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A constructive interior penalty method for optimal control problems with state and input constraints.

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Abstract— **This paper exposes a methodology which allows us to address constrained optimal control of non linear systems by interior penalty methods. A constructive choice for the penalty functions that are introduced to account for the constraints is established in the article. It is shown that it allows us to approach the solution of the non linear optimal control problem using a sequence of unconstrained problems, whose solutions are readily characterized by the simple calculus of variations. An illustrative example is given. The paper extends recent contributions, originally focused on linear dynamics.**

I. INTRODUCTION

This paper exposes a methodology allowing us to solve a constrained optimal control problem (COCP) for a general single-input single-output (SISO) with non linear dynamics. This methodology belongs to the class of *interior point methods* (IPMs) which consists in approaching the optimum by a path lying strictly inside the constraints. In the interior, optimality conditions are much easier to characterize and to make explicit. A penalty function approach commonly considered in finite dimensional optimization problem is employed.

An augmented performance index is generally considered in penalty methods for both finite optimization problem and optimal control problem. It is constructed as the sum of the original cost function and so-called *penalty functions* that have some diverging asymptotic behavior when the constraints are approached by any tentative solution. This augmented performance index can then be optimized in the absence of constraints, yielding a biased estimate of the solution of the original problem. The weight of the penalty functions is gradually reduced to provide a converging sequence, hopefully diminishing the bias.

The penalty function methods are computationally appealing, as they yield *unconstrained* problems for which a vast range of highly effective algorithms are available. In finite dimensional optimization, outstanding algorithms have resulted from the careful analysis of the choice of penalty functions and the sequence of weights. In particular, the *interior points methods* which are nowadays implemented in successful software packages such as KNITRO [1], OOQP [2] have their foundations in these approaches. We refer the interested reader to [3] for a historical perspective on this topic. In this article, we apply similar penalty methods to solve COCPs. COCPs represent a very handy formulation of objectives in numerous applications, especially because constraints are very natural in problems of engineering interest. Unfortunately, these constraints induce some serious difficulties [4], [5], [6]. In particular, it is a well known fact [6] that constraints bearing on state variables are difficult to characterize, as they generate both constrained and unconstrained arcs along the optimal trajectory. To determine optimality conditions, it is usually necessary to know or to a-priori postulate the sequence and the nature of the arcs constituting the desired optimal trajectory. Active or inactive parts of the trajectory split the optimality system in as many coupled subsets of algebraic and differential equations. Yet, not much is known on this sequence, and this often results in a high complexity. Therefore, it is often preferred to use a discretization based approach to this problem, and to treat it, e.g. through a collocation method [7], as a finite dimensional problem [8], [9], [10], [11], [12], [13], [14]. In this context, IPMs have been applied to optimal control problems by Wright [15], Vicente [16], Leibfritz and Sachs [17], Jockenhövel, Biegler and Wächter [18]. This is not the path that we explore, as we wish to use indirect methods (a.k.a. adjoint methods) to take advantage of their accuracy as we can approach the continuous time solution as precisely as we want through a mesh refinement.

Although there is a well-established literature on the mathematical foundations of IPMs for finite-dimensional mathematical programming [19], this is not yet the case for optimal control problems. These methods are of particular interest since each solution of the sequence of optimal control problem is easily computed using classical stationarity conditions of the solution. The main difficulty is to guarantee that the sequence of solution is strictly interior. This point is critical since interiority is a requirement to avoid ill-posedness and computational failure of implemented algorithms. The problem of interiority in infinite dimensional optimization has been addressed in [20] for input-constrained optimal control, and in [21] for a state and input constrained optimal control problem with linear time varying dynamics. Both contributions provide penalty functions guaranteeing the interiority of the solutions. As shown in [21], a constructive choice of the penalty functions for *linear* systems guarantees that the state constraint is *strictly* satisfied. Moreover, depending on the behavior of the control in the vicinity of the saturation, the control constraint can be guaranteed to be also *strictly* satisfied. The purpose of this article is to generalize the results obtained in the case of linear systems [21] to *non linear* dynamics. A new element of proof is introduced

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to circumvent the impossibility to build explicit control deviations to desaturate a constrained trajectory (whereas it is possible in both [20], [21]). Considering converging sequences and elementary topological properties of well-chosen subspaces is the main tool of Section IV. This new view-point impacts the proofs but not the spirit of the results formulated in [21]. The algorithm of [21] is thus relevant here.

This paper is organized as follows: in Section II, the COCP is presented together with two penalized optimal control problems (POCP): a state and input constrained one, and an input constrained one, respectively POCP1 and POCP2. POCP2 is the easiest to solve. We give sufficient conditions for these two POCPs to be equivalent. In Section III a sufficient condition on the state penalty is derived such that this condition holds. In Section IV, a sufficient condition on the control penalty is given such that the second condition holds as well. In Section V, a constructive choice of the penalty is given such that the two aforementioned conditions hold and a completely unconstrained algorithm converging to the solution of the COCP is given. The proposed algorithm is tested on an illustrative example in Section VI. Conclusions and perspectives are given in Section VII.

II. Notations, problem statement and penalty method.

A. Constrained optimal control problem and notations

In this article, we investigate the following state and input constrained COCP

$$\min_{u\in U^{\mathrm{ad}}}\left[J(x^u,u)=\int_0^T \ell(x^u,u)dt\right] \tag{1}$$

where $\ell : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$ is a Lipschtiz function of its arguments with Λ a Lipschitz constant, $x^u(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}$ are the state and the control of the following SISO non linear dynamics

$$\dot{x} \;=\; f(x,u),\;\; x(0)=x_0 \tag{2}$$

Further, over the time interval $[0,T]$, $T>0$ given, it is assumed that f is C^1 and that there exists a constant $0<C<+\infty$ such that the following inequality holds:

$$\| f(x,u) \| \leq C(1+\| x \|),\;\forall x, \forall |u|\leq 1 \tag{3}$$

This assumption allows one to guarantee that finite time trajectories remain bounded. The control u is constrained to belong to the following set

$$\mathcal{U}=\{u \text{ s.t. } |u(t)|\leq 1 \text{ a.e. } t\in[0,T]\} \tag{4}$$

which is the unit closed ball of Lebesgue essentially bounded measurable functions $[0,T]\mapsto\mathbb{R}$. The set U^{ad} in (1) is the following

$$U^{\mathrm{ad}} \triangleq \{u\in\mathcal{U} \text{ s.t. } g^- \leq g(x^u(t)) \leq g^+, \forall t\in[0,T]\} \tag{5}$$

where $g:\mathbb{R}^n\mapsto\mathbb{R}$ is assumed to be of class C^1. The state constraints are given in equation (5). Let us define the functions $G:\mathcal{U}\times[0,T]\mapsto\mathbb{R}$ and $\psi:\mathcal{U}\mapsto\mathbb{R}$ as follows

$$G(u,t) \triangleq \max\left\{g(x^u(t))-g^+, g^- - g(x^u(t))\right\} \tag{6}$$

$$\psi(u) \triangleq \sup_{t\in[0,T]} G(u,t) \tag{7}$$

From this definition, the function ψ is such that

$$u\in U^{\mathrm{ad}} \Leftrightarrow \psi(u)\leq 0 \tag{8}$$

For the analysis developed in the rest of the paper, we define two useful subsets of U^{ad}

$$V^{\mathrm{ad}} \triangleq \{u\in\mathcal{U} \text{ s.t. } g^- < g(x^u(t)) < g^+, \forall t\} \tag{9}$$

$$W^{\mathrm{ad}} \triangleq \{u\in\overset{\circ}{\mathcal{U}} \text{ s.t. } g^- < g(x^u(t)) < g^+, \forall t\} \tag{10}$$

where $\overset{\circ}{\mathcal{U}}$ denotes the interior of $\mathcal{U}$ w.r.t. the L^∞ norm. In the following we consider that the set V^{ad} is not empty. These sets satisfy

$$W^{\mathrm{ad}}\subset V^{\mathrm{ad}}\subset U^{\mathrm{ad}}$$

B. Presentation of the penalized problems

Following the approach of interior methods in their application to optimal control [20], we introduce two penalty functions

$$\gamma_g(.):\quad [g^-,g^+] \;\to [0,+\infty)$$
$$\gamma_u(.):\quad [-1,1] \;\to [0,+\infty)$$

which are assumed to be strictly convex, symmetric, and go to infinity as their argument approaches one of the bounds of the definition interval. These functions serve to define the following POCPs

1) POCP1: note $\epsilon>0$, solve:

$$\min_{u\in U^{\mathrm{ad}}}\left[K(u,\epsilon)=\int_0^T \ell(x^u,u)+\epsilon\left[\gamma_g\circ g(x^u)+\gamma_u(u)\right]dt\right] \tag{11}$$

under the dynamics (2). At this stage, not much has been gained since the POCP1 is just as difficult to solve as the COCP (1). The main difficulty is the state constraint. This is a well-known fact in optimal control, as discussed in the introduction, stemming from the difficulty to handle the calculus of variations in this case. Interestingly, this point can be alleviated as will be shown.

2) POCP2: note $\epsilon>0$, solve:

$$\min_{u\in\overset{\circ}{\mathcal{U}}}\left[K(u,\epsilon)=\int_0^T \ell(x^u,u)+\epsilon\left[\gamma_g\circ g(x^u)+\gamma_u(u)\right]dt\right] \tag{12}$$

under the dynamics (2).

C. Sufficient conditions for equivalence of POCPs.

In fact, POCP1 and POCP2 are not equivalent. In POCP1 the control is constrained to belong to U^{ad}, while, on the other hand, in POCP2 it belongs to $\overset{\circ}{\mathcal{U}}$. In more details, the output constraint used to define U^{ad} is not present in the formulation of POCP2. This is precisely what makes (12) appealing as it is easy to solve. In the following, we wish to show that, provided γ_g and γ_u are suitably chosen, these two problems have the same solution. To establish this point, we introduce two preliminary assumptions and prove a result that establishes the relation between POCP1 and POCP2.

Assumption 1 (existence, uniqueness): There exists an unique global solution u^* for POCP1.

This assumption can be easily satisfied by adding a strong convexity assumption on the cost (1) and linearity of the dynamics with respect to u. Under Assumption 1 one obtains the following lemma.

Proposition 1: Assume that the following holds

(C1) For any $u \in \mathcal{U} \setminus V^{\text{ad}}$, $K(u, \epsilon) = +\infty$ for all $\epsilon > 0$,

(C2) For all $\epsilon > 0$, for any $u_1 \in V^{\text{ad}} \setminus W^{\text{ad}}$ there exists $u_2 \in W^{\text{ad}}$ such that $K(u_1, \epsilon) > K(u_2, \epsilon)$,

then, there exists a unique solution $u^\sharp$ for POCP2 and one has

$$u^\sharp = u^*$$

Proof: Condition (C1) implies that

$$\min_{u \in U^{\text{ad}}} K(u, \epsilon) = \min_{u \in U^{\text{ad}} \setminus V^{\text{ad}} \cup V^{\text{ad}}} K(u, \epsilon) = \min_{u \in V^{\text{ad}}} K(u, \epsilon)$$

and

$$\min_{u \in V^{\text{ad}}} K(u, \epsilon) = \min_{u \in V^{\text{ad}} \cup (\mathring{\mathcal{U}} \setminus V^{\text{ad}})} K(u, \epsilon) = \min_{u \in \mathring{\mathcal{U}}} K(u, \epsilon)$$

which shows the existence of a solution to POCP2. Then, using condition (C2) one has

$$\min_{u \in \mathring{\mathcal{U}}} K(u, \epsilon) = \min_{u \in U^{\text{ad}}} K(u, \epsilon) = \min_{u \in W^{\text{ad}}} K(u, \epsilon) \tag{13}$$

To conclude, one now has to prove uniqueness. Let us consider an optimal control $u^\sharp$ for POCP2. From (C1), this control belongs to $\mathring{\mathcal{U}} \cap V^{\text{ad}} = W^{\text{ad}}$. Then it is admissible for POCP1 and is such that $K(u^\sharp, \epsilon) = K(u^*, \epsilon)$ from (13). By uniqueness of u^*, one has $u^* = u^\sharp$. ∎

III. Interiority of the optimal constrained state

In this section, we study how the penalty function $\gamma_g(.)$ can be used to guarantee that Condition (C1). To do so, we recall the following result

Lemma 1 ([21]): In POCP1, if the penalty function γ_g is such that

$$\lim_{\alpha \downarrow 0} \gamma_g(g^+ - \alpha)\mu_g(\alpha) = +\infty \tag{14}$$

where

$$\mu_g(\alpha) \triangleq \text{meas}\left(\{t \text{ s.t. } G(u, t) \geq -\alpha\}\right) \tag{15}$$

with meas(.) is the Lebesgue measure of its argument, then (C1) holds.

Since the measure μ_g appears in equation (14), it is handy to give a lower bound on it. This will be used in Section V, in the explicit construction of suitable penalty functions. A lower bound is given by the following result.

Lemma 2: Considering an input $u \in \mathcal{U}$, and assuming that $\psi(u) = 0$. Then, there exists a constant $K < +\infty$ such that the measure $\mu_g(\alpha)$ defined in equation (15) is lower-bounded under the form

$$\mu_g(\alpha) \geq \frac{\alpha}{K} \tag{16}$$

Proof: The proof is given in Appendix A together with the expression of K. ∎

Using Lemmas 1 and 2, one finally obtains

Proposition 2: If the state penalty γ_g is such that

$$\lim_{\alpha \downarrow 0} \alpha \gamma_g(g^+ - \alpha) = +\infty \tag{17}$$

then Condition (C1) holds.

IV. Interiority of the optimal constrained control

In this section, we determine sufficient conditions on the penalty functions $\gamma_u(.)$ and $\gamma_y(.)$ such that Condition (C2) holds. Consider $u_1 \in V^{\text{ad}} \setminus W^{\text{ad}}$, it serves to build another control $u_2 \in W^{\text{ad}}$ by using a density argument detailed below. This density allows us to approach any control in V^{ad} by a sequence of controls in W^{ad}. Section IV-B exposes the existence of a control $u_2 \in W^{\text{ad}}$ arbitrary close of u_1 in the L^∞ sense. In Section IV-C, the conditions on the penalties are exhibited and the main result is given in Proposition 5.

A. Density of W^{ad} in V^{ad}

The main purpose of this section is to prove that the control sets V^{ad} and W^{ad} have the same closure in the L^∞ sense (Proposition 3).

Proposition 3: The sets V^{ad} and W^{ad} satisfy

$$\overline{W^{\text{ad}}} = \overline{V^{\text{ad}}}$$

Proof: First, $W^{\text{ad}} \subset V^{\text{ad}}$, thus $\overline{W^{\text{ad}}} \subseteq \overline{V^{\text{ad}}}$. Now let us prove the inverse inclusion. Consider any $v \in V^{\text{ad}} \setminus W^{\text{ad}}$. Define $-\beta \triangleq \psi(v) < 0$. One can build a sequence $(u_n)_{n \in \mathbb{R}}$ such that $u_n = (1 - \epsilon_n)v$, where $(\epsilon_n)_{n \in \mathbb{N}}$ is a sequence converging to 0, with $\epsilon_n > 0$. The sequence $(u_n)_{n \in \mathbb{N}}$ converges to v in the topology of L^∞. From equation (3) and using Grönwall Lemma [22], $\parallel x^u \parallel$ is bounded for all $u \in \mathcal{U}$, moreover $f(.,.)$ being C^1 this implies that f is Lipschitz with respect to its arguments. Thus $\parallel \dot{x}^{u_n}(t) - \dot{x}^v(t) \parallel \leq \lambda(\parallel x^{u_n}(t) - x^v(t) \parallel + \parallel u_n(t) - v(t) \parallel)$, $\lambda < +\infty$. Using Grönwall Lemma, there exists $K < \infty$ such that $\parallel x^{u_n} - x^v \parallel_{L^\infty} \leq K \parallel u_n - v \parallel_{L^1}$. Thus, if u_n converges to v in the L^∞ sense, it converges in the L^1 sense and x^{u_n} uniformly converges to x^v. Using the continuity of g, the sequence $(g(x^{u_n}))_{n \in \mathbb{N}}$ uniformly converges to $g(x^v)$. Then, there exists N such that $\forall n > N$, $\parallel g(x^{u_n}) - g(x^v) \parallel_{L^\infty} < \frac{\beta}{2}$. Then, the sequence $(u_n)_{n > N}$ belongs to W^{ad}. Therefore, v is an adherent point to W^{ad} and $V^{\text{ad}} \subseteq \overline{W^{\text{ad}}}$. Eventually, this yields $\overline{W^{\text{ad}}} = \overline{V^{\text{ad}}}$. ∎

B. Construction of u_2

Let us consider any control $u_1 \in V^{\text{ad}} \setminus W^{\text{ad}}$ and note $\psi(u_1) = -2\beta_0 \leq 0$. From Proposition 3 we have the following existence result: there exists $u_2 \in W^{\text{ad}}$ and $\alpha > 0$ such that

$$\parallel u_2 \parallel_{L^\infty} = 1 - \alpha, \ \ \parallel u_1 - u_2 \parallel_{L^\infty} \leq \alpha, \ \ \psi(u_2) \leq -\beta_0 \tag{18}$$

C. Condition guaranteeing the strict interiority of the optimal trajectory

The following result gives an upper estimate on the difference $K(u_2,\epsilon) - K(u_1,\epsilon)$. This estimate is the sum of three terms, representing respectively

(i) the integral variation of the original cost (1)
(ii) the integral variation of the state penalty $\epsilon\gamma_g \circ g$
(iii) the integral variation of the input penalty $\epsilon\gamma_u$

Proposition 4: For any u_2 satisfying (18), for any $\epsilon > 0$ one has

$$K(u_2,\epsilon) - K(u_1,\epsilon) \le \alpha \left[U_\ell + U_g(\epsilon) - L(\epsilon,\alpha)\right] \quad (19)$$

with

$$\begin{aligned} U_\ell &\triangleq \Lambda T\left[K_E + 1\right] \\ U_g(\epsilon) &\triangleq \epsilon T K_g K_E \gamma_g'(g^+ - \beta_0) \\ L(\epsilon,\alpha) &\triangleq \epsilon\mu_{u_1}(\alpha)\gamma_u'(1-2\alpha) \end{aligned}$$

where K_E and K_g are positive constant (definied in Appendix B) and, for any measurable function u_1

$$\mu_{u_1}(s) \triangleq \text{meas}\left(\{t \quad \text{s.t.} \quad |u_1| \ge 1 - s\}\right) \quad (20)$$

where meas(.) is the Lebesgue measure of its argument.

Proof: See Appendix B. ■

Finally, using (19), the following result holds.

Proposition 5: If for all $\epsilon > 0$, there exists $\alpha > 0$ such that

$$L(\epsilon,\alpha) > U_\ell + U_g(\epsilon) \quad (21)$$

then

$$K(u_2,\epsilon) < K(u_1,\epsilon), \quad \forall\epsilon > 0$$

and Condition (C2) holds.

V. Main results and algorithm

In Section III and IV, conditions have been given, under the form of Proposition 2 and Proposition 5 respectively, such that the Conditions (C1)-(C2) required in the statement of Proposition 1 hold. These propositions are given under the form of an equation (17) and an inequality (21). In this section, a class of penalty functions γ_g and γ_u are given such that these actually hold.

A. Penalty design

The inequality (21) is now studied. Depending on the nature of the optimal trajectory of (11), the desired strict positivity of $L(\epsilon,.) - U_\ell - U_g(\epsilon)$ stems from the term $L(\epsilon,.)$. Thus, our study requires that an assumption on the behavior on the measure $\mu_{u^*}(.)$ is formulated.

Assumption 2 (touching of input constraint): Define

$$m_{u^*}(\alpha) = \text{meas}\left(\{t \text{ s.t. } |u^*(t)| \le \| u^* \|_{L^\infty} - \alpha\}\right) \quad (22)$$

There exists $M > 0$ and $q \ge 0$ such that the asymptotic behavior close to zero of the measure m_{u^*} defined in equation (22) satisfies:

$$m_{u^*}(\alpha) \ge M\alpha^q \quad (23)$$

We are now ready to state our main result.

Theorem 1 (Main Result): Under Assumptions 1 and 2, there exists penalty functions $\gamma_g(.)$ and $\gamma_u(.)$ such that POCP1 and POCP2 are equivalent: their respective unique solutions are equal. A particular choice of penalty is:

$$\gamma_g(g) = \left[\frac{1}{2}\left(\frac{g^+ - g^-}{\sqrt{(g^+ - g)(g - g^-)}} - 1\right)\right]^{n_g} \quad (24)$$

$$\gamma_u(u) = \left[\frac{1}{2}\left(\frac{2}{\sqrt{1-u^2}} - 1\right)\right]^{n_u} \quad (25)$$

with $n_g > 2$ and $n_u > \max\{1, 2(q-1)\}$, q being given in (23)

Proof: The existence is proven by showing that (24) and (25) are suitable penalties. The penalty (24) is such that equation (17) is satisfied; therefore Condition (C1) holds. Now, let us prove that if the optimal solution u^* of (11) belongs to V^{ad}, then it belongs to W^{ad}. The proof considers two mutually exclusive cases.

- If $\| u^* \|_{L^\infty} < 1$, then $u^* \in W^{\text{ad}}$ which proves (C2).
- If $\| u^* \|_{L^\infty} = 1$, then using equations (20) and (22), one has $m_{u^*} = \mu_{u^*}$. This implies $\gamma_u'(1 - 2\alpha)m_{u^*}(\alpha) = \gamma_u'(1-2\alpha)\mu_{u^*}(\alpha) \ge \gamma_u'(1-2\alpha)M\alpha^q$. The control penalty (25) is such that $\lim_{\alpha\downarrow 0} L(\epsilon,\alpha) \ge \lim_{\alpha\downarrow 0}\gamma_u'(1-2\alpha)M\alpha^q = +\infty$, $U_\ell < +\infty$ and $U_g(\epsilon) < +\infty$. Moreover, γ_u' is a continuous function of α and m_{u^*} is lower bounded by a continuous function of α (see (23)). As a consequence, there always exists $\alpha > 0$ such that Proposition 5 holds. Then (C2) holds and $u^* \in W^{\text{ad}}$. This contradiction shows that this case is impossible and $u^* \in W^{\text{ad}}$.

We have proven that (C1) and (C2) are always satisfied provided that the penalty functions are appropriately chosen. This implies that problems POCP1 and POCP2 are equivalent. ■

B. Investigation of convergence

Theorem 1 allows us to solve POCP2 instead of POCP1. Our ultimate goal is to solve (1), which as announced earlier in Section II, is approached by a sequence of POCP1, or much simpler, thanks to the equivalence of Theorem 1, a sequence of POCP2. One such algorithm is presented below. Now, let us mention a few facts on convergence of the constructed sequence $(u_{\epsilon_n},\epsilon_n)_{n\in\mathbb{N}}$ where $(\epsilon_n)_{n\in\mathbb{N}}$ is a decreasing sequence converging to zero, and $u^*_{\epsilon_n}$ the solution of (12) for $\epsilon = \epsilon_n$. The proof of convergence of the cost $\lim_{n\to+\infty} K(u^*_{\epsilon_n},\epsilon_n) = J^*$ follows along the same lines as the proof in [23] and [24]. To prove the convergence of $u^*_{\epsilon_n}$ an assumption on the strong convexity of J can be used. More details can be found in [24]. The only difference is that we do not need the assumption of the interiority of the solution of the POCP anymore. Nevertheless, the sequence $(u^*_{\epsilon_n})$ of solutions of POCP2 converges to a solution in $\overline{V^{\text{ad}}}$ which can be different from U^{ad}. Then, to ensure that (1) is actually solved, assuming that this problem has a unique solution, one has to ensure that $\overline{V^{\text{ad}}} = U^{\text{ad}}$. A necessary and sufficient condition such that it is true is given in Appendix C.

C. Algorithm

1) Change of variables: First, the following change of variable is used

$$u \triangleq \phi(\nu) = \tanh(k\nu) \tag{26}$$

Where $k \neq 0$ is a factor allowing to set the slope of the function about zero, ν is an unconstrained variable such that $\tanh(k\nu) \in \mathcal{U}$, and such that the corresponding POCP

$$\min_{\nu} \left[P(\nu, \epsilon) = \int_0^T \ell(x, \phi(\nu)) + \epsilon[\gamma_g \circ g(x) + \gamma_u \circ \phi(\nu)]dt \right] \tag{27}$$

is defined with the penalty functions from (24) and (25).

Corollary 1: Under Assumptions 1 and 2, and from Theorem 1, POCP1 and (27) are equivalent in the sense that there exists an optimal solution ν^* of (27) such that

$$u^* = \tanh(k\nu^*)$$

where u^* is the optimal solution of POCP1.

Proof: The proof is exactly the same as Theorem 5 in [21]. ■

2) Solving algorithm: The purpose of the main result of this paper, i.e. Theorem 1 (and Corollary 1 which stems from it), is to allow one to solve a simple OCP (Problem (27)) instead of POCP1 because they are equivalent. Each problem (27) penalized by ϵ from a sequence (ϵ_n) can be solved using the calculus of variations. Define the Hamiltonian of the penalized problem (27) as follows

$$\begin{aligned} H_\epsilon(x, \nu, p) \quad \triangleq \quad & \ell(x, \phi(\nu)) + \epsilon\left[\gamma_g \circ g(x) + \gamma_u \circ \phi(\nu)\right] \\ & + p^T f(x, \phi(\nu)) \end{aligned} \tag{28}$$

where $p \in \mathbb{R}^n$ is the adjoint state of Pontryagin solution of $\frac{dp}{dt} = -\frac{\partial H_\epsilon}{\partial x}$ and where the penalty functions are chosen according to Theorem 1. The choice of n_u can be made by trial and error which solely depend on the nature of the desired (but a-priori unknown) optimal solution u^*. Now, defining a positive decreasing sequence, one can approach the solution of (1).

- Step 1: Initialize the continuous functions $x(t)$ and $p(t)$ such that the initial $g^- < g(x(t)) < g^+$ for all $t \in [0, T]$, and set $\epsilon = \epsilon_0$. Note that $x(t)$ and $p(t)$ need not satisfy any differential equation at this stage, even if it is better if they do.
- Step 2: Solve for each time $\frac{\partial H_\epsilon}{\partial \nu} = 0$, and note ν_ϵ^* the solution.
- Step 3: Solve the $2n$ differential equations $\frac{dx}{dt} = f(x, \phi(\nu_\epsilon^*))$ and $\frac{dp}{dt} = -\frac{\partial H_\epsilon}{\partial x}(x, \nu_\epsilon^*, p)$ forming a two point boundary values problem using bvp4c (see [25]), with the following boundary constraints $x(0) = x_0$ and $p(T) = 0$.
- Step 4: Decrease ϵ, initialize $x(t)$ and $p(t)$ with the solutions found at Step 3 and restart at Step 2.

Convergence of the state in $L^\infty([0, T]; \mathbb{R}^n)$ and convergence of the control in $L^2([0, T]; \mathbb{R})$ for COCP (11) ([24], [23]) can be established as well.

VI. Numerical Example

To illustrate the proposed methodology, we consider the following simple example of COCP with control affine non linear dynamics

$$\ddot{x}(t) = x(t) + x(t)^3 - \dot{x}(t) + 10x(t)^2 u(t) \tag{29}$$

with the constraints $|u(t)| \leq 1$ and $-.05 \leq x(t)^3 + \dot{x}(t)/2 \leq g^+$, with $g^+ = 0.3$ if $1 \leq t \leq 1.5$ and $g^+ = 0.4$ everywhere else. The criterion to minimize is $J(x, u) = \int_0^2 -\frac{x(s)^2}{2} ds$. We set $u = \tanh(\nu/2)$. The state penalty γ_g is chosen according to (24) with $n_g = 2.1 > 2$. Since the cost does not depend on u and since the system is a control-affine system, we chose the derivative of the control penalty $\gamma_u'(.)$ such that $\gamma_u' \circ \tanh(\nu/2) = \sinh(\nu)$. It is convenient (but not required) as it allows one to analytically solve the step 2 of our algorithm. Besides, this choice is such that $\lim_{\alpha \downarrow 0} L(\epsilon, \alpha) = +\infty$ (see (21)) if $\mu_{u^*}(\alpha) > K\alpha$. In our case, u^* is a succession of bang-bang control and constrained arcs, so $\mu_{u^*}(0) > 0$, $q = 0$ and the penalty is well designed since it makes (21) hold. Another equivalent possibility would have been to use the penalty (25) with $n_u = 2$ and to numerically solve the Step 2 of our algorithm. The initial state is $x_0 = (.3, 0)^T$. The algorithm has been initialized with $x(t) = x_0$ and $p(t) \equiv 0$ for all $t \in [0, T]$. The sequence $(\epsilon_n)_{n \in [1,36]}$ is a logarithmic decreasing sequence from 1 to 10^{-7}. By construction, the solver produces a sequence of feasible solutions, that are simple suboptimal with respect to the original cost (1). The optimal cost is $J(x^*, u^*) = -0.34476$.

TABLE I

ITERATIONS

♯ iter	ϵ	Cost
1	1	$J = -0.16312$
6	10^{-1}	$J = -0.18571$
11	10^{-2}	$J = -0.28331$
16	10^{-3}	$J = -0.33055$
21	10^{-4}	$J = -0.34113$
26	10^{-5}	$J = -0.34382$
31	10^{-6}	$J = -0.34456$
36	10^{-7}	$J = -0.34476$

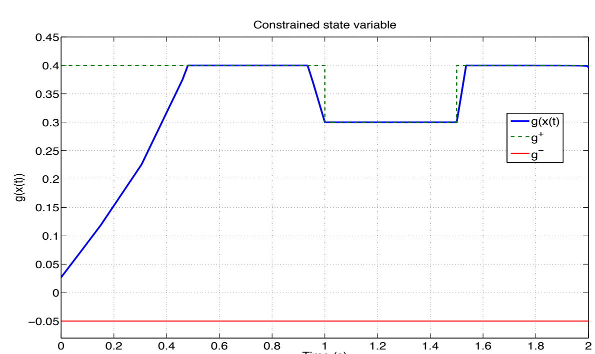

Fig. 1. Optimal state constraint for $\epsilon = 10^{-8}$.

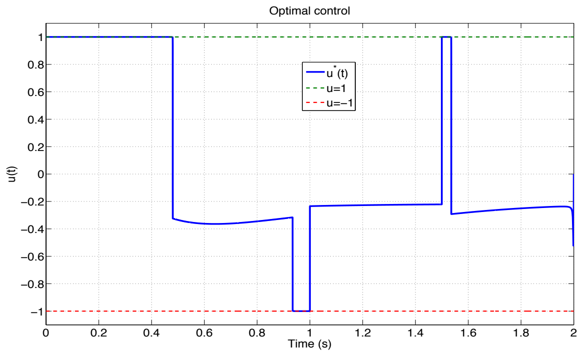

Fig. 2. Optimal control for $\epsilon = 10^{-8}$.

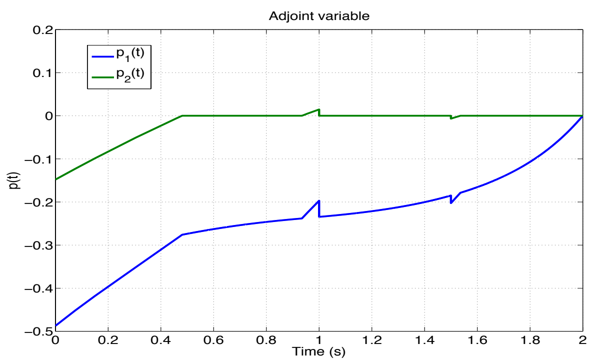

Fig. 3. Adjoint vector $p(t)$ for $\epsilon = 10^{-8}$. One can see that both adjoint variables exhibit discontinuities at some junction points (i.e. at the transition between an unconstrained and a constrained arc).

VII. CONCLUSIONS

As a result of the proposed study, a practical method to solve constrained optimal control problems for non linear systems has been given. It solely requires the mathematical formulation of a suitably penalized OCP. A constructive choice has been given. This unconstrained problem can then be handled using a classic two-point boundary value problem solver. The presented iterative algorithm using an off-the-shelf routine is quite easy to implement and provides satisfactory results.

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APPENDIX

A. Proof of Lemma 2

First, using equation (3) together with Grönwall Lemma, one has $\| x \| \leq e^{CT}(1+ \| x_0 \|) - 1 \triangleq K_T$. Now, let us define:

$$K_x \triangleq \sup_{\|x\| \leq K_T, |u| \leq 1} \| f(x,u) \| \tag{30}$$

$$K_g \triangleq \sup_{\|x\| \leq K_T} \| \frac{\partial g(x)}{\partial x} \| \tag{31}$$

The continuity of f and $\frac{\partial g}{\partial x}$ yields $K_x, K_g < +\infty$. Let us recall that $x(t) - x(s) = \int_s^t f(x(\tau), u(\tau))d\tau$. Now, let us consider that $G(u,s) = g^+ - g(x^u(s)) = \alpha$ and $G(u,t) = g^+ - g(x^u(s)) = 0$. This yields: $g(x(t)) - g(x(s)) = \alpha \leq K_g \| x(t) - x(s) \| \leq K_g K_x (t-s)$. This yields $t - s \geq \alpha(K_x K_g)^{-1}$. Since the measure μ_g cannot be lower than the minimal time needed to reach the constraint g^+ starting from $g(x(s)) = g^+ - \alpha$, we finally obtain: $\mu_g(\alpha) \geq t - s \geq \frac{\alpha}{K_x K_g}$. Note $K \triangleq K_x K_g$. The same argument holds when replacing g^+ by g^-.

B. Proof of Proposition 4

To prove Proposition 4, we need to exhibit an upper bound on $\| x^{u_2} - x^{u_1} \|_{L^\infty}$. From equation (3) and using Grönwall Lemma [22], $\| x^u \|$ is bounded for all $u \in \mathcal{U}$, moreover $f(.,.)$ being C^1 this implies that f is Lipschitz with respect to its arguments. Thus $\| \dot{x}^{u_2}(t) - \dot{x}^{u_1}(t) \| \leq \lambda(\| x^{u_2}(t) - x^{u_1}(t) \| + \| u_2(t) - u_1(t) \|)$, $\lambda < +\infty$. Using Grönwall Lemma, there exists $K < +\infty$ such that $\| x^{u_2} - x^{u_1} \|_{L^\infty} \leq K \| u_2 - u_1 \|_{L^1}$. Noting $u_2 = u_1 + \alpha v$, $v \in \mathcal{U}$, there exists $K_E < +\infty$ such that the following holds

$$\| x^{u_2} - x^{u_1} \|_{L^\infty} \leq K_E \alpha \tag{32}$$

Now, we can prove Proposition 4: let us study the difference $K(u_2, \epsilon) - K(u_1, \epsilon)$ which can be decomposed as follows $K(u_2, \epsilon) - K(u_1, \epsilon) = K^+ + K^-$. Where $K^+ \geq 0$ (resp. $K^- \leq 0$) represents the possible increase (resp. decrease) on the penalized cost (11) when compared to u.

1) An upper bound on the possible increase K^+: To exhibit an upper bound on the possible increase, K^+ is split into two parts itself: the possible increase of the original cost $\int \ell(x,u,t)dt$ and the possible increase due to the state penalty, separately.

a) Possible increase of the original cost: There, an upper bound on the possible increase of $\int_0^T |\ell(x^{u_2}, u_2)| - |\ell(x^{u_1}, u_1)|dt$ is exhibited. Let us call K_ℓ this upper bound. Now, let us consider that the cost function $\int \ell(x,u,t)dt$ is Lipschitz with constant Λ, then from equations (18) and (32), one has

$$\begin{aligned} K_\ell &\leq \Lambda \int_0^T \| x^{u_2} - x^{u_1} \|_{L^\infty} + \| u_2 - u_1 \|_{L^\infty} \, dt \\ &\leq \Lambda T \left[K_E \alpha + \alpha \right] \end{aligned}$$

We define this upper bound as follows:

$$\alpha U_\ell \triangleq \alpha \Lambda T \left[K_E + 1 \right] \tag{33}$$

b) Possible increase due to the state penalty: Note $K_{\gamma_g} \triangleq \epsilon \int_0^T \gamma_g \circ g(x^{u_2}) - \gamma_g \circ g(x^{u_1}) dt$. The integrand is positive when $G(u_2,.) \geq G(u_1,.)$. But, from the construction of u_2 and equation (18), one has $G(u_2,.) \leq -\beta_0$. Using the convexity and symmetry properties of the penalties, and equation (31) one obtains

$$\begin{aligned} K_{\gamma_g} &\leq \epsilon \int_0^T K_g \| x^{u_2} - x^{u_1} \|_{L^\infty} \gamma_g'(g^+ - \beta_0) dt \\ K_{\gamma_g} &\leq \epsilon T K_g K_E \alpha \gamma_g'(g^+ - \beta_0) \end{aligned}$$

We define this upper bound as follows:

$$\alpha U_g(\epsilon) \triangleq \alpha \epsilon T K_g K_E \gamma_g'(g^+ - \beta_0) \tag{34}$$

Finally, using equations (33) and (34), we have:

$$K^+ \leq \alpha \left[U_\ell + U_g(\epsilon) \right] \tag{35}$$

2) A lower bound on the possible decrease K^-: The aim of this part is to exhibit a lower bound on $|K^-|$. Here, we consider that the decrease can only be provided by the control penalty. Let us define $K_u \triangleq \epsilon \int_0^T \gamma_u(u_2) - \gamma_u(u_1) dt$. Equation (18) yields that the integrand of the previous equation is never negative since $|u_2(t)| \leq |u_1(t)|$. Using convexity and symmetry properties of the penalty functions and equation (20) one has

$$\begin{aligned} K^- &\leq \epsilon \int_{|u_1| \geq 1-\alpha} \gamma_u(u_2) - \gamma_u(u_1) dt \\ K^- &\leq -\epsilon \int_{|u_1| \geq 1-\alpha} \| u_2 - u_1 \|_{L^\infty} \gamma_u'(|u_2(t)|) dt \\ K^- &\leq -\epsilon \alpha \gamma_u'(1 - 2\alpha) \mu_{u_1}(\alpha) \end{aligned} \tag{36}$$

We define this lower bound as follows:

$$K^- \leq -\alpha L(\epsilon, \alpha) \triangleq -\alpha \epsilon \gamma_u'(1 - 2\alpha) \mu_{u_1}(\alpha) \tag{37}$$

3) An upper bound on $K(u_2, \epsilon) - K(u_1, \epsilon)$: Gathering equations (35) and (37), one finally obtains

$$K(u_2, \epsilon) - K(u_1, \epsilon) \leq \alpha \left[U_\ell + U_g(\epsilon) - L(\epsilon, \alpha) \right] \tag{38}$$

This concludes the proof of Proposition 4.

C. Well-posedness

Proposition 6: The subset U^{ad} is a closed subset of $\mathcal{U}$.

Proof: Consider a sequence $(u_n)_{n \in \mathbb{N}}$, $u_n \in U^{\mathrm{ad}}$, which converges (uniformly) to $u_f \in \mathcal{U}$: $\lim_{n \to +\infty} \| u_n - u_f \|_{L^\infty} = 0$. the sequence $(g(x^{u_n}))$ uniformly converges to $g(x^{u_f})$. Thus $\lim_{n \to \infty} \psi(u_f) \leq 0$. From (8), $u_f \in U^{\mathrm{ad}}$. This concludes the proof. ■

Definition 1: Considering $\psi : \mathcal{U} \mapsto \mathbb{R}$ defined in (7), then one says that $u_0 \in \mathcal{U}$ is a minimum of ψ if

$$\psi(u_0) = 0 \tag{39}$$

and if there exists a neighborhood $\mathcal{V}$ of u_0 such that for all $v \in \mathcal{V} \cap \mathcal{U}$

$$\psi(v) \geq \psi(u_0) \tag{40}$$

Theorem 2: Consider ψ defined by (7), the following propositions are equivalent

(i) ψ has no minimum.
(ii) $U^{\mathrm{ad}} = \overline{W^{\mathrm{ad}}}$

Proof: First, we need the following Lemma:

Lemma 3: Considering ψ defined in (7), the following propositions are equivalent:

(i) ψ has no minimum (in the sense of Definition 1).
(ii) $U^{\mathrm{ad}} = \overline{V^{\mathrm{ad}}}$

Proof: The proof is inspired by [30]. $(i) \Rightarrow (ii)$. By contraposition: From Proposition 6, the set U^{ad} is closed, so $\overline{V^{\mathrm{ad}}} \subseteq U^{\mathrm{ad}}$. Now, let us suppose $\overline{V^{\mathrm{ad}}} \neq U^{\mathrm{ad}}$, then there exists $u_0 \in U^{\mathrm{ad}} \setminus \overline{V^{\mathrm{ad}}}$. Thus, there exists a neighborhood $\mathcal{V}(u_0)$ such that $\mathcal{U} \cap \mathcal{V}(u_0) \cap \overline{V^{\mathrm{ad}}} = \mathcal{V}(u_0) \cap \overline{V^{\mathrm{ad}}} = \emptyset$. This yields, $\forall u \in \mathcal{U} \cap \mathcal{V}(u_0)$, $\psi(u) \geq \psi(u_0) = 0$. Then u_0 is a minimum.
$(ii) \Rightarrow (i)$. By contraposition: Let u_0 be a minimum of ψ. Then, one has $\psi(u_0) = 0$ and there exists a neighborhood $\mathcal{V}(u_0)$ such that $\forall v \in \mathcal{U} \cap \mathcal{V}(u_0)$, $\psi(v) \geq \psi(u_0) = 0$. Thus, u_0 is such that $\mathcal{U} \cap \mathcal{V}(u_0) \cap V^{\mathrm{ad}} = \mathcal{V}(u_0) \cap V^{\mathrm{ad}} = \emptyset$. This yields that, u_0 is not an adherent point of V^{ad}, and one has $u_0 \in U^{\mathrm{ad}} \setminus \overline{V^{\mathrm{ad}}}$ which yields $U^{\mathrm{ad}} \neq \overline{V^{\mathrm{ad}}}$. This concludes the proof. ∎

From Proposition 3, one has $\overline{V^{\mathrm{ad}}} = \overline{W^{\mathrm{ad}}}$. Substituting $\overline{V^{\mathrm{ad}}}$ by $\overline{W^{\mathrm{ad}}}$ in (ii) from Lemma 3 yields the result. ∎