Angular velocity nonlinear observer from vector measurements

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ABSTRACT
This paper proposes a technique to estimate the angular velocity of a rigid body from vector measurements. Compared to the approaches presented in the literature, it does not use attitude information nor rate gyros as inputs. Instead, vector measurements are directly filtered through a nonlinear observer estimating the angular velocity. Convergence is established using a detailed analysis of the linear-time varying dynamics appearing in the estimation error equation. This equation stems from the classic Euler equations and measurement equations. A high gain design allows to establish local uniform exponential convergence. Simulation results are provided to illustrate the method.

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1. Introduction
This article considers the question of estimating the angular velocity of a rigid body from embedded sensors. This broad question has applications in various fields of engineering and applied science. Some specific examples are as follows. In aerospace, the deployment phase of spinning satellites starts by a detumbling maneuver during which the angular velocity is controlled in an active way until it reaches a zero value (Bošković, Li, & Mehra, 2000). The control strategy employs an estimation of this variable, in closed-loop. High velocity spinning objects are very common in ballistics. The XM25 air-burst rifle (smart-weapon) fires smart shells which estimate their rotation to determine the traveled distance (so that explosion of the projectile can be activated at any user-defined distance). Finally, the problem of angular velocity estimation can also be found in the emerging field of smart devices for sport such as the on-board football camera (Kitani, Horita, & Hideki, 2012) as it is important for athletes in many sports to train their skills to spin a ball.

In the literature, several types of methods have been proposed to address this question. On the one hand, the straightforward solution is to use a strap-down rate gyro (Titterton & Weston, 2004), which directly provides measurements of the angular velocities. However, rate gyros being relatively fragile and expensive components, prone to drift, other types of solutions are often preferred. Instead, a two-step approach is commonly employed. The first step is to determine attitude from vector measurements, i.e. on-board measurements of reference vectors being known in a fixed frame. Vector measurements play a central role in the problem of attitude determination as discussed in a recent survey (Crassidis, Markley, & Cheng, 2007). In a nutshell, when two or more independent vectors are measured with vector sensors attached to a rigid body, its attitude can be simply defined as the solution of the classic Wahba problem (Wahba, 1965) which formulates a minimization problem having the rotation matrix from a fixed frame to the body frame as unknown. The second step is to reconstruct angular velocities from the attitude. At any instant, full attitude information can be obtained (Bar-Itzhack, 1996; Choukroun, 2003; Shuster, 1978, 1990). In principle, once the attitude is known, angular velocity can be estimated from a time-differentiation. The survey (Bar-Itzhack, 2001) names this approach the derivative method. However, noise disturbs this process. To address this issue, introducing a priori information in the estimation process is a valuable technique to filter-out noise from the estimates. For this reason, numerous observers using Euler's equations for a rigid body have been proposed to estimate angular velocity (or angular momentum, which is equivalent) from full attitude information (Jorgensen & Gravdahl, 2011; Salcudean, 1991; Sunde, 2005; Thienel & Sanner, 2007). Besides this two-step approach, a more direct solution can be proposed. In this paper, we expose an algorithm that directly uses the vector measurements and reconstructs the angular velocity in a simple manner.
The contribution of this paper is a nonlinear observer reconstructing the angular velocity of a rotating rigid body from vector measurements directly, namely by bypassing the relatively heavy first step of attitude estimation. Variants and extensions of this approach can be found in Magnis (2015), Magnis and Petit (2013) and Magnis and Petit (2015a,b). The proposed method allows one to estimate the angular velocity without any gyroscope. Contrary to the method presented in Oshman and Dellus (2003), it does not employ time differentiation of the measurements.

This paper is organized as follows. In Section 2, we introduce the notations and the problem statement. We analyze the attitude dynamics (rotation and Euler equations) and relate it to the measurements. In Section 3, we define a nonlinear observer with extended state and output injection. To prove its convergence, the error equation is identified as a linear time-varying (LTV) system perturbed by a linear–quadratic term. The dominant part of the error equation is identified as a linear time-varying (LTV) system. In turn, this property reveals instrumental to conclude on the exponential uniform convergence of the error dynamics. Illustrative simulation results are given in Section 4. Conclusions and perspectives are given in Section 5.

2. Notations and problem statement

2.1. Notations

Norms. The Euclidean norms in \( \mathbb{R}^3 \) and in \( \mathbb{R}^3 \) are denoted by \( |\cdot| \). The induced norm on \( 9 \times 9 \) matrices is denoted by \( \|\cdot\| \). Namely, \( \|M\| = \max_{x \in \mathbb{R}^9, \|x\|=1} |MX| \).

The cross-product matrix associated with a vector \( x \in \mathbb{R}^3 \) is denoted by \( [x_x] \), i.e. \( \forall y \in \mathbb{R}^3, \ [x_x]y = x \times y \). Namely,

\[
[x_x] \triangleq \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}
\]

where \( x_1, x_2, x_3 \) are the coordinates of \( x \) in the standard basis of \( \mathbb{R}^3 \).

2.2. Problem statement

Consider a rigid body rotating with respect to an inertial frame \( \mathcal{R}_\infty \). Note \( \mathcal{R} \) the rotation (orthogonal) matrix representing the linear mapping from \( \mathcal{R}_\infty \) to a body frame \( \mathcal{R}_b \) attached to the rigid body, expressed in \( \mathcal{R}_\infty \). \( \mathcal{R} \) satisfies the differential equation

\[
\dot{\mathcal{R}} = \mathcal{R}[\omega_x]
\]

where \( \omega \) is the angular velocity of the rigid body expressed in the body frame. The dynamics of \( \omega \) itself is governed by the famed Euler’s equations (Landau & Lifchitz, 1982)

\[
\dot{\omega} = J^{-1}(J\omega \times \omega + \tau)
\]

where \( J = \text{diag}(J_1, J_2, J_3) \) is the matrix of inertia and \( \tau \) is the external torque applied to the rigid body.

Consider two reference vectors \( \hat{a}, \hat{b} \) expressed in the inertial frame. Then, the expressions of \( a, b \) in the body frame at time \( t \) are

\[
a(t) = R(t)\hat{a}, \quad b(t) = R(t)\hat{b}.
\]

The variables \( a, b \) are called vector measurements. For implementation, they can be produced by direction sensors such as e.g. accelerometers, magnetometers or Sun sensors to name a few (Magnis & Petit, 2014).

We now formulate some assumptions.

**Assumption 1.** \( \hat{a}, \hat{b} \) are constants and linearly independent.

**Assumption 2.** \( J \) and \( \tau \) are known.

**Assumption 3.** \( \omega \) is bounded: \( |\omega(t)| \leq \omega_{\text{max}} \) at all times.

The problem we address in this paper is the following.

**Problem 1.** From measurements of the type (3), find an estimate \( \hat{\omega} \) of the angular velocity \( \omega \) appearing in (1), assuming it satisfies (2).

**Remark 1.** Without loss of generality, we assume \( \hat{a}^T\hat{b} \geq 0 \) (if not, one can simply consider \( -\hat{a} \) instead of \( \hat{a} \)). We denote \( p \triangleq \hat{a}^T\hat{b} \geq 0 \).

Assumption 1 implies that \( p \) is constant and \( p \in [0, 1) \). Note that, for all time \( t \)

\[
a(t)^Tb(t) = \hat{a}^T(R(t)R(t)^Tb) = \hat{a}^T\hat{b} = p.
\]

3. Observer definition and analysis of convergence

3.1. Observer definition

From Assumption 1, we have \( \frac{\partial}{\partial t} \hat{a} = 0 \). Hence, the time derivative of the measurement \( a \) is

\[
\dot{a} = R^T\hat{a} = -[\omega_x]R^T\hat{a} = \hat{a} \times \omega
\]

and the same holds for \( \dot{b} = b \times \omega \). To solve Problem 1, the main idea of the paper is to consider the reconstruction of the extended 9-dimensional state \( X \) by its estimate \( \hat{X} \)

\[
X = [a^T \ b^T \ \omega^T]^T, \quad \hat{X} = [\hat{a}^T \ \hat{b}^T \ \hat{\omega}^T]^T.
\]

The state is governed by

\[
\dot{\hat{X}} = \begin{pmatrix} a \times \omega \\
\ b \times \omega \\
E(\omega) + J^{-1}\tau \end{pmatrix}
\]

and the following observer is proposed

\[
\dot{\hat{X}} = \begin{pmatrix} a \times \hat{\omega} - ak(\hat{a} - a) \\
\ b \times \hat{\omega} - ak(\hat{b} - b) \\
E(\hat{\omega}) + J^{-1}\tau + k^2(a \times \hat{a} + b \times \hat{b}) \end{pmatrix}
\]

where \( a \in (0, 2\sqrt{T - p}) \) and \( k > 0 \) are constant (tuning) parameters. Denote

\[
\tilde{X} \triangleq X - \hat{X} \triangleq [\hat{a}^T \ \hat{b}^T \ \hat{\omega}^T]^T
\]

the error state. We have

\[
\dot{\tilde{X}} = \begin{pmatrix} -ak \hat{a} & 0 & 0 \\
0 & -ak \hat{b} & 0 \\
k^2[a_x] & k^2[b_x] & 0 \end{pmatrix} \tilde{X} + \begin{pmatrix} 0 \\
0 \\
E(\hat{\omega}) - E(\omega) \end{pmatrix}.
\]

In Section 3.4 we will exhibit, for each value \( 0 < \alpha < 2\sqrt{T - p} \), a threshold value \( k^* \) such that for \( k > k^* \), \( \tilde{X} \) converges locally uniformly exponentially to zero.
3.2. Preliminary change of variables and properties

The study of the dynamics (8) employs a preliminary change of coordinates. Denote

$$Z(t) = \left( \begin{array}{c} \hat{t}^T \\ \tilde{b}_T^T \\ \tilde{\omega}^T \end{array} \right)$$

yielding a reformulation of (8)

$$\dot{Z} = kA(t)Z + \left( \begin{array}{c} 0 \\ 0 \\ \frac{E(\omega)^T - E(\dot{\omega})}{k} \end{array} \right)^T$$

(10)

with

$$A(t) \triangleq \left( \begin{array}{ccc} -\alpha I & 0 & [a(t) x] \\ 0 & -\alpha I & [b(t) y] \\ [a(t) x] & [b(t) y] & 0 \end{array} \right)$$

(11)

which we will analyze as an ideal linear time-varying (LTV) system

$$\dot{Z} = kA(t)Z$$

(12)

disturbed by the input term

$$\xi \triangleq \left( \begin{array}{c} 0 \\ 0 \\ \frac{E(\omega)^T - E(\dot{\omega})}{k} \end{array} \right)^T$$

(13)

The idea is that for sufficiently large values of $k$, the rate of convergence of (12) will ensure stability of system (10). We start by upper-bounding $A(t)$ and the disturbance (13).

**Proposition 1** (Bound on the Unforced LTV System). $A(t)$ defined in (11) is upper-bounded by

$$A_{\text{max}} \triangleq \max\left( \sqrt{2 + 2a^2}, \sqrt{3 + a^2} \right).$$

**Proof.** Let $Y \in \mathbb{R}^3$ such that $|Y| = 1$. For convenience, denote

$$Y = (Y_1, Y_2, Y_3)^T$$

with $Y_i \in \mathbb{R}^3$. One has

$$|A(t)Y|^2 = | -\alpha Y_1 + a \times Y_3 |^2 + | -\alpha Y_2 + b \times Y_3 |^2 + |a \times Y_1 + b \times Y_2|^2 \leq (1 + \alpha^2)(|Y_1|^2 + |a \times Y_3|^2 + |Y_2|^2 + |b \times Y_3|^2) + 2(|a \times Y_1|^2 + |b \times Y_2|^2) \leq \max(2 + 2a^2, 3 + a^2)|Y|^2 = A_{\text{max}}^2.$$  

Hence, $\|A(t)\| = \max_{|Y| = 1} |A(t)Y| \leq A_{\text{max}}$.

**Proposition 2** (Bound on the Disturbance). For any $Z$, $\xi$ is bounded by

$$\|\xi\| \leq \sqrt{2\omega_{\text{max}}}|Z| + k|Z|^2.$$  

(14)

**Proof.** We have

$$\|\xi\| = \frac{1}{k}|E(\omega) - E(\dot{\omega})|^2$$

with, due to the quadratic nature of $E(\cdot)$,

$$E(\omega) - E(\dot{\omega}) = J^{-1}(J\dot{\omega} \times \omega + J\omega \times \dot{\omega} - J\dot{\omega} \times \dot{\omega})$$

Using $J = \text{diag}(J_1, J_2, J_3)$, we have

$$J^{-1}(J\dot{\omega} \times \omega) = \frac{1}{J_1}(J_1\dot{\omega}_2\omega_3 - J_3\dot{\omega}_2\omega_3), \frac{1}{J_2}(J_2\dot{\omega}_3\omega_1 - J_3\dot{\omega}_3\omega_1), \frac{1}{J_3}(J_3\dot{\omega}_1\omega_2 - J_2\dot{\omega}_1\omega_2).$$

Using similar expressions for $J^{-1}(J\omega \times \dot{\omega})$ and $J^{-1}(J\dot{\omega} \times \dot{\omega})$ yields

$$E(\omega) - E(\dot{\omega}) = \begin{pmatrix} \frac{J_2 - J_3}{J_1} (\dot{\omega}_2\omega_3 - \dot{\omega}_3\omega_2) \\ \frac{J_3 - J_1}{J_2} (\dot{\omega}_3\omega_1 - \dot{\omega}_1\omega_3) \\ \frac{J_1 - J_2}{J_3} (\dot{\omega}_1\omega_2 - \dot{\omega}_2\omega_1) \end{pmatrix} \leq \delta_1 - \delta_2.$$  

As $J_1, J_2, J_3$ are the main moments of inertia of the rigid body, we have (Landau & Lifchitz, 1982, Section 32.9) $J_1 \leq J_j + J_k$ for all permutations $i, j, k$ and hence

$$|J_1 - J_2|, \quad |J_2 - J_3|, \quad |J_1 - J_3| \leq 1.$$  

As a straightforward consequence $|\delta_2| \leq |\omega|^2$. Moreover, by the Cauchy-Schwarz inequality

$$(\dot{\omega}_2\omega_3 + \dot{\omega}_3\omega_2)^2 \leq (\omega_2^2 + \omega_3^2)(\omega_2^2 + \omega_3^2) \leq (\omega_2^2 + \omega_3^2)^2.$$  

Using similar inequalities for all the coordinates of $\delta_1$ yields

$$|\delta_1|^2 \leq 2|\omega|^2|\omega| \leq 2|\omega|^2$$

Hence,

$$\|\xi\| \leq \frac{|\delta_1| + |\delta_2|}{k} \leq \sqrt{2\omega_{\text{max}}}|\omega| + k|\omega|^2 \leq \sqrt{2\omega_{\text{max}}}|Z| + k|Z|^2.$$  

3.3. Analysis of the LTV dynamics $\dot{Z} = kA(t)Z$

We will now use a result on the exponential stability of LTV systems. The claim of Hill and Ilchmann (2011, Theorem 2.1), which is instrumental in the proof of the next result, is as follows: consider a LTV system $\dot{Z} = M(t)Z$ such that: (i) $M(\cdot)$ is $l$-Lipschitz, with $l > 0$, (ii) there exist $K \geq 1, c \geq 0$ such that for any $t$ and any $\gamma \geq 0$, $\|e^{lM(t)}\| \leq Ke^{-c}$ (frozen-time exponential stability). Then, for any $t_0, Z_0$, the solution of $\dot{Z} = M(t)Z$ with initial condition $Z(t_0) = Z_0$ satisfies, for any $t \geq t_0$,

$$|Z(t)| \leq Ke^{(\sqrt{l^2 + c} - e^{-ct})t_0}|Z_0|.$$  

Using this result, we will show that the convergence of (12) can be tailored by choosing $k$ to arbitrarily increase the rate of convergence, while keeping the overshoot constant.

**Theorem 1.** Let $\alpha \in (0, 2\sqrt{1-p})$ be fixed. There exists a continuous function $\gamma(k) = \infty$ such that the solution of (12) satisfies

$$|Z(t)| \leq Ke^{-\gamma(k)(t-t_0)}|Z(t_0)|$$

with

$$K \triangleq \left( 1 + \frac{a}{\sqrt{1-p}} \right) \sqrt{1 - \frac{a}{\sqrt{1-p}}}$$

(15)

for any initial condition $t_0, Z(t_0)$ and any $t \geq t_0$.

**Proof.** Consider any fixed value of $t$. We start by studying the frozen-time matrix $A(t)$. Denote

$$\mu \triangleq \sqrt{8(1-p^2)}.$$
Introduce the following (time-varying) matrices

\[
P_1 = \begin{pmatrix}
a & 0 & \frac{b - pa}{\sqrt{2(1 - p^2)}} \\
0 & b & \frac{a - pb}{\sqrt{2(1 - p^2)}} \\
0 & 0 & 0
\end{pmatrix}
\]

\[
P_2 = \frac{1}{\mu} \begin{pmatrix}
2(pa - b) & 0 & 0 \\
2(a - pb) & -\sqrt{8} & a \times b \\
a \times b & 0 & 0
\end{pmatrix}
\]

\[
P_3 = \frac{1}{\mu} \begin{pmatrix}
2a \times b & 0 & 0 \\
2a \times b & \sqrt{4(1 + p) - a^2} & a - b \\
a(b - a) & 0 & 0
\end{pmatrix}
\]

\[
P_4 = \frac{1}{\mu} \begin{pmatrix}
2a \times b & 0 & 0 \\
2a \times b & -\sqrt{4(1 - p) - a^2} & a - b \\
a\alpha(a - b) & 0 & 0
\end{pmatrix}
\]

and

\[
P = (P_1 P_2 P_3 P_4) \in \mathbb{R}^{4 \times 9}.
\]

We have

\[
p^{-1}A(t) p = \begin{pmatrix}
M_1 & 0 & 0 & 0 \\
0 & M_2 & 0 & 0 \\
0 & 0 & M_3 & 0 \\
0 & 0 & 0 & M_4
\end{pmatrix}
\]

with

\[
M_i = -\alpha I, \quad M_i = \frac{1}{2} \begin{pmatrix}
-\alpha & \sqrt{\alpha - \alpha^2} & -\alpha \\
\sqrt{\alpha - \alpha^2} & -\alpha & \sqrt{\alpha - \alpha^2} \\
-\alpha & \sqrt{\alpha - \alpha^2} & -\alpha
\end{pmatrix}
\]

for \( i = 2, 3, 4 \) with

\[
\alpha_2 \pm 2\sqrt{2} > \alpha_3 \pm 2\sqrt{1 + p} \geq \alpha_4 \pm 2\sqrt{1 - p} > \alpha.
\]

For all \( s \geq 0 \)

\[
\|e^{A(t)s}\| \leq \|p\| \|p^{-1}\| e^{-\frac{\alpha}{2} s}.
\]

Moreover

\[
\|p\| \|p^{-1}\| = \sqrt{\frac{\lambda_{\text{max}}(p^T p)}{\lambda_{\text{min}}(p^T p)}}
\]

where \( \lambda_{\text{max}}(\cdot) \), \( \lambda_{\text{min}}(\cdot) \) respectively designate the maximum and minimum eigenvalues. Besides,

\[
p^T p = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & Q_2 & 0 & 0 \\
0 & 0 & Q_3 & 0 \\
0 & 0 & 0 & Q_4
\end{pmatrix}
\]

with, for \( i = 2, 3, 4 \)

\[
Q_i = \begin{pmatrix}
1 + \frac{\alpha^2}{\alpha_i^2} & \frac{\alpha}{\alpha_i} \sqrt{1 - \frac{\alpha}{\alpha_i}} \\
\alpha \sqrt{1 - \frac{\alpha}{\alpha_i}} & 1 - \frac{\alpha^2}{\alpha_i}
\end{pmatrix}
\]

yielding the eigenvalues

\[
eig(p^T p) = \left\{ 1, 1 \pm \frac{\alpha}{2\sqrt{1 + p}}, 1 \pm \frac{\alpha}{2\sqrt{1 - p}} \right\}.
\]

Thus, for all \( s \geq 0 \)

\[
\|e^{A(t)s}\| \leq Ke^{-\frac{\alpha}{2} s}.
\]
We can now state the main result of the paper.

**Theorem 2 (Solution to Problem 1).** For any \( \alpha \in (0, 2\sqrt{1 - \bar{p}}) \), there exists \( k' \) defined by (19) such that for \( k > k' \), the observer (6) defines an error dynamics (8) for which the equilibrium 0 is locally uniformly exponentially stable. The basin of attraction of this equilibrium contains the ellipsoid

\[
\left\{ \dot{X}(t_0), \quad \lfloor \tilde{a}(t_0) \rfloor^2 + \lfloor \tilde{b}(t_0) \rfloor^2 + \lfloor \tilde{\omega}(t_0) \rfloor^2 < r(k)^2 \right\}
\]

(20)

where \( r(k) \) is defined by (18).

**Proof.** Let \( k > k' \). Consider the candidate Lyapunov function

\[
V(t, Z) \triangleq Z^T \left( \int_t^{\infty} \phi(t, \tau)^T \phi(t, \tau) d\tau \right) Z
\]

where \( \phi \) is the transition matrix of system (12). Let \( (t, Z) \) be fixed. From Proposition 1, \( kA(\cdot) \) is bounded by \( kA_{\text{max}} \). Thus (see for example Khalil, 2000, Theorem 4.12)

\[
V(t, Z) \geq \frac{1}{2kA_{\text{max}}} |Z|^2 \triangleq c_1 |Z|^2 \triangleq W_1(Z).
\]

Moreover, Theorem 1 implies that for all \( \tau \geq t \)

\[
|\phi(t, \tau) Z| \leq K e^{-r(k)(\tau-t)} |Z|
\]

which gives

\[
V(t, Z) \leq K^2 \int_t^{\infty} e^{-2r(k)(\tau-t)} |Z|^2 \leq \frac{K^2}{2\gamma(k)} |Z|^2
\]

\[
\triangleq c_2 |Z|^2 \triangleq W_2(Z).
\]

By construction, \( V \) satisfies

\[
\frac{dV}{dt}(t, Z) + \frac{\partial V}{\partial Z}(t, Z) kA(t) Z = -|Z|^2.
\]

Hence, the derivative of \( V \) along the trajectories of (10) is

\[
\frac{d}{dt} V(t, Z) = -|Z|^2 + \frac{\partial V}{\partial Z}(t, Z) \xi.
\]

Using

\[
\left| \frac{\partial V}{\partial Z}(t, Z) \xi \right| \leq \frac{K^2}{\gamma(k)} \left( \sqrt{2} \omega_{\text{max}} |Z|^2 + k |Z|^2 \right).
\]

Hence

\[
\frac{d}{dt} V(t, Z) \leq -|Z|^2 \left( 1 - \frac{K^2}{\gamma(k)} \sqrt{2} \omega_{\text{max}} |Z|^2 + \frac{kk^2}{2\gamma(k)} |Z|^2 \right)
\]

\[
\triangleq -W_3(Z).
\]

As \( k > k' \), we have \( 1 - \frac{K^2}{\gamma(k)} \sqrt{2} \omega_{\text{max}} |Z|^2 > 0 \). We proceed as in Khalil (2000, Theorem 4.9). If the initial condition of (10) satisfies \( |Z(t_0)| < r(k) \), or equivalently

\[
|Z(t_0)| < \frac{\gamma(k)}{kk^2} \left( 1 - \frac{K^2}{\gamma(k)} \sqrt{2} \omega_{\text{max}} \right) \frac{1}{\sqrt{c_1}} \frac{1}{c_2}
\]

then \( W_3(Z(t_0)) > 0 \) and, while \( W_3(Z(t)) > 0 \), \( Z(t) \) remains bounded by

\[
|Z(t)|^2 \leq \frac{V(t)}{c_1} \leq \frac{V(t_0)}{c_1} \leq \frac{c_2}{c_1} |Z(t_0)|^2
\]

which shows that

\[
W_3(Z) \geq \left( 1 - \frac{K^2}{\gamma(k)} \sqrt{2} \omega_{\text{max}} \frac{c_2}{c_1} |Z(t_0)| \right) |Z|^2
\]

\[
\triangleq c_3 |Z|^2
\]

so that

\[
\frac{d}{dt} V(t, Z(t)) \leq -c_3 |Z(t)|^2 \leq -\frac{c_2}{c_1} V(t, Z(t))
\]

and in the end

\[
|Z(t)|^2 \leq \frac{1}{c_1} e^{-\frac{c_2}{c_1} t} |Z(t_0)|^2 \leq \frac{c_2}{c_1} e^{-\frac{c_2}{c_1} t} |Z(t_0)|^2.
\]

This shows that the 0 equilibrium of (10) is locally uniformly exponentially stable and that its basin of attraction contains the ball of equation

\[
|Z(t_0)|^2 < r(k)^2.
\]

Using (9) to go back to the \( \tilde{a}, \tilde{b}, \tilde{\omega} \) coordinates allows one to deduce that the basin of attraction contains the ellipsoid (20).

**Remark 3.** The limitations imposed on \( \tilde{a}(t_0) \) and \( \tilde{b}(t_0) \) in (20) are not truly restrictive, as the actual values \( \alpha(t_0), \beta(t_0) \) are assumed known, so the observer may be initialized with \( \tilde{a}(t_0) = 0, \tilde{b}(t_0) = 0 \). What matters is that the error on the unknown quantity \( \omega(t_0) \) can be large in practice. Interestingly, when \( k \) goes to infinity \( r(k) \) tends to the limit \( \frac{1}{\sqrt{A_{\text{max}}}} \left( \omega_\text{max} \right)^{\frac{2}{3}} > 0 \) and arbitrarily large initial error \( \tilde{\omega}(t_0) \) is thus allowed from (20).

**Remark 4.** The threshold \( k' \) depends linearly on \( \omega_{\text{max}} \), which gives helpful hint in the tuning of observer (6).

**Remark 5.** As \( \frac{c_2}{c_1} \) goes to infinity with \( k \), the convergence rate is arbitrary fast for \( k \) large enough.

### 4. Simulation results

In the simulations reported here, we consider that the rigid body is a parallelepiped of size \( 10 \times 10 \times 20 \) [cm\(^3\)] having a homogeneously distributed mass of 2 [kg]. The resulting moments of inertia are

\[
J_1 = 88 \quad [\text{kg cm}^2], \quad J_2 = 88 \quad [\text{kg cm}^2], \quad J_3 = 33 \quad [\text{kg cm}^2].
\]

No torque is applied to the system, which is thus in free-rotation.

#### 4.1. Sensitivity to parameters

We illustrate the dependence of the observer with respect to three parameters: (i) \( p \) which quantifies the linear independence of \( (\tilde{a}, \tilde{b}) \), (ii) \( \omega_{\text{max}} \), the maximal rotation rate of the rigid body, (iii) the tuning gain \( k \). To this end, we chose the reference vectors as

\[
\tilde{a} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T, \quad \tilde{b} = \begin{pmatrix} p & \sqrt{1 - p^2} & 0 \end{pmatrix}^T.
\]

For sake of accuracy in the implementation, reference dynamics (5) and state observer (6) were simulated using Runge–Kutta 4 method with sample period 0.1 [s] for various values of \( p \) and \( \omega(0) \) and with \( \alpha = \sqrt{T - \bar{p}} \). Additive white Gaussian noise with standard deviation \( \sigma_\alpha = 0.025 \) [Hz\(^{-\frac{1}{2}}\)] (respectively \( \sigma_\beta = 0.02 \) [Hz\(^{-\frac{1}{2}}\)]) was added to each measurement coordinate of \( a \) (respectively \( b \)). Typical measurements are given in Fig. 1.
Fig. 1. Vector measurements: a (top, three coordinates) and b (bottom, three coordinates).

Fig. 2. $k = 0.25, \omega_{\text{max}} = 6 [\degree/s]$. The rate of convergence degrades when $p$ increases.

Fig. 3. $k = 0.25, p = 0.2$. Impact of $\omega_{\text{max}}$ on the convergence rate.

Fig. 4. $\omega_{\text{max}} = 62 [\degree/s], p = 0.2$. When $k$ increases, the convergence is faster but the measurement noise filtering degrades.

Fig. 5. Variations of the normalized reference magnetic field along the orbit.

4.2. Orbital trajectory

We now challenge Assumption 1. We investigate the behavior of the observer in the case where $b$ is time-varying, which might happen in real-life applications. In the following, the rigid body is a satellite in orbit. It is rotating about its center of gravity which trajectory is an orbit. The two reference vectors are the Sun direction $\hat{a}$ and the normalized magnetic field $\hat{b}$. The satellite is equipped with 6 Sun sensors providing at all times a measure of the Sun direction $y_a$ in a Sun sensor frame $R_s$, and 3 magnetometers able to measure the normalized magnetic field $y_b$ in a magnetometer frame $R_m$. It shall be noted that the sensor frames $R_s$ need not coincide with $R_m$ and can also differ from the body frame $R_b$ (defined along the principal axes of inertia) through a constant rotation $R_{m,b}$, respectively $R_{s,b}$. With these notations, we have

$$a = R_{m,b}^T y_a, \quad b = R_{s,b}^T y_b$$

which is a simple change of coordinates of the measurements.

We simulate 50 min (approx. half a period) of the trajectory of the satellite on a circular orbit with altitude 765 [km] and
inclination $30^\circ$ with respect to a polar orbit. Leaving out the
consideration of Earth shadowing (which does not occur for this
half orbit), the Sun direction $\hat{a}$ is considered as constant. In the
simulation its value is

$$\hat{a} = (0.3977 \quad 0.3445 \quad 0.1989)^T$$

This is not the case for the normalized magnetic field $\hat{b}$ which varies
significantly with the latitude. As a result, the state model (5) is not
valid. To accurately simulate the values of $b(t)$, we compute the full
rotation dynamics (1)–(2) and use the measurements Eq. (3). $\hat{b}$ is
determined by the position of the satellite using the IGRF12 model.
Its values are reported in Fig. 5. Additive white Gaussian noise with
standard deviation $0.02$ [Hz$^{-2}$] was added to each coordinate of
$a$, $b$. Again, we use a Runge–Kutta 4 method with sampling period
$0.1$ [s]. The initial condition on $\omega$ is

$$\omega(0) = (0.5 - 2.5)^T \text{[°/s]}.$$  

The performance of the observer is represented in Fig. 6. An
artificial reset is performed every $300$ [s] to stress the convergence
properties of the error. As can be expected the convergence rate
and the peaking depend on the initial value, hence on the value of
$b$. However, the variations of these convergence parameters
remain small. A residual error of about $0.3$ [°/s] remains on each
coordinate of $\omega - \hat{\omega}$. This is partly due to measurement noise and
partly to the fact that $\hat{b}$ varies with time. In this simulation, the
contribution of these two phenomena to the error are of similar
magnitude. With less noisy sensors, it would be possible to increase
the gain $k$ further to reduce the sum of these two contributions to
the error.

5. Conclusions and perspectives

A new method to estimate the angular velocity of a rigid
body has been proposed in this article. The method uses on-
board measurements of constant and independent vectors. The
estimation algorithm is a nonlinear observer which is very simple
to implement and induces a very limited computational burden.
At this stage, an interesting (but still preliminary) conclusion is
that, in the cases considered here, rate gyro could be replaced
with an estimation software employing cheap, rugged and resilient
sensors. In fact, any set of sensors producing vector measurements
such as e.g., Sun sensors, magnetometers, could constitute one
such alternative. Assessing the feasibility of this approach requires
further investigations including experiments.

More generally, this observer should be considered as a first
element of a class of estimation methods which can be developed
to address several cases of practical interest. In particular, the
introduction of noise in the measurement and uncertainty on the
input torque (assumed here to be known) will require extensions
such as optimal filtering to treat more general cases. White or
colored noises will be good candidates to model these elements.
Also, slow variations of the reference vectors $\hat{a}$, $\hat{b}$ should deserve
particular care, because such drifts naturally appear in some cases.
For example, Earth’s magnetic field measured on-board satellites
varies according to the position along the orbit.

On the other hand, one can also consider that this method can
be useful for other estimation tasks. Among the possibilities are
the estimation of the inertia $J$ matrix which we believe is possible
from the measurements considered here. This could be of interest
for the recently considered task of space debris removal (Bonnal,
Ruault, & Desjean, 2013).

Finally, recent attitude estimation techniques have favored the
use of vector measurements together with rate gyro measurements
as inputs. Among these approaches, one can find Extended Kalman
Filters (EKF)-like algorithms e.g. Choukroun, Bar-Itzhack, and
Oshman (2006), Choukroun, Weiss, Bar-Itzhack, and Oshman
(2012) and Schmidt, Ravandoor, Kurz, Busch, and Schilling (2008),
nonlinear observers (Grip, Fossen, Johansen, & Saberi, 2012; Mah-
only, Hamel, & Pflimlin, 2008; Martin & Salaün, 2010; Taiyebi,
Roberts, & Benallegue, 2011; Trumpf, Mahony, Hamel, & Lageman,
2012; Vasconcelos, Silvestre, & Oliveira, 2008), Wahba-based fil-
tering, unscented Kalman filtering, particle filtering. This contribu-
tion suggests that, here also, the rate gyro could be replaced with
more in-depth analysis of the vector measurements.

References

determination. In Proceedings of the national technical meeting of the institute
of navigation (pp. 699–706).


control in the presence of sensor bias. In Proceedings of the IEEE aerospace
conference (pp. 505–511).


estimation from vector observations using a matrix Kalman filter. IEEE
Transactions on Aerospace and Electronic Systems, 48(4).


biased gyro and vector measurements with time-varying reference vectors.

L. Magnis, N. Petit / Automatica 75 (2017) 46–53


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