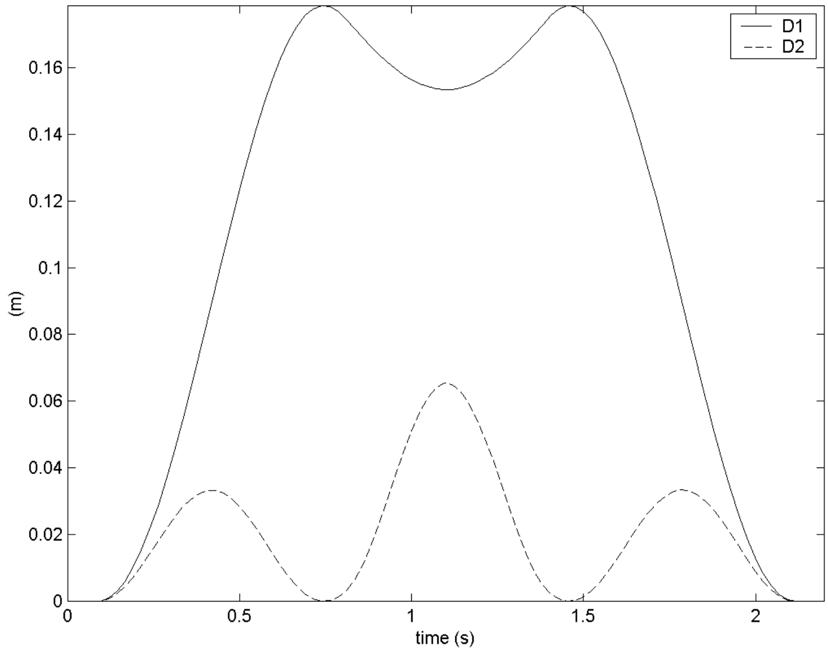

Fig. 5. The tank position D_1 and D_2 associated to the contours sequence of figure 4

B. Translation of the circular tank: the tumbler

Model 6: When $Z \equiv 0$, $\vec{\omega} \equiv 0$ and $(\vec{\imath}_1, \vec{\imath}_2, \vec{k}) \equiv (\vec{I}_1, \vec{I}_2, \vec{K})$, model 2 becomes for a straight bottom ($\bar{h}$ constant):

$$\begin{cases} \ddot{H} = g\bar{h}\Delta H \\ g\nabla H \cdot \vec{n} = -u.\vec{n} \quad \text{on } \partial\Omega \\ \ddot{D} = u \end{cases} \tag{39}$$

Assume that Ω is the disk of radius l and D its center. We denote by (r, θ) the polar coordinates with respect to the center of Ω. The following lemma provides a simple positive and constructive answer to the motion planing problem.

Lemma 6: Take two arbitrary C^3 time functions y_1 and y_2. Then D and H defined by

$$\begin{cases} D_1(t) = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} y_1 \left(t - \dfrac{l \cos\varphi}{\sqrt{g\bar{h}}} \right) \cos^2\varphi \, d\varphi \\ D_2(t) = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} y_2 \left(t - \dfrac{l \cos\varphi}{\sqrt{g\bar{h}}} \right) \cos^2\varphi \, d\varphi \\ H(r,\theta,t) = \dfrac{\cos\theta}{\pi} \sqrt{\dfrac{\bar{h}}{g}} \displaystyle\int_0^{2\pi} \dot{y}_1 \left(t - \dfrac{r}{\sqrt{g\bar{h}}} \cos\varphi \right) \cos\varphi \, d\varphi \\ \qquad + \dfrac{\sin\theta}{\pi} \sqrt{\dfrac{\bar{h}}{g}} \displaystyle\int_0^{2\pi} \dot{y}_2 \left(t - \dfrac{r}{\sqrt{g\bar{h}}} \cos\varphi \right) \cos\varphi \, d\varphi \end{cases} \tag{40}$$

satisfy automatically (39).
When y_1 and y_2 are constant, $D_1 = y_1$, $D_2 = y_2$ and $H = 0$. Steering from steady position (p_1, p_2) to steady position (q_1, q_2) can then be solved as for the rectangular tank.

Figure 6 shows the shape of the free surface during a transition between two steady points. This snapshot was computed using lemma 6. The corresponding `Matlab` code can be obtained upon request to the authors.

Proof: The direct proof which consists in verifying (39) is left to the reader. The proof given below is much more instructive. It explains the method used to obtain (40). Moreover it can be generalized to any variable depth profile $\bar{h}$ depending only on r. This proof uses some classical computations detailed in [28].

Let us perform a Laplace transform with respect to the time variable (with zero initial conditions)

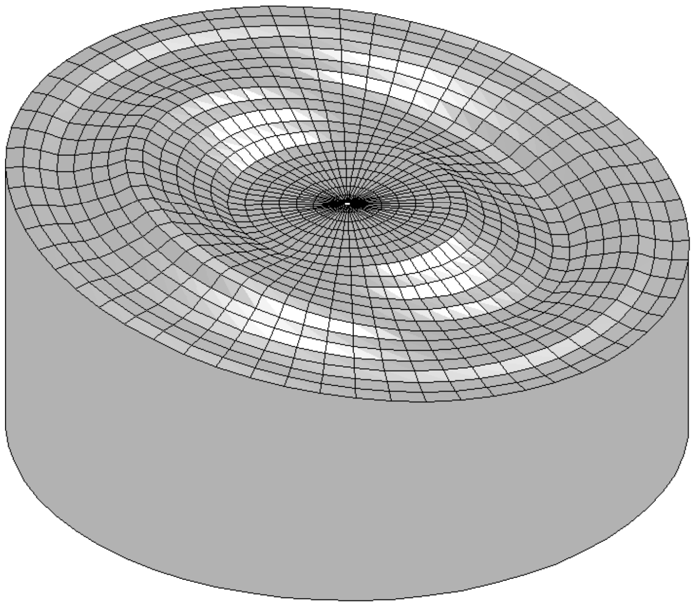

Fig. 6. The tumbler in movement. Snapshot of an animation computed using the explicit parameterization (40).

$$\Delta\hat{H}(x,y,s) - \frac{s^2}{g\bar{h}}\hat{H}(x,y,s) = 0 \tag{41}$$

where s can be considered as a parameter. Let us find a solution of equation (41) in cylindrical coordinates in the form $\bar{H}(r,\theta) = R(r)\Theta(\theta)$. By differentiation we get

$$\frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} - \frac{s^2}{g\bar{h}} = 0. \tag{42}$$

The variable θ appears only in the second term of this equation. So the term $\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}$ is independent of r and θ. It just depends on s and it can be denoted by $-\nu^2(s)$, $\nu \in \mathbf{C}$. Thus

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta} = -\nu^2(s)$$

and

$$\Theta = A(s)\cos(\nu(s)\theta) + B(s)\sin(\nu(s)\theta)$$

for some integration constant $A(s)$ and $B(s)$. Then (42) writes

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + R\left(-\frac{s^2}{g\bar{h}} - \frac{\nu(s)^2}{r^2}\right) = 0.$$

This is a Bessel equation. Its general solution is a combination of $J_{\nu(s)}\left(\frac{isr}{\sqrt{g\bar{h}}}\right)$ and $Y_{\nu(s)}\left(\frac{is\rho}{\sqrt{g\bar{h}}}\right)$. Since $Y_{\nu(s)}\left(\frac{isr}{\sqrt{g\bar{h}}}\right)$ is not bounded for $r = 0$, we only consider solutions involving $J_{\nu(s)}$. A set of bounded solution of (41) is given by

$$\hat{H}(r,\theta,s) = J_{\nu(s)}\left(\frac{isr}{\sqrt{g\bar{h}}}\right)(A(s)\cos(\nu(s)\theta) + B(s)\sin(\nu(s)\theta)) \tag{43}$$

The boundary conditions in the Laplace domain is

$$g\frac{\partial \hat{H}}{\partial r} = -\hat{u}_1(s)\cos\theta - \hat{u}_2(s)\sin\theta \text{ for } r = l.$$

Via (43) the boundary conditions also read

$$\frac{\partial \hat{H}}{\partial r} = \frac{is}{\sqrt{g\bar{h}}} {J'}_1\left(\frac{isr}{\sqrt{g\bar{h}}}\right)(A(s)\cos\theta + B(s)\sin\theta) \text{ for } r = l.$$

By identification we have $\nu(s) = 1$ and

$$\begin{cases} \hat{u}_1(s) = -\, isA(s)\sqrt{\frac{g}{\bar{h}}}\ {J'}_1\left(\frac{isr}{\sqrt{g\bar{h}}}\right) \\ \hat{u}_2(s) = -\, isB(s)\sqrt{\frac{g}{\bar{h}}}\ {J'}_1\left(\frac{isr}{\sqrt{g\bar{h}}}\right) \\ \hat{H}(r,\theta,s) = (A(s)\cos\theta + B(s)\sin\theta)\ J_1\left(\frac{isr}{\sqrt{g\bar{h}}}\right). \end{cases}$$

Transforming these equations back into the time domain using the Poisson integral representations

$$J_1\left(\frac{isr}{\sqrt{g\bar{h}}}\right) = \frac{1}{2i\pi}\int_0^{2\pi} e^{-\frac{sr\cos\varphi}{\sqrt{g\bar{h}}}}\cos\varphi\; d\varphi$$
$${J'}_1\left(\frac{isr}{\sqrt{g\bar{h}}}\right) = \frac{1}{2\pi}\int_0^{2\pi} e^{-\frac{sr\cos\varphi}{\sqrt{g\bar{h}}}}\cos^2\varphi\; d\varphi$$

with

$$A(s) = 2is\sqrt{\frac{\bar{h}}{g}}\ \hat{y}_1, \quad B(s) = 2is\sqrt{\frac{\bar{h}}{g}}\ \hat{y}_2,$$

yields (40). ■

C. Translation and rotation

We consider a general tank with an arbitrary domain Ω and assume that the dynamics are described by model 2. We will prove that the method used for the one-dimensional tank can be extended to the two-dimensional one.

Assume that $t \mapsto D = (D_1, D_2, Z)$ is a given smooth time function. We can adjust the tank rotations such that the term

$$(\ddot{D} + g\vec{K})\cdot(x_1\vec{\imath}_1 + x_2\vec{\imath}_2)$$

appearing in (25) vanishes identically. With the three Euler angles (φ, θ, ψ) (see, e.g.,[29, pages 10,16]) this gives the following two equations

$$\begin{aligned} -(g+\ddot{Z})\cos\phi\sin\theta =& \ddot{D}_1(\cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi) \\ & - \ddot{D}_2(\cos\phi\cos\theta\sin\psi + \sin\phi\cos\psi) \\ -(g+\ddot{Z})\sin\phi\sin\theta =& \ddot{D}_1(\sin\phi\cos\theta\cos\psi + \cos\phi\sin\psi) \\ & + \ddot{D}_2(-\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi) \end{aligned}$$

that must be completed by the non-holonomic constraint $\vec{\omega}\cdot\vec{k} = 0$ (the tank cannot spin around $\vec{k}$)

$$\dot{\psi} + \dot{\phi}\cos\theta = 0.$$

Simple computations give $\psi \in [0, 2\pi[$ and $\theta \in]-\pi/2, \pi/2[$ directly

$$\cos\psi = \frac{\ddot{D}_1}{\sqrt{\ddot{D}_1^2 + \ddot{D}_2^2}}, \quad \sin\psi = -\frac{\ddot{D}_2}{\sqrt{\ddot{D}_1^2 + \ddot{D}_2^2}}, \quad \tan\theta = -\frac{\sqrt{\ddot{D}_1^2 + \ddot{D}_2^2}}{g + \ddot{Z}}.$$

The remaining angle $\phi \in [0, 2\pi[$ is then obtained by integrating

$$\dot{\phi} = -\dot{\psi}/\cos\theta.$$

This method is just a compensation of accelerations by tank rotations. With such rotations the vector $\vec{k}$ that is orthogonal to the liquid surface at rest always remains co-linear to the total acceleration $\ddot{D} + g\vec{K}$. As for the one-dimensional tank, we can move the tank from one steady-state position to another one. Expected for simple motions $t \mapsto D(t)$ such as straight line ones, the orientation of the tank is not preserved between two steady-state positions: θ always returns to 0 after the motion, whereas the net rotation around the vertical axis $\vec{K}$, i.e., the total variation of $\phi + \psi$, does not. This results from the non-holonomic constraint $\dot{\psi} + \dot{\phi}\cos\theta = 0$.

Notice that, if the problem is to steer the tank from $D_0 = (p_1, p_2)$ at time 0 to $D_T = (q_1, q_2)$ at time $T > 0$ and to preserve its initial and final orientations, such method works when we take the straight trajectory $D(t) = (1 - \sigma(t))D_0 + \sigma(t)D_T$ with $[0, T] \ni t \mapsto \sigma(t) \in [0, 1]$ a smooth function such that $\sigma(0) = 0$, $\sigma(T) = 1$ and $\dot{\sigma} = \ddot{\sigma} = 0$ at $t = 0$ and $t = T$.

D. Open problem: beyond rectangular and circular shapes

For special geometries of the fluid domain Ω (namely rectangle and disk) and bottom profile ($\bar{h}$ constant) we have seen that

$$\begin{aligned} \frac{\partial^2 H}{\partial t^2} &= \nabla \cdot \left(\bar{h} \left(\ddot{D} + g\nabla H \right) \right) \quad \text{on } \Omega \\ g\nabla H \cdot \vec{n} &= -u \cdot \vec{n} \quad \text{on } \partial\Omega \\ \ddot{D} &= u \end{aligned}$$

is steady-state controllable with the two controls $\ddot{D}_1 = u_1$ and $\ddot{D}_2 = u_2$. We have also seen that in the one-dimensional case it is steady-state controllable for arbitrary bottom profile $\bar{h}(x)$. Is-it still true in the two dimensional case with an arbitrary domain Ω? As far as we know the ellipsoidal case is problematic. Using the technique we detailed for the circular case, we are left with Mathieu equations instead of Bessel equations. The fundamental solutions of these Mathieu equations do not have handy integral representations that would give a constructive proof of controllability when turned back into the time-domain. Up to now this seems a major obstruction to our method.

IV. Conclusion

The results presented in this paper are all based on linear control models deduced from shallow water approximations. This is a major restriction but we would like to emphasize the difficulties one would encounter dealing with a non-horizontal fluid velocity. For an arbitrary liquid height, a correct description of the dynamics around steady-states could be obtained as follows. For the translation of the tank of figure 1 with irrotational 2D flows, we linearize the Euler equations and the free boundary conditions. Following [14, page 436] the system is described by a scalar potential $\phi(x, z, t)$ depending on the horizontal coordinate x, the vertical one z and the time t, that satisfies

$$\begin{aligned} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} &= 0 \quad \text{for } (x, z) \in [-a, a] \times [0, \bar{h}] \\ \frac{\partial \phi}{\partial z}(x, 0, t) &= 0 \quad \text{for } x \in [-a, a] \\ g\frac{\partial \phi}{\partial z}(x, \bar{h}, t) &= -\frac{\partial^2 \phi}{\partial t^2}(x, \bar{h}, t) \quad \text{for } x \in [-a, a] \\ \frac{\partial \phi}{\partial x}(-a, z, t) &= \dot{D}(t) \quad \text{for } z \in [0, \bar{h}] \\ \frac{\partial \phi}{\partial x}(a, z, t) &= \dot{D}(t) \quad \text{for } z \in [0, \bar{h}] \end{aligned}$$

where $D(t)$ is the control, the horizontal tank position. The fluid velocity with respect to the tank admits two components, v the horizontal one and w the vertical one given by

$$v(x, z, t) = \frac{\partial \phi}{\partial x} - \dot{D}(t), \quad w(x, z, t) = \frac{\partial \phi}{\partial z}.$$

The liquid height is also derived from ϕ via

$$h(x, t) = \bar{h} - \frac{1}{g}\frac{\partial \phi}{\partial t}(x, \bar{h}, t).$$

This implicit formulation of the dynamics is very similar to differential-algebraic systems of index 1 [30], [31], [32]

$$\frac{dX}{dt} = f(X, Y, U), \quad 0 = g(X, Y, U)$$

often encountered for finite dimensional systems ($\frac{\partial g}{\partial Y}$ invertible). Set

$$X \equiv (\phi(x, \bar{h}, t))_{-a \leq x \leq a}, \quad Y \equiv \phi, \quad U \equiv D.$$

Then the algebraic part $g(X, Y, U) = 0$ reads

$$\begin{cases} \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial z^2} = 0 \quad \text{for } (x, z) \in [-a, a] \times [0, \bar{h}] \\ \dfrac{\partial \phi}{\partial z}(x, 0, t) = 0 \quad \text{for } x \in [-a, a] \\ \phi(x, \bar{h}, t) = X(x, t) \quad \text{for } x \in [-a, a] \\ \dfrac{\partial \phi}{\partial x}(-a, z, t) = \dot{U} \quad \text{for } z \in [0, \bar{h}] \\ \dfrac{\partial \phi}{\partial x}(a, z, t) = \dot{U} \quad \text{for } z \in [0, \bar{h}]. \end{cases} \tag{44}$$

The differential part $dX/dt = f$ corresponds to

$$\frac{\partial^2 \phi}{\partial t^2}(x, \bar{h}, t) = -g \frac{\partial \phi}{\partial z}(x, \bar{h}, t) \quad \text{for } x \in [-a, a]$$

The system is of "index 1" since the "algebraic part" is invertible with respect to the "algebraic variables" Y: ϕ is a linear function of X and U by solving (44). Such implicit formulations of "index one" remain valid when the fluid is irrotational and described by the following nonlinear Euler equations (see, e.g., [14, pp:431–436]):

$$\begin{aligned} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} &= 0 \quad \text{for } -a \leq x \leq a, \quad 0 \leq z \leq h(x, t) \\ \frac{\partial \phi}{\partial z}(x, 0, t) &= 0 \quad \text{for } x \in [-a, a] \\ \left[\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right)\right]_{(x, h(x,t), t)} + g h(x, t) &= 0 \quad \text{for } x \in [-a, a] \\ \frac{\partial \phi}{\partial z}(x, h(x, t), t) - \frac{\partial h}{\partial x}(x, t) \frac{\partial \phi}{\partial x}(x, h(x, t), t) - \frac{\partial h}{\partial t}(x, t) &= 0 \quad \text{for } x \in [-a, a] \\ \frac{\partial \phi}{\partial x}(-a, z, t) &= \dot{D}(t) \quad \text{for } z \in [0, h(-a, t)] \\ \frac{\partial \phi}{\partial x}(a, z, t) &= \dot{D}(t) \quad \text{for } z \in [0, h(a, t)] \end{aligned}$$

with $z = h(x, t)$ the free surface equation (the profiles $\zeta(x, t) = \phi(x, h(x, t), t)$ and $h(x, t)$ corresponding then to the "differential variables" X).

Very few results (see [33] for a first result on a closely related problem) are available concerning the controllability and stabilization of such implicit systems of infinite dimension. Are such systems steady-state controllable?

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Appendix

Technical lemma

Lemma 7: Take $\mathbf{R} \ni x \mapsto c(x)$ a strictly positive smooth function and consider for each $s \in \mathbf{C}$, $x \mapsto A(x, s)$, the solution of

$$\begin{aligned} &\frac{\partial}{\partial x}\left(c^2(x)\frac{\partial A}{\partial x}\right) = s^2 A \\ &A(0, s) = a \\ &\frac{\partial A}{\partial x}(0, s) = b \end{aligned}$$

with $(a,b) \in \mathbf{R}^2$. Set $\sigma(x) = \displaystyle\int_0^x \frac{d\xi}{c(\xi)}$. Then for each x, there exists an L^2 function $[-\sigma(x),\sigma(x)] \ni \xi \mapsto \mathcal{B}(x,\xi) \in \mathbf{R}$ such that

$$A(x,s) = a\sqrt{\frac{c(0)}{c(x)}}\cosh(s\sigma(x)) + \int_{-\sigma(x)}^{\sigma(x)} \mathcal{B}(x,\xi)\exp(\xi s)d\xi.$$

Proof: The proof of this result is organized as follows

1. A Liouville transform, $x \mapsto z$ and $A \mapsto u$, is performed.
2. Using a majoring series we prove that, for each z, $s \mapsto u(z,s)$ is an entire functions of exponential kind.
3. We show that for any given $z \in [0,1]$, $\imath\mathbf{R} \ni s \mapsto u(z,s)$ is, up to some addition of exponentials, in L^2
4. We conclude thanks to the Paley-Wiener theorem.

Remark 6: $\mathcal{B}$ depends on a and b as detailed below. $\mathcal{B} = 0$ if $(a,b) = (0,0)$.

Liouville transform

The Liouville transform

$$(x,A) \mapsto (z,u)$$

(see for instance [34, page 110]) turns the equations

$$\frac{d}{dx}\left(p(x)\frac{dA}{dx}\right) + (\lambda r(x) - q(x))\,A = 0,$$

where $p(x) > 0$ into the following form

$$\frac{d^2u}{dz^2} + (\rho^2 - h(z))u = 0$$

where ρ depends only on λ and can be considered as a parameter.

Here

$$p(x) = c^2(x), \quad \lambda = -s^2, \quad r(x) = 1, \quad q(x) = 0, \quad x \in [0,L].$$

With the change of variables

$$z = \int_0^x \frac{1}{c}, \quad u(z,s) = (c(x))^{1/2}A(x,s)$$

we obtain

$$H(z) = \frac{F''(z)}{F(z)} \quad \text{with } F(z) = \sqrt{c(x)}.$$

We have turned

$$\left\{\begin{aligned} \frac{\partial}{\partial x}\left(c^2(x)\frac{\partial A}{\partial x}\right) &= s^2A \\ A(0,s) &= a \\ \frac{\partial A}{\partial x}(0,s) &= b \end{aligned}\right. \tag{45}$$

into

$$\left\{\begin{aligned} \frac{d^2u}{dz^2} - (h(z) + s^2)u &= 0 \\ u(0,s) &= \alpha \\ \frac{du}{dz}(0,s) &= \beta \end{aligned}\right. \tag{46}$$

with

$$\alpha = u(0,s) = a(c(0))^{1/2}, \quad \beta = \frac{du}{dz}(0,s) = \frac{c'(0)(c(0))^{1/2}}{2}a + c(0)^{3/2}b$$

Proving that $\mathbf{C} \ni s \mapsto u(z,s)$ is an entire function of exponential type

Let $W(z,s)$ the 2×2 matrix solution of

$$\frac{dW}{dz} = \begin{pmatrix} 0 & 1 \\ h(z)+s^2 & 0 \end{pmatrix} W \text{ with } W(0,s) = I$$

Then $u(z,s) = \begin{pmatrix} 1 & 0 \end{pmatrix} W(z,s) \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Let us show that W is entire with respect to s. By the classical fixed point technique $W(z,s) = \sum_{i\geq 0} W_i(z,s)$ with the following recurrence

$$W_0(z,s) = I, W_{i+1}(z,s) = \int_0^z \begin{pmatrix} 0 & 1 \\ h(z)+s^2 & 0 \end{pmatrix} W_i(\xi,s) d\xi$$

Each $W_i(z,s)$ is a polynomial in s^2. Its degree is $2i$ and its coefficients depend only on z. Reordering all the terms we get

$$\sum_{0\leq i\leq k} W_i(z,s) = \sum_{0\leq j\leq k} W^{j,k}(z) s^{2j}.$$

From step k to $k+1$ we have

$$W^{j,k+1}(z) = W^{j,k}(z) + \mathcal{W}^{j,k+1}(z)$$

where $\mathcal{W}^{j,k+1}$ is the coefficient of s^{2j} in W_{k+1}.

Take $K>0$ and $z\in[0,K]$. Set $m = \sup_{[0,K]} \mid h \mid$ and define the following majoring series by the recurrence

$$M_0(z,s) = I, M_{i+1}(z,s) = \int_0^z \begin{pmatrix} 0 & 1 \\ m+s^2 & 0 \end{pmatrix} M_i(\xi,s)\; d\xi$$

As previously we define

$$\sum_{0\leq i\leq k} M_i(z,s) = \sum_{0\leq j\leq k} M^{j,k}(z) s^{2j}, M^{j,k+1}(z) = M^{j,k}(z) + \mathcal{M}^{j,k+1}(z)$$

By classical matrix computations we get

$$M(z,s) = \begin{pmatrix} \cosh(z\sqrt{m+s^2}) & \sinh(z\sqrt{m+s^2})/\sqrt{m+s^2} \\ \sinh(z\sqrt{m+s^2})\sqrt{m+s^2} & \cosh(z\sqrt{m+s^2}) \end{pmatrix}.$$

For each j, the matrices $M^{j,k} = \sum_{j\leq l\leq k-1} \mathcal{M}^{j,l}$ converge as k tends to ∞. Denote by M^j the limit. By construction, $M = \sum_{j\geq 0} M^j(z)\;\rho^{2j}$ and this series has an infinite radius of convergence in ρ, since, for each z, the functions $s\mapsto \cosh(z\sqrt{m+s^2})$, $s\mapsto \sinh(z\sqrt{m+s^2})/\sqrt{m+s^2}$ and $s\mapsto \sinh(z\sqrt{m+s^2})\sqrt{m+s^2}$ are entire functions of s^2.

But, for each i, j and k, the matrices $M^{j,k}$ and $\mathcal{M}^{j,k+1}$ whose entries are always non-negative, dominate the absolute values of the entries of $W^{j,k}$ and $\mathcal{W}^{j,k+1}$, respectively. Thus for each j, the matrices $W^{j,k} = \sum_{j\leq l\leq k-1} \mathcal{W}^{j,l}$ converge as k tends to ∞. Denote by W^j the limit. By construction, $W = \sum_{j\geq 0} W^j(z)\rho^{2j}$ and this series has an infinite radius of convergence in ρ, since M has one. In other words, W is an entire function of ρ. Moreover the entries of M are upper bounds of the entries of W. Thus W is of exponential type in ρ: for each $z\in[0,K]$, there exists $E>0$ such that

$$\forall s\in\mathbf{C},\quad |W(z,s)| \leq E\exp(z|s|).$$

We have proven that, for each $z\in[0,\pi]$, $u(z,s)$ is an entire function of s with exponential type :

$$\forall s\in\mathbf{C},\quad u(z,s) \leq b(z)\exp(z|s|) \tag{47}$$

for some $b(z)>0$ well chosen.

Proving that "a part" of $i\mathbf{R} \ni s \mapsto u(z,s)$ belongs to L^2

From the Volterra equation of the second kind satisfied by u (see for instance [34, p. 111]),

$$u(z,s) = \alpha \cosh(sz) + \beta \frac{\sinh(sz)}{s} + \frac{1}{s}\int_0^z \sinh(s(z-\xi))h(\xi)u(z,s)d\xi \tag{48}$$

Denote

$$w(z,s) = u(z,s) - \alpha \cosh(sz) \tag{49}$$

From (48) we deduce

$$w(z,s) = \phi(z,s) + \frac{1}{s}\int_0^z \sinh(s(z-\xi))h(\xi)w(z,s)d\xi \tag{50}$$

with

$$\phi(z,s) = \beta \frac{\sinh(sz)}{s} + \frac{1}{s}\int_0^z \sinh(s(z-\xi))h(\xi)\alpha\cosh(s\xi)d\xi.$$

Clearly, there exists D such that for all $z \in [0,K]$ and $s \in \imath\mathbf{R}$,

$$\mid \phi(z,s) \mid \leq \frac{D}{1+\mid s \mid}$$

(h is bounded). Let us show that for any given z, $i\mathbf{R} \ni s \mapsto w(z,s)$ is in L^2. To prove this we use the following classical majoring arguments (see [34, p. 112] for instance). Denote

$$\mu(z,s) = \sup_{0\leq\xi\leq z} \mid w(\xi,s) \mid$$

We deduce from (50) that

$$\mu(z,s) \leq \frac{D}{1+\mid s \mid} + \frac{1}{\mid s \mid} m\mu(z,s)K$$

for $m = \sup_{[0,K]} \mid h \mid$. So

$$\mu(z,s) \leq \frac{D}{(1+\mid s \mid)} \frac{1}{1-\frac{mK}{|s|}}.$$

And for $\mid s \mid \geq 2mK$

$$\mu(z,s) \leq \frac{2D}{1+\mid s \mid}$$

which proves that $i\mathbf{R} \ni s \mapsto w(z,s)$ is in L^2.

Use of the Paley-Wiener theorem

$i\mathbf{R} \ni s \mapsto w(z,s)$ is in L^2 and is an entire function of s of exponential type such that $\mid w(z,s) \mid \leq d(z)\exp(z \mid s \mid)$. Thanks to the Paley-Wiener theorem we can conclude that for each z there exists $[-z,z] \ni \xi \mapsto \mathcal{K}(z,\xi)$ in $L^2[-z,z]$ such that

$$w(z,s) = \int_{-z}^{z} \mathcal{K}(z,\xi)\exp(\xi s)\ d\xi.$$

Then

$$u(z,s) = \alpha\cosh(sz) + \int_{-z}^{z} \mathcal{K}(z,\xi)\exp(\xi s)d\xi.$$

and

$$
\begin{aligned}
A(x,s) &= \frac{1}{\sqrt{c(x)}} u\left(\sigma(x), s\right) \\
&= \frac{\alpha}{\sqrt{c(x)}} \cosh(s\sigma(x)) + \frac{1}{\sqrt{c(x)}} \int_{-\sigma(x)}^{\sigma(x)} \mathcal{K}(\sigma(x),\xi) \exp(\xi s) d\xi \\
&= a\sqrt{\frac{c(0)}{c(x)}} \cosh(s\sigma(x)) + \int_{-\sigma(x)}^{\sigma(x)} \mathcal{B}(x,\xi) \exp(\xi s) d\xi
\end{aligned}
\tag{51}
$$

with $z = \sigma(x) = \int_0^x \frac{1}{c}$ and $\mathcal{B}(x,\xi) = \mathcal{K}(\sigma(x), s)/\sqrt{c(x)}$. ■

I. Simulations

In this section, we report some results of [18]. They correspond to numerical simulations (Godunov scheme) of the $1D$ nonlinear Saint-Venant equations 12 with $\theta = 0$ with the open-loop control $u = \ddot{D}$ of formula (31) and based on the linear tangent equations. This simulation indicates that, when the tank motion is not to fast the neglected nonlinearity are not very important. Several other simulation show that our open-loop control design is effective when $\sup|\ddot{D}|/ga \ll \bar{h}$.

In the following, Δ, which is the required time for a wave to meet a boundary starting from the opposite one, is equal to 1. The vertical scale of the figures has been enlarged by a factor 3 for the reader to see the details.

A. *Transfer time T=4.0*

The prediction of a slow move is rather close to the numerical results of a Godunov scheme simulation. Results are shown on figure 7.

B. *Transfer time T=2.5*

Yet as the move speeds up the prediction results get more different from the numerical simulation. Results are shown on figure 8.

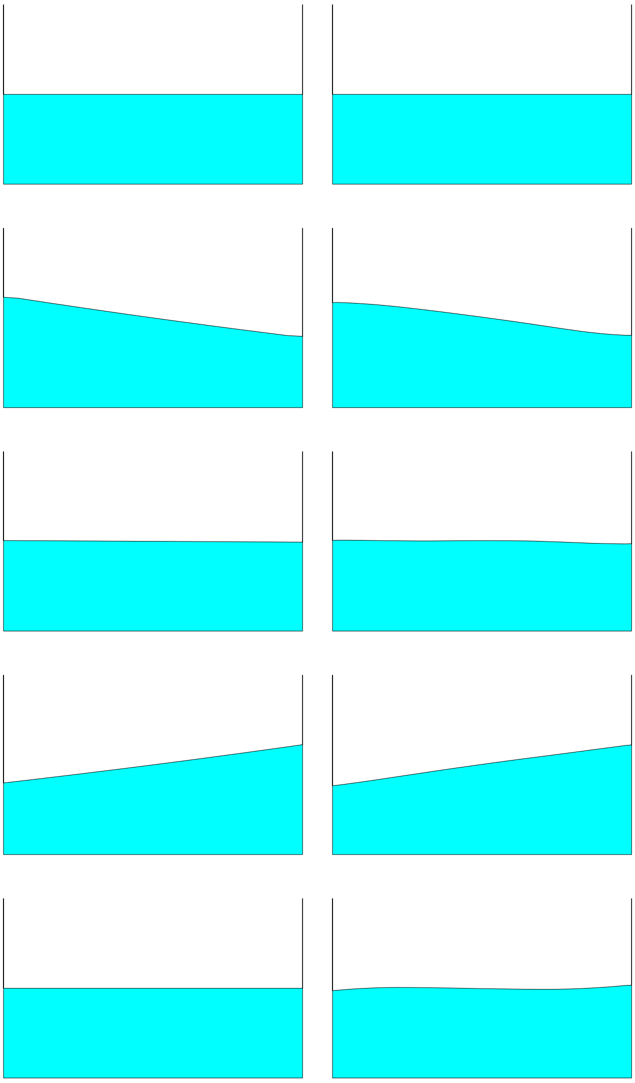

Fig. 7. T=4.0; snapshots at t=0, t=T/4, t=T/2, t=3T/4 and t=T. Left: linear prediction. Right: nonlinear simulation.

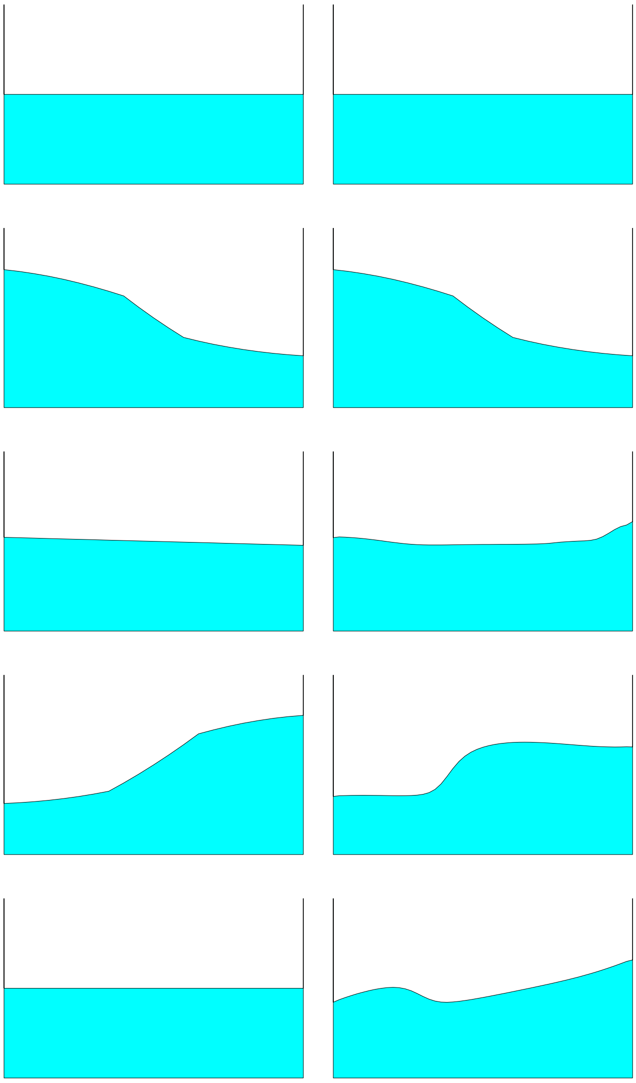

Fig. 8. T=2.5; snapshots at t=0, t=T/4, t=T/2, t=3T/4 and t=T. Left: linear prediction. Right: nonlinear simulation.