Mathematical Programming Approach to Hybrid Systems
Analysis and Control

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Drivers for Control Research

Novel Applications
enabled by
• new computer power
• new actuators
• new sensors

Novel Theory
motivated by
• system integration
• system failures
Hybrid Systems

\[ \begin{aligned}
S & \triangleq (X, U, \varphi), \\
X & = \{1, 2, 3, 4, 5\}, \\
U & = \{a, b, c\}, \\
\varphi : & \quad X \times U \to X \\
\end{aligned} \]

\[ \begin{aligned}
\frac{dx(t)}{dt} & = Ax(t) + Bu(t) \\
y(t) & = Cx(t) + Du(t) \\
\end{aligned} \]

with

\[ x \in \mathbb{R}^n, u \in \mathbb{R}^m \]

\[ y \in \mathbb{R}^p \]
Hybrid Systems in Control - Motivation

• Switches *occurring naturally*
  
because plant operates in different modes

• Switches *introduced by controller*
  
to accommodate constraints: anti-windup, MPC
  to implement sequence: PLC
Switches introduced by controller: Model Predictive Control (MPC)

**Theorem:** The solution of the MPC problem yields a piecewise affine state feedback law.
(Bemporad, Morari, Dua, Pistikopoulos, 2000)

Example: \[ y = \frac{s + 1}{s^2 + s + 2} u \quad T_s = 0.2 \]
\[ -1 \leq u \leq 1 \quad x_1, x_2 \geq -0.5 \]
• Switches introduced by controller: MPC

• Explicit MPC = PWA controller

\[
\begin{cases}
-1.0000 & \text{if } \begin{bmatrix} 0.2425 & 0.0000 \\ 0.0000 & 0.2425 \\ -2.5336 & -1.3548 \\ -2.4411 & 0.5570 \\ 0.0000 & -2.0000 \end{bmatrix} x \begin{bmatrix} 1.0000 \\ 1.0000 \\ 1.0000 \\ 1.0000 \end{bmatrix} \\
[-4.3528, 1.0000] x - 2.7954 & \text{if } \begin{bmatrix} 0.0000 & 0.2425 \\ -2.0000 & 0.0000 \\ 0.6615 & -0.8424 \\ -1.1548 & 0.2635 \\ 2.4411 & -0.5570 \end{bmatrix} x \begin{bmatrix} 1.0000 \\ 1.0000 \\ -1.0000 \\ 1.0000 \end{bmatrix} \\
[-2.5336, -1.3548] x & \text{if } \begin{bmatrix} -0.6615 & 0.8424 \\ -2.5336 & -1.3548 \\ 2.5336 & 1.3548 \\ -2.0000 & 0.0000 \\ 0.0000 & -2.0000 \end{bmatrix} x \begin{bmatrix} 1.0000 \\ 1.0000 \\ 1.0000 \\ 1.0000 \end{bmatrix} \\
1.0000 & \text{if } \begin{bmatrix} 0.0000 & -2.0000 \\ -0.6615 & -1.7922 \\ -2.0000 & 0.0000 \\ 0.0000 & -2.0000 \end{bmatrix} x \begin{bmatrix} 1.0000 \\ 1.0000 \\ -1.0000 \\ 1.0000 \end{bmatrix} 
\end{cases}
\]

• Closed-loop MPC
Hybrid Systems in Control - Motivation

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• Switches *introduced by model simplification*
  
to realize model hierarchy: approximate
lower level dynamics by switches
Hybrid Systems in Control - Motivation

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Modeling Framework

- Complex enough to be practically important
- Simple enough to allow analysis and synthesis

"Things should be made as simple as possible but not simpler" ….Einstein

Discrete Time Piecewise Affine Systems

\[
\begin{align*}
  x(t + 1) &= A_i x(t) + B_i u(t) + f_i \\
  y(t) &= C_i x(t) + g_i \\
  L_i x(t) + M_i u(t) &\leq N_i
\end{align*}
\]
Modeling Framework

Too restrictive?

Piecewise Affine Systems, equivalent to:

- Mixed Logic Dynamical Systems
- (Extended) Linear Complementarity Systems
- Max-Min-Plus -Scaling Systems

(Heemels, De Schutter, Bemporad, 2000)

(Each framework has its advantages)

May be too general, …..
Hybrid Systems

Hybrid Control Systems

symbols $\delta_i$ \rightarrow automaton / logic \rightarrow symbols $\delta_0$

A/D \rightarrow continuous dynamical system \rightarrow D/A

continuous states \rightarrow inputs

Sastry, Lygeros, Tomlin, Godbole, Pappas
Alur, Pnueli, Maler, Henzinger, Krogh, ...

Switched (PWA) Systems

\[
\dot{x} = \begin{cases} 
  f_i(t, x, u) & \text{if } x \in \mathcal{R}_i \\
  & i = 1, \ldots, N 
\end{cases}
\]

\[
\dot{x} = \begin{cases} 
  A_i x + B_i u + f_i & \text{if } H_i x \leq K_i \\
  & i = 1, \ldots, N 
\end{cases}
\]

Sontag, Branicky, Johansson, Rantzer,
Morse, Hespanha, van der Schaft, Tsitsiklis,
Blondel, ...

ETH Eidgenössische Technische Hochschule Zürich
### From Algebraic Equalities to Mixed-Integer Linear Inequalities

<table>
<thead>
<tr>
<th>Propositional logic</th>
<th>Mixed product</th>
<th>Threshold condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algebraic equalities</td>
<td>MI linear inequalities</td>
<td></td>
</tr>
<tr>
<td>Truth value operator: $[X] \in {0, 1}$</td>
<td>$\bar{\delta} = [\delta_1, \ldots, \delta_n]' \in {0, 1}^n$</td>
<td>$[x \leq 0] = \delta$</td>
</tr>
<tr>
<td>$[P(X_1, \ldots, X_n)] = 1$</td>
<td>$A\bar{\delta} \leq B$</td>
<td>$x \leq M(1 - \delta)$</td>
</tr>
<tr>
<td>$z = \delta x$</td>
<td>$\begin{cases} z \leq M\delta \ z \geq m\delta \ z \leq x - m(1 - \delta) \ z \geq x - M(1 - \delta) \end{cases}$</td>
<td>$x \geq \epsilon + (m + \epsilon)\delta$</td>
</tr>
<tr>
<td>$\delta \in {0, 1}$</td>
<td>$x \in [m, M]$</td>
<td></td>
</tr>
<tr>
<td>$x \in [m, M]$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(Williams, 1977) (Glover, 1975) (Witsenhausen, 1966)
**MLD Hybrid Models**

**Mixed Logical Dynamical (MLD) form** (Bemporad, Morari, *Automatica*, March 1999)

\[
\begin{align*}
x(t + 1) &= Ax(t) + B_1 u(t) + B_2 \delta(t) + B_3 z(t) \\
y(t) &= Cx(t) + D_1 u(t) + D_2 \delta(t) + D_3 z(t) \\
E_2 \delta(t) + E_3 z(t) &\leq E_4 x(t) + E_1 u(t) + E_5
\end{align*}
\]

\[
x, y, u = \begin{bmatrix} x^c \\ x^l \end{bmatrix}, \quad x^c \in \mathbb{R}^{n_c}, x^l \in \{0, 1\}^{n_l}, \quad z \in \mathbb{R}^{r_c}, \quad \delta \in \{0, 1\}^{r_l}
\]

**Well-Posedness Assumption:**

\[
\begin{align*}
\delta(t) &= F(x(t), u(t)) \\
z(t) &= G(x(t), u(t))
\end{align*}
\]

\[
\{x(t), u(t)\} \rightarrow \{x(t + 1)\} \quad \{x(t), u(t)\} \rightarrow \{y(t)\}
\]

are single valued

Well posedness allows defining trajectories in \(x\)- and \(y\)-space

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**ifa**
Major Advantage of PWA/MLD Framework

All problems of analysis:
• Stability
• Verification
• Controllability / Reachability
• Observability

All problems of synthesis:
• Controller Design
• Filter Design / Fault Detection & Estimation

can be expressed as (mixed integer) mathematical programming problems for which many algorithms and software tools exist.

However, all these problems are NP-hard.
Research Topics
(Bemporad, Borrelli, Ferrari-Trecate, Mignone, Torrisi, Morari)

Synthesis
- Control (MPC)
- Explicit PWA MPC controllers
- State estimation (MHE)/fault detection

Analysis
- Reachability / Verification
- Stability
- Observability

MLD/PWA Hybrid Systems

Modeling
- HYSDEL
- Identification

Applications
- Car suspension system
- Gas supply system
- Hydroelectric power plant
...
HYbrid System Description LAnguage (HYSDEL)

- Planned integration with CHECKMATE (CMU)
Identification of Hybrid systems

Model and datapoints

$\begin{align*}
y_{k+1} &= \begin{cases}
[0.9, 0.2, 0] [y_k, u_k, 1]' & [y_k, u_k] \in C_1 \\
[0.5, 0.4, 2] [y_k, u_k, 1]' & [y_k, u_k] \in C_2 \\
[0.3, -0.3, -5] [y_k, u_k, 1]' & [y_k, u_k] \in C_3
\end{cases}
\end{align*}$

**Problem:**
Identify a piecewise ARX model from a finite set of noisy measurements.

Useful when the switches between different submodels cannot be measured

*The estimation of the submodels cannot be separated from the problem of estimating the regions*
Identification Algorithm

Exploit the combined use of

- **clustering** ⇒ “K-means” like procedure
- **linear identification** ⇒ weighted least squares
- **classification** ⇒ linear support vector machines

G. Ferrari-Trecate, M. Muselli, D. Liberati, M. Morari,
*A Clustering Technique for the Identification of Piecewise Affine Systems*, HS2001, Section FA
Dialysis Therapy

- Blood urea concentration is measured

- Bi-exponential dynamic
  (Liberati et. al., 1993)
  - First part (30-40 minutes)
    *Fast decrease*
  - Second part (3-4 hours)
    *Slow decrease*

An early estimation of both the time constants and the switching time allows the assessment of the total duration of the therapy
Dialysis Therapy

Take the log of the data

\[ \downarrow \]

Piecewise Affine approximation

\[ \downarrow \]

Estimation of the time constants

The switching time cannot be measured directly

Depends on both the patient physiology and the clearance rate of the dialyzer
EEG Analysis

Problem: discriminate the presence of different mental tasks from EEG

Proposition: EEG in a single mental "state" $\approx$ AR model of low order

(C. Anderson et al., 1995)

The switch between mental states cannot be measured

Hybrid identification
Application of EEG Analysis:

Brain computer interfacing

- High inter-subjects and intra-subjects variability of EEG
  
  Need to update models easily

- Biofeedback: the subject can be forewarned that he is changing mental state

Epileptic patients: Early seizure detection

- Prompt intervention against epilepsy crisis
The MIT's Technology Review magazine recently listed brain-machine interfaces as one of the 10 emerging technologies that will "soon have a profound impact on the economy and on how we live and work."
Research Topics
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- **Synthesis**
  - Control (MPC)
  - Explicit PWA MPC controllers
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- **Analysis**
  - Reachability / Verification
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  - Observability

- **MLD/PWA Hybrid Systems**

- **Modeling**
  - HYSDEL
  - Identification

- **Applications**
  - Car suspension system
  - Gas supply system
  - Hydroelectric power plant
  ...

(Bemporad, Borrelli, Ferrari-Trecate, Mignone, Torrisi, Morari)
**Analysis vs. Synthesis**

Control of stable system with input constraints

- Analysis of closed loop stability
  conservative / difficult

- Synthesis of feedback controllers with stability guarantee
  industrial routine

**Laptop Computer**

- Analysis: $10^{10^{20}}$ states
- $10^{100}$ atoms in universe (Wm. A. Wulf, President NAE)
Hybrid Systems Control Review

• Piecewise Quadratic/Linear Lyapunov functions
  Linear Matrix Inequalities to characterize stability and performance

• Pontryagin Maximum Principle

• Hamilton Jacobi Bellman equation

• Parametric Programming

• …….

Bemporad, Berardi, Boyd, Borrelli, Branicky, Buss, Burlirsch, De Santis, Di Benedetto, Hassibi, Hedlund, Johansson, Kratz, Lygeros, Mitter, Piccoli, Rantzer, Riedinger, Sastry, Styrk, Sussmann, Tomlin, Zann
Receding Horizon Control

- Optimize at time \( t \) (new measurements)
- Only apply the first optimal move \( u(t) \)
- Repeat the whole optimization at time \( t + 1 \)
- Advantage of on-line optimization \( \Rightarrow \) FEEDBACK
Model Predictive Control

\[ J_{opt} = \min_{U} J(U, x(t)) \triangleq \sum_{k=0}^{N} \| Q(x(t + k|t) - \bar{x}) \| + \| Ru(t + k) \| \]

subject to \( u_{\min} \leq u(t + k) \leq u_{\max} \)
\( x_{\min} \leq x(t + k|t) \leq x_{\max} \)

system dynamics
\( U \triangleq \{ u(t), u(t + 1), \ldots, u(t + N_u) \} \)

- **Objective**: determine the optimal input sequence \( u(t), \ldots, u(t + k) \) driving the system from \( x(t) \) to \( \bar{x} \), compatibly with the limits on \( u(t + k), x(t + k|t) \)
- **Apply** the first input move \( u(t) \) according to RHC
- **Repeat** the optimization at time \( t + 1 \)
Model Predictive Control

\[ J_{opt} = \min_U J(U, x(t)) \triangleq \sum_{k=0}^{N} \|Q(x(t + k|t) - \bar{x})\| + \|Ru(t + k)\| \]

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\( x_{\min} \leq x(t + k|t) \leq x_{\max} \)
system dynamics
\( U \triangleq \{u(t), u(t + 1), \ldots, u(t + N_u)\} \)

- **Linear** Model and 2-norm performance index
  \( \Rightarrow \) Quadratic Program
- **Linear** Model and \( \infty \)-norm performance index
  \( \Rightarrow \) Linear Program
- **Hybrid** Model and \( \infty \)-norm performance index
  \( \Rightarrow \) Mixed Integer Linear Program
Theorem: The solution of the MPC problem yields a piecewise affine state feedback law

\[ J_{opt} = \min_U J(U, x(t)) \triangleq \sum_{k=0}^{N} \| Q(x(t+k|t) - \bar{x}) \| + \| R u(t+k) \| \]
subject to \[ u_{\min} \leq u(t+k) \leq u_{\max} \]
\[ x_{\min} \leq x(t+k|t) \leq x_{\max} \]
\[ \text{system dynamics} \]
\[ U \triangleq \{ u(t), u(t+1), \ldots, u(t+N_u) \} \]

(Bemporad et. Al., 2000; Bemporad, Borrelli, Morari, CDC 2000 TuM05-1, WeM01-1)
The solution of the MPC can be computed explicitly

\[ J_{opt} = \min_U J(U, x(t)) \triangleq \sum_{k=0}^{N} \| Q(x(t + k|t) - \bar{x}) \| + \| Ru(t + k) \| \]

subject to \( u_{\text{min}} \leq u(t + k) \leq u_{\text{max}} \)
\( x_{\text{min}} \leq x(t + k|t) \leq x_{\text{max}} \)

system dynamics
\( U \triangleq \{u(t), u(t + 1), \ldots, u(t + N_u)\} \)

- The solution of the MPC can be computed explicitly
- Off-line optimization: optimize for all \( x(t) \)

\[ \Rightarrow \text{Multi-parametric Program} \]

Explicit solvers for QP, LP and MILP are available

(Bemporad et. Al., 2000; Bemporad, Borrelli, Morari, CDC 2000 TuM05-1, WeM01-1)
MPC for MLD Systems

Example: Alternate Heating of Two Furnaces

(Heidlund, Rantzer CDC1999)

- Objective:
  - Control the Temperature of Two Furnaces

- Constraints:
  - Switching Control between three operation modes:
    1- Heat only the first furnace
    2- Heat only the second furnace
    3- Do not heat any furnaces

Amount of energy $u_0$ fixed

Can be parametrized!

\[
\begin{align*}
x &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x(t) + B_i u_0 \\
B_i &= \begin{cases} 
\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} & \text{if first furnace heated} \\
\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} & \text{if second furnace heated} \\
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \text{if no heating}
\end{cases}
\end{align*}
\]
Alternate Heating of Two Furnaces

- MLD system
  \[ u(k) = \begin{cases} 
  [1 \ 0 \ 0] & \text{if first furnace heated} \\
  [0 \ 1 \ 0] & \text{if second furnace heated} \\
  [0 \ 0 \ 1] & \text{if no heating} 
  \end{cases} \]

- mp-MILP optimization problem
  \[ \min \left\{ v_0^2 \right\} \quad J(v_0^2, x(t)) \triangleq \sum_{k=0}^{2} \| R(v(k+1) - v(k)) \|_{\infty} + \| Q(x(k|t) - x_e) \|_{\infty} \]

\[ \begin{align*}
1 & \leq x_1 \leq 1 \\
1 & \leq x_2 \leq 1 \\
0 & \leq u_0 \leq 1
\end{align*} \]

- Computational complexity of mp-MILP

<table>
<thead>
<tr>
<th>State ( x(t) )</th>
<th>3 variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input ( u(t) )</td>
<td>3 variables</td>
</tr>
<tr>
<td>Aux. binary vector ( \delta(t) )</td>
<td>0 variables</td>
</tr>
<tr>
<td>Aux. continuous vector ( z(t) )</td>
<td>9 variables</td>
</tr>
</tbody>
</table>

| linear constraints | 168 |
| continuous variables | 33 |
| binary variables | 9 |
| parameter variables | 3 |
| time to solve the mp-MILP | 5 min |
| Number of regions | 105 |
mp-MILP Solution

\[ u_0 = 0.4 \]

\[ u_0 = 0.8 \]
Characteristics of the Solution

\[ u(k) = F_k x(k) + G_k \iff x(k) \in \mathcal{X}_k, \; \mathcal{X}_k = \{ x \mid L_k x \leq M_k \} \; \; k = 0, \ldots, N. \]

- Piecewise affine control law, polyhedral regions
- Simultaneous and automatic partitioning and control law synthesis
- Stability guarantee (PWL Lyapunov function)
- On-line implementation does not require storage of all \( \mathcal{X}_k \)
Research Topics
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Applications

• Traction control (Ford Research Center)
• Gas supply system (Kawasaki Steel)
• Batch evaporator system (Esprit Project 26270)
• Anesthesia (Hospital Bern)
• Hydroelectric power plant
• Power generation scheduling (ABB)
Analgesia Control during Anesthesia
Clinical Goals

- $P$ must have priority
- $P$ in range rather than tracked
- aggressive action upon constraint violation
- constraints prioritization:
  1. hypotension $P > P_{low}$
  2. overdosing $C < C_{high}$
  3. hypertension $P < P_{high}$
  4. underdosing $C > C_{low}$

$C = \text{Drug concentration}$
$P = \text{Blood Pressure}$
Controller Implementation

- Explicit MPC
  - Control horizon: 3
  - Dimension: 8
  - Prediction horizon: 10
  - Weight $P$: 150
  - Number of regions: 127
  - Weight $C$: 1
Case Study I

[Graph showing time series data with annotations: No overdosing, Dynamic trade-off, Strong stimulus, Artifacts, and flow rate.]
In the Operating Room

1. Induction
2. Artifact detection
3. Intense stimulation
Applications

• **Traction control** (Ford Research Center)

• **Gas supply system** (Kawasaki Steel)

• **Batch evaporator system** (Esprit Project 26270)

• **Anesthesia** (Hospital Bern)

• **Hydroelectric power plant**

• **Power generation scheduling** (ABB)
Optimization of Combined Cycle Power Plants

(ETH-ABB joint project)

Deregulated energy market
- electricity/gas demands and prices change rapidly

Optimize the plant hourly

Optimization

Maximize the profit while fulfilling the operating constraints
Topologies of Combined Cycle Power Plants

Simple plant:

- Two turbines and two binary inputs (on/off commands)

Other plants:

- Several gas/steam turbines, firings ...

The complexity of the example is scalable!
Optimization of Combined Cycle Power Plants

Logic
- Gas/Steam turbines can be switched on/off
- Different types of startup procedures
- Minimum up/down times

Continuous
- Normal operation: Dynamics of the energy/steam production
- Constraints on the production capabilities

Hybrid model

Economic optimization ⇒ Predictive control for hybrid systems
Conclusions

• Hybrid system models are important

• Control Theory ⇔ Computer Science

• Computations must be an inherent part of any new theory