Robust state-feedback stabilization of an underactuated network of interconnected $n+m$ hyperbolic PDE systems

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Abstract

We detail in this article the development of a delay-robust stabilizing state feedback control law for an underactuated network of two subsystems of heterodirectional linear first-order hyperbolic Partial Differential Equations interconnected through their boundaries. Only one of the two subsystems is actuated. The proposed approach is based on the backstepping methodology. A backstepping transform allows us to construct a first feedback to tackle in-domain couplings present in the actuated PDE subsystem. Then, we introduce a predictive tracking controller to stabilize the second PDE subsystem. The stabilization of this subsystem implies the stabilization of the whole network. Finally, the proposed control law is combined with a low-pass filter to become robust with respect to small delays in the control signal and uncertainties on the system parameters.

Key words: Hyperbolic Partial Differential Equations; difference systems; backstepping; interconnection; predictor; robustness.

1 Introduction

The ability to manipulate flow properties (concentration, temperature, density, etc.) is a question of major technological importance. Such a situation occurs for flow regulation in mining [43] or hydraulic networks [19], for control of after-treatment devices in exhaust lines [20] and for blending in liquid or solid networks [14], to name a few. Remarkably, in all these examples, transportation of matter occurs across space, and propagation phenomena have to be taken into account to adequately represent their dynamical behavior [19,23,38]. One natural mathematical representation of these transport phenomena is through hyperbolic Partial Differential Equations (PDEs).

The control and estimation of coupled hyperbolic PDEs is an active research topic [2,16,22,31,39,44]. Unlike results for linear Ordinary Differential Equations (ODEs), constructive control designs, even for linear hyperbolic PDEs, are harder to find and often require specific controllability results [17], or many independent actuators to be available [41].

In this paper, we explore the design of a stabilizing state feedback control law for an underactuated network. More precisely, we consider a network of two subsystems of $n+m$ heterodirectional linear first-order hyperbolic PDEs interconnected through their boundaries, with only one actuated system. This kind of interconnected systems may appear when considering oil production systems made of networks of pipes (whose principal line is known as the manifold), each pipe being subject to torsional and axial oscillations [38]. Similar kind of interconnections can also appear when modeling traffic network systems with different types of vehicles [13], ventilation in buildings [43], density-flow systems [27], open canals [19], communication networks [24] or the case of the Rijke tube [18] (even if, in this case an ODE is sandwiched between two PDEs systems). Among the different possible approaches to deal with these networks, PI boundary controllers have been considered in [9,27] for fully actuated networks (i.e., with one control per set of heterodirectional PDEs). The authors obtained explicit stability conditions using appropriate quadratic Lyapunov functions. In [29], the authors consider a flatness-based design of a feedforward control of tree-like transmission networks. Similar cases of interconnected problems have been considered in [40], with a velocity recirculation in a wave equation. The exact boundary controllability of nodal profile for quasilinear hyperbolic systems with interface conditions in a tree-like networks has been assessed in [42] using the method of characteristics. However, even if the proposed method

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is straightforward to implement, it may require solving a set of PDEs online, which is computationally expensive. Underactuated PDEs have been the source of several contributions during these last years. Recently in [39], the authors have considered the output feedback control of series interconnections of 2×2 semilinear hyperbolic system using the dynamics on the characteristic lines. Regarding networks of ODEs and PDEs, some results have been obtained recently using the backstepping approach [1,21]. Interestingly, in [25], the author proposes a systematic design for the stabilization of ODE-PDE-ODE based on successive backstepping transformations. Finally, the backstepping approach has been used in [4] to design a robust stabilizing output feedback control law for an underactuated cascade network of n systems of two heterodirectional linear first-order hyperbolic PDEs interconnected through their boundaries.

In this paper, to construct a stabilizing controller, inspired by [8,7], we propose to use backstepping techniques to reformulate the system at stake under the form of difference equations that are more suitable for control design. We follow a two-steps procedure. First, we consider a virtual input acting on the non-actuated PDE subsystem and design a novel prediction-based control law [3,10,12] for difference equations, which guarantees the stabilization of this subsystem. Then, we focus on the actuated PDE subsystem and guarantee the tracking of the previously designed virtual law by constructing a backstepping transformation tackling the in-domain couplings of this PDE subsystem. This two-steps procedure can be performed under an invertibility assumption on the boundary matrix between the two subsystems. As a final ingredient, we propose an adequate low-pass filter that can be added to the control design and renders the final control law strictly proper and thus robust to delays in the loop as studied in [5,11]. The resulting feedback law guarantees robust stabilization of the entire network. This is the main contribution of the paper.

The paper is organized as follows. In Section 2, we present the problem under consideration and the control approach. Then, Section 3 focuses on the stabilization of the first PDE subsystem reformulated as time-delay difference equations (comparison system) and the design of a prediction-based control law, while Section 4 presents a backstepping procedure for the second PDE subsystem. After presenting in Section 5 the resulting stabilizing control law for the PDE network, Section 6 details the design of a low-pass filter guaranteeing delay-robust stabilization. Finally, the merits of our design are illustrated in numerical simulations in Section 7.

2 Problem Formulation

2.1 Definitions and notations

In this section we detail the notations used through this paper. For any distinct real numbers a and b, we denote $L^2([a,b],\mathrm{R})$ the space of real-valued square-integrable functions defined on $[a,b]$ with the standard L^2 norm, *i.e.*, for any $f \in L^2([a,b],\mathrm{R})$, $||f||^2_{L^2([a,b])} = \int_a^b f^2(x)\mathrm{d}x$. The letters n_p, n_d, m_p, m_d designate non-zero integers. For two functions u_1 in $L^2([a_1,b_1],\mathrm{R})$, u_2 in $L^2([a_2,b_2],\mathrm{R})$ (where $a_1 \neq b_1$ and $a_2 \neq b_2$ are real numbers), we use the following notation

$$||(u_1,u_2)||^2_{L^2} = ||u_1||^2_{L_2([a_1,b_1])} + ||u_2||^2_{L_2([a_2,b_2])}.$$

It represents the sum of the square of the L^2-norm of each function. This notation can be straightforwardly generalized to an arbitrary finite number of L^2-functions. The set $L^\infty([0,1],\mathrm{R})$ denotes the space of bounded real-valued functions defined on $[0,1]$ with the standard L^∞ norm, *i.e.*, for any $f \in L^\infty([0,1],\mathrm{R})$, $||f||_{L^\infty} = \underset{x\in[0,1]}{\text{ess sup}}|f(x)|$. The sets $\mathcal{T}_b$ and $\mathcal{U}$ are defined as

$$\mathcal{T}_b = \{(x,\xi) \in [0,1]^2 \text{ s.t. } \xi \leq x\}, \tag{1}$$
$$\mathcal{U} = \{(x,\xi) \in [0,1] \times [-1,0]\}, \tag{2}$$

The first domain corresponds to the lower triangular part of the unit square. The second domain corresponds to a square. The space $L^\infty(\mathcal{T}_b)$ (resp. $L^\infty(\mathcal{U})$) stands for the space of real-valued L^∞ functions on $\mathcal{T}_b$ (resp. $\mathcal{U}$). For any integer $m > 0$ and any real delay $\tau > 0$, we denote $L^2([-\tau,0],\mathrm{R}^m)$ the Banach space of L^2 functions mapping the interval $[-\tau,0]$ into R^m. For a function $\phi : [-\tau,\infty) \mapsto \mathrm{R}^m$, we define its partial trajectory $\phi_{[t]}$ by $\phi_{[t]} : \phi(t+\theta)$, $-\tau \leq \theta \leq 0$. This maximum delay τ will be related to the transport velocities of the considered PDE system. The associated norm is given by

$$||\phi_{[t]}|| = \left(\int_{-\tau}^0 \phi^T(t+s)\phi(t+s)ds\right)^{\frac{1}{2}}. \tag{3}$$

For every $\tau > r > 0$, we define

$$||\phi_{[t]}||_r = \left(\int_{-r}^0 \phi^T(t+\theta)\phi(t+\theta)d\theta\right)^{\frac{1}{2}}. \tag{4}$$

The variable s denotes the Laplace variable. The space C^+ corresponds to the complex right half plane: $\mathrm{C}^+ = \{s \in \mathrm{C},\ \mathrm{Re}(s) \geq 0\}$, where Re denotes the real part of a complex number. Provided it is defined, the Laplace transform of a function f will be denoted $\hat{f}$. For all $p \in \mathrm{N}$, we denote Id_p the identity matrix of dimension p (or Id if no confusion arises). Finally, for any proper and stable transfer matrix $G(s)$, we denote $\bar{\sigma}(G(s))$ the largest singular value of $G(s)$ ($s \in \mathbb{C}^+$) and $\underline{\sigma}(G(s))$ its lowest singular value.

2.2 System under consideration

In this paper, we consider a system composed of two independent subsystems coupled through their boundaries, as schematically represented in Fig. 1. Each subsystem is composed of an arbitrary number of linear hyperbolic PDEs. Only one subsystem is actuated at its boundary. The first subsystem is defined by the following set of PDEs

$$\begin{aligned} \partial_t u^p(t,x) + \Lambda_p^+ \partial_x u^p(t,x) &= \Sigma_p^{++}(x) u^p(t,x) \\ &\quad + \Sigma_p^{+-}(x) v^p(t,x), \quad (5) \\ \partial_t v^p(t,x) - \Lambda_p^- \partial_x v^p(t,x) &= \Sigma_p^{-+}(x) u^p(t,x) \\ &\quad + \Sigma_p^{--}(x) v^p(t,x), \quad (6) \end{aligned}$$

evolving in $\{(t,x) \text{ s.t. } t > 0,\ x \in [0,1]\}$, where $u^p = \begin{pmatrix} u_1^p & \dots & u_{n_p}^p \end{pmatrix}^T$ and $v^p = \begin{pmatrix} v_1^p & \dots & v_{m_p}^p \end{pmatrix}^T$ are the PDE states. The matrices Λ_p^+ and Λ_p^- are diagonal ($\Lambda_p^+ = \text{diag}\ (\lambda_i^p)$, $\Lambda_p^- = \text{diag}\ (\mu_i^p)$) and their coefficients satisfy

$$-\mu_{m_p}^p \leq \dots \leq -\mu_1^p < 0 < \lambda_1^p \leq \dots \leq \lambda_{n_p}^p.$$

The spatially-varying inside domain coupling matrices are defined as follows

$$\begin{aligned} \Sigma_p^{++}(x) &= \{(\sigma_p^{++})_{ij}(x)\}_{1\leq i,j\leq n_p}, \\ \Sigma_p^{+-}(x) &= \{(\sigma_p^{+-})_{ij}(x)\}_{1\leq i\leq n_p, 1\leq j\leq m_p}, \\ \Sigma_p^{-+}(x) &= \{(\sigma_p^{-+})_{ij}(x)\}_{1\leq i\leq m_p, 1\leq j\leq n_p}, \\ \Sigma_p^{--}(x) &= \{(\sigma_p^{--})_{ij}(x)\}_{1\leq i,j\leq m_p}. \end{aligned}$$

Their coefficients are assumed to be continuous functions. We assume that the diagonal terms of the matrices Σ_p^{++} and Σ_p^{--} are equal to zero (such coupling terms can be removed using exponential change of coordinates).The subsystem (5)-(6) will be called the *proximal subsystem* since it will be the one actuated. The second subsystem will be called the *distal subsystem* since it will not be directly actuated. It satisfies a similar set of equations

$$\begin{aligned} \partial_t u^d(t,x) + \Lambda_d^+ \partial_x u^d(t,x) &= \Sigma_d^{++}(x) u^d(t,x) \\ &\quad + \Sigma_d^{+-}(x) v^d(t,x), \quad (7) \\ \partial_t v^d(t,x) - \Lambda_d^- \partial_x v^d(t,x) &= \Sigma_d^{-+}(x) u^d(t,x) \\ &\quad + \Sigma_d^{--}(x) v^d(t,x), \quad (8) \end{aligned}$$

evolving in $\{(t,x) \text{ s.t. } t > 0,\ x \in [-1,0]\}$, where $u^d = \begin{pmatrix} u_1^d & \dots & u_{n_d}^d \end{pmatrix}^T$ and $v^d = \begin{pmatrix} v_1^d & \dots & v_{m_d}^d \end{pmatrix}^T$ are the PDE states. The matrices Λ_d^+ and Λ_d^- are diagonal ($\Lambda_d^+ = \text{diag}\ (\lambda_i^d)$, $\Lambda_d^- = \text{diag}\ (\mu_i^d)$) and their coefficients satisfy

$$-\mu_{m_d}^d \leq \dots \leq -\mu_1^d < 0 < \lambda_1^d \leq \dots \leq \lambda_{n_d}^d. \quad (9)$$

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Their coefficients are assumed to be continuous functions. The two, somehow independent, subsystems are coupled through their boundaries. More precisely, the boundary conditions of (5)-(6) and (7)-(8) verify

$$\begin{aligned} u^d(t,-1) &= R_d v^d(t,-1), \quad (10) \\ v^d(t,0) &= Q_{dp} v^p(t,0) + Q_{dd} u^d(t,0), \quad (11) \\ u^p(t,0) &= Q_{pp} v^p(t,0) + Q_{pd} u^d(t,0), \quad (12) \\ v^p(t,1) &= R_p u^p(t,1) + V(t), \quad (13) \end{aligned}$$

where the matrices R_p, R_d, $Q_{pp}, Q_{dd}, Q_{dp}, Q_{pd}$ are constant matrices of appropriate dimensions. The function V is a vector input function (control law with m_p components) that takes real values. The initial conditions of the systems (5)-(6) and (7)-(8) are given as $u_0^p(\cdot) = u^p(0,\cdot)$, $v_0^p(\cdot) = v^p(0,\cdot)$, $u_0^d(\cdot) = u^d(0,\cdot)$ and $v_0^d(\cdot) = v^d(0,\cdot)$. These initial conditions are L^2 functions. The interconnected system (5)-(13) is depicted in Figure 1.

Finally, we denote τ_p the maximum transport delay for the proximal system, τ_d the maximum transport delay for the distal system and $\tau = \tau_p + \tau_d$:

$$\tau_p = \sup_{\substack{1\leq j\leq m_p \\ 1\leq i\leq n_p}} \left(\frac{1}{\lambda_i^p} + \frac{1}{\mu_j^p}\right), \tau_d = \sup_{\substack{1\leq j\leq m_d \\ 1\leq i\leq n_d}} \left(\frac{1}{\lambda_i^d} + \frac{1}{\mu_j^d}\right). \quad (14)$$

Remark 1 *The velocity matrices $\Lambda_\cdot^+$, and $\Lambda_\cdot^-$ are assumed to be constant but the presented results can be extended to the case of velocities which are $C^1([0,1],\mathbb{R})$-functions.*

Note that, as we consider weak solutions, the system (5)-(13) is well-posed. This can be shown using [9, Theorem A.6, page 254] and a reformulation of the system under consideration.

Indeed, in details, the interconnected system (5)-(13) can be rewritten in a more condensed form as a general $n+m$ hyperbolic system (with $n = m_d + n_p$ and $m = n_d + m_p$) by performing the change of variables $\bar{x} = -x$

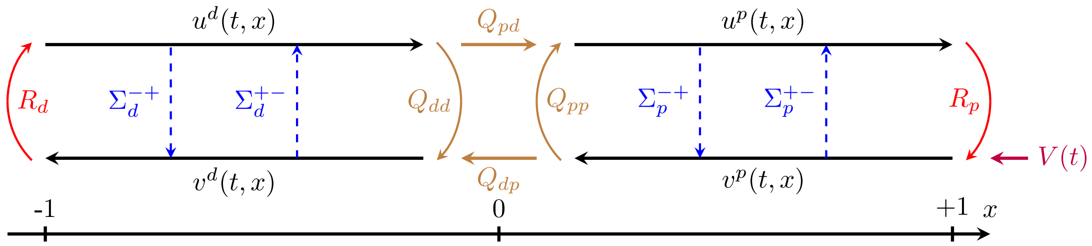

Fig. 1. Schematic representation of the interconnected system (5)-(6), (7)-(8) with the boundary conditions (10)-(13).

on the distal subsystem. This change of variables has sometimes been referred to as folding. More precisely, let us consider the concatenated states $u(t,x)$ and $v(t,x)$ defined on for all $t>0$ and all $x\in[0,1]$ by

$$u(t,x)=\begin{pmatrix}u^p(t,x)\\ v^d(t,-x)\end{pmatrix},\quad v(t,x)=\begin{pmatrix}v^p(t,x)\\ u^d(t,-x)\end{pmatrix}. \quad (15)$$

The states u and v satisfy the following hyperbolic system

$$\begin{aligned}\partial_t u(t,x)+\Lambda^+\partial_x u(t,x)&=\Sigma^{++}(x)u(t,x)\\ &\quad+\Sigma^{+-}(x)v(t,x), \quad (16)\\ \partial_t v(t,x)-\Lambda^-\partial_x v(t,x)&=\Sigma^{-+}(x)u(t,x)\\ &\quad+\Sigma^{--}(x)v(t,x), \quad (17)\end{aligned}$$

with the boundary conditions

$$u(t,0)=Qv(t,0),\quad v(t,1)=Ru(t,1)+AV(t), \quad (18)$$

where

$$\begin{aligned}&\Lambda^+=\begin{pmatrix}\Lambda_p^+ & 0\\ 0 & \Lambda_d^-\end{pmatrix},\ \Lambda^-=\begin{pmatrix}\Lambda_p^- & 0\\ 0 & \Lambda_d^+\end{pmatrix},\\ &\Sigma^{++}(x)=\mathrm{diag}(\Sigma_p^{++}(x),\Sigma_d^{--}(-x)),\\ &\Sigma^{+-}(x)=\mathrm{diag}(\Sigma_p^{+-}(x),\Sigma_d^{-+}(-x)),\\ &\Sigma^{-+}(x)=\mathrm{diag}(\Sigma_p^{-+}(x),\Sigma_d^{+-}(-x)),\\ &\Sigma^{--}(x)=\mathrm{diag}(\Sigma_p^{--}(x),\Sigma_d^{++}(-x)),\\ &Q=\begin{pmatrix}Q_{pp} & Q_{pd}\\ Q_{dp} & Q_{dd}\end{pmatrix},\ R=\begin{pmatrix}R_p & 0\\ 0 & R_d\end{pmatrix},A=\begin{pmatrix}\mathrm{Id}_{m_p}\\ 0_{n_d,m_p},\end{pmatrix}\end{aligned}$$

The diagonal components of the matrix Λ^+ are denoted λ_i $(1\leq i\leq n_p+m_d)$ while the ones of the matrix Λ^- are denoted μ_i $(1\leq i\leq n_d+m_p)$. System (16)-(18) is under-actuated as, due to the matrix A, only a part of the boundary $x=1$ is actuated. Although several stabilization results can be found in the literature for the stabilization of $(n+m)$ hyperbolic system fully actuated at one boundary (see for instance [16,28]), only few results exist about underactuated hyperbolic systems [7,15]. However, we must be aware that (16)-(18) is a specific case of underactuated PDE system since it has a cascade structure due to the sparsity of the in-domain matrices $\Sigma^{\cdot\cdot}$. This cascade structure naturally appears when considering equations (5)-(13). Thus, to design a stabilizing control law, it seems relevant to consider this form rather than the condensed form (16)-(18). However, the form (16)-(18) will be helpful for the robustness analysis.

2.3 Problem formulation

In order to design a control law that stabilizes the interconnected system (5)-(13), we make the following assumption.

Assumption 1 *Let us consider $z\in L^2([-\tau,0],\mathrm{R}^{m_p+n_d})$ (where $\tau=\tau_p+\tau_d$ is defined in (14)) that satisfies the difference system defined for all $1\leq i\leq m_p+n_d$ by*

$$z_i(t)=\sum_{\substack{1\leq l\leq m_p+n_d\\ 1\leq k\leq n_p+m_d}} R_{ik}Q_{kl}z_l\left(t-\frac{1}{\lambda_k}-\frac{1}{\mu_l}\right), \quad (19)$$

We assume that (19) is exponentially stable in the sense of the norm (3).

It has been shown in [32] that a necessary condition to guarantee the existence of robustness margins for an arbitrary closed-loop system is that the open-loop transfer function must have a finite number of poles on the closed right half-plane. For the system (16)-(18) (and consequently (5)-(13)), [8, Theorem 4] proved that this implies that the open-loop system without in-domain couplings must be exponentially stable. Using the method of the characteristics, it is straightforward to show that this open-loop system without in-domain couplings has equivalent stability properties to those of the difference system (19) (with $z(t)\equiv v(t,1)$, see [8] for details). Therefore, Assumption 1 constitutes a reasonable assumption as it is necessary for the existence of robustness margins for the closed-loop system. In other terms, if Assumption 1 is not fulfilled, then, for any feedback law, the introduction of any arbitrarily small delay in the actuation will destabilize the closed-loop system [32]. However, this condition is always satisfied for physical applications, due to the presence of damping terms in the

PDE system (or neglected dynamics involving diffusion terms) since these terms slightly modify equation (19). Note that equation (19) can be expressed in a condensed vector form as $z(t) = \sum_p^N C_p z(t - \eta_p)$, where N, the matrices C_p and the delays η_p are obtained from (19). In [26], it is proved that, if the delays are rationally independent [1], this equation is exponentially stable if and only if

$$\sup_{\theta_p \in [0,2\pi]^N} \mathrm{Sp}\,(\sum_p^N C_p \exp(i\theta_p)) < 1, \tag{20}$$

where Sp denotes the spectral radius. As condition (20) may be computationally expensive to check, alternatively, some numerically tractable sufficient conditions have been proposed using Lyapunov-Krasovskii theory [33,36].

To stabilize the distal subsystem through the proximal subsystem, we make the following assumption.

Assumption 2 *The rank of the matrix Q_{dp} is equal to m_d.*

This assumption means that the matrix Q_{dp} admits a right inverse. A possible choice is given by the Moore-Penrose right inverse: $Q_{dp}^T(Q_{dp}Q_{dp}^T)^{-1}$. This assumption will be used to design a virtual actuation for the distal subsystem. However, one must be aware that it is a conservative assumption [2] but, to the best of our knowledge, only specific results ([7] for instance) currently exist in the literature for the stabilization of underactuated systems that do not have any specific cascade structure.

Remark 2 *Assumption 2 implies that $m_d \leq m_p$.*

A controller ensuring a PDE system's exponential stability may exhibit poor closed-loop behavior and even instability in practice due to vanishing delay margins. For this reason, several concepts of robust stability have been introduced to ensure that the stability holds even in the presence of (possibly small) uncertainties on the delays. We recall here several definitions relevant to our control problem.

Definition 3 (Delay-robust stabilization [32]) *Consider a plant transfer function G and a feedback controller K such that GK is regular[3] and K stabilizes G. The closed-loop system is robustly stable with respect to delays if and only if there exists ϵ_0 such that, for all $\epsilon \in [0, \epsilon_0]$ the closed-loop transfer function in the presence of a delay ϵ in the actuation (i.e. $GK\,(I + e^{-\epsilon s}GK)^{-1}$) is stable. Let us denote $H = GK$ and $\mathfrak{P}_H$ the (discrete) set of its poles in $\mathrm{C}_1 = \{s \in \mathrm{C} \mid \mathrm{Re}(s) > 0\}$. Let us define $\gamma = \lim_{\substack{|s| \to \infty \\ s \in \mathrm{C}_1 \setminus \mathfrak{P}_H}} Sp(H(s))$, where Sp stands for the spectral radius. If $\gamma < 1$, then the closed-loop system is robustly stable with respect to delays. If $\gamma > 1$, then the closed-loop system is not robustly stable with respect to delays.*

The more general concept of w-stability proves more useful in this context.

Definition 4 (w-stability [17]) *Consider a plant transfer function G and a feedback controller K such that GK is regular. The closed-loop system is* w-stable *if and only if for any approximate identity I_δ, the closed-loop transfer function $GK\,(I + I_\delta GK)^{-1}$ is stable. An approximate identity is a family of transfer functions I_δ such that*

(1) $\|I_\delta\|_\infty < 1$, $I_0 = I$;
(2) On every compact set of the open Right-Half Plane, I_δ converges to I when δ goes to zero.

Suppose that (G, K) is input-output stable. Then (G,K) is w-stable *if there exists a $\rho > 0$ such that*

$$\lim_{\{s \in \bar{\mathrm{C}}^+ \mid\ |s| > \rho\}} ||G(s)K(s)|| < 1. \tag{21}$$

Approximate identities may include more general transfer functions than the ones stemming from uncertainties on the delays. Thus, *w*-stability implies delay-robust stability. Moreover it is easy to show that the condition $\gamma < 1$ implies (21). Hence the conditions for delay-robustness and w-stability are the same for input-output stable systems, possibly except for the case $\gamma = 1$.

In this paper, we show that if Assumption 1 and Assumption 2 are satisfied, then it is possible to explicitly design a state feedback control law V that **robustly stabilizes** the system (5)-(13), i.e.:

- the state (u^p, v^p, u^d, v^d) of the resulting feedback system (5)-(13) exponentially converges to its zero equilibrium (**stabilization problem**), *i.e.* there exist $\kappa_0 \geq 1$ and $\nu > 0$ such that for any initial condition $(u_0^p, v_0^p, u_0^d, v_0^d) \in (L^2[0,1])^{m_p+n_p} \times (L^2[-1,0])^{m_d+n_d}$

$$||(u^p, v^p, u^d, v^d)||_{L^2} \leq \kappa_0 \mathrm{e}^{-\nu t} ||(u_0^p, v_0^p, u_0^d, v_0^d)||_{L^2}.$$

- the resulting closed-loop system (5)-(13) is w-stable.

[1] Extending the variable z, it is always possible to rewrite the system in a situation where the delays are rationally independent.

[2] Less restrictive conditions could be obtained if controllability conditions were available for underactuated PDEs systems. This is a direction of future works.

[3] i.e. GK is bounded on some Right-Half plane and $\lim_{\lambda \to \infty} G(\lambda)K(\lambda)$ exists (where λ is real), see [32, Section 2] for details.

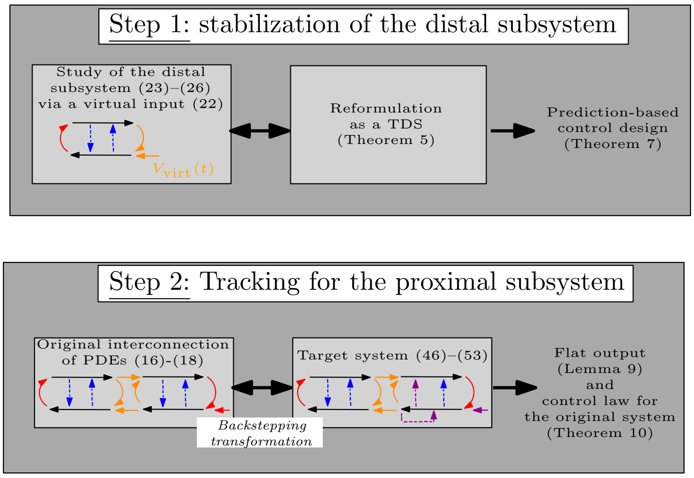

Fig. 2. Schematic representation of the different steps of the control design.

2.4 Control strategy

The distal subsystem can only be controlled through the proximal subsystem. Due to the hyperbolic structure of the proximal subsystem, the effect of the actuation on the distal subsystem will be delayed by the transport time $\frac{1}{\mu_1^p}$ but also modified by the different in-domain couplings that are present in equation (6). Using a backstepping transformation, it is however, possible to construct a first feedback that tackles in-domain couplings present in the proximal subsystem. Then, using the right invertibility of Q_{dp} (Assumption 2), it becomes possible to track the state $Q_{dp}v^p(t,0)$ entering in the distal subsystem. However, due to the transport time between the boundaries of the proximal subsystem and to guarantee a causal control law, we choose to track $Q_{dp}v^p(t+\frac{1}{\mu_1^p},0)$, by considering $V_{\mathrm{virt}}(t) = Q_{dp}v^p(t+\frac{1}{\mu_1^p},0)$ as a virtual input acting on the distal subsystem. This leads us to the stabilization of a hyperbolic system with a delayed actuation. This problem has already been evoked in [6] for an interconnection between an ODE and a PDE. To tackle this problem, we design a new kind of predictor. To summarize, our approach, illustrated in Fig. 2 is described as follows.

(1) We consider the distal subsystem with the virtual actuation $V_{\mathrm{virt}}(t) = Q_{dp}v^p(t+\frac{1}{\mu_1^p},0)$. We find a particular flat output [35,30] for the distal subsystem. Using a state predictor for this flat-output, we design a virtual actuation $V_{\mathrm{virt}}^{\mathrm{ref}}(t)$ that stabilizes the distal subsystem. This is done in Section 3
(2) Using a backstepping transformation, we design in Section 4.1 a first feedback law that tackles in-domain couplings in the proximal subsystem.
(3) Then, we propose a flatness-based feedforward tracking control design so that the output of the proximal subsystem tracks the function $V_{\mathrm{virt}}^{\mathrm{ref}}(t)$. This allows the stabilization of the distal subsystem. Note that the use of the backstepping transformation considerably simplifies this feedforward controller design. This is the purpose of Section 4.2.
(4) Finally, we show in Section 5 that the proposed control law stabilizes the global system (5)-(13).

The approach we develop in this paper corresponds to a first step towards a *recursive interconnected dynamics framework*. Such a framework is natural to design an explicit state-feedback control law that stabilizes a network of interconnected linear hyperbolic systems (potentially coupled with other classes of systems). Roughly speaking, the proposed control law is recursively obtained by considering stabilizing virtual inputs for the last subsystem and ensuring the proximal subsystem's output converges to this desired virtual inputs. This approach can be extended to a higher number of subsystems. The control design becomes more straightforward and is based on simple assumptions that can be independently verified for each subsystem. This new framework allows for a "plug-and-play"-like approach to control design since additional subsystems, satisfying similar conditions, can be added to the network using the same procedure.

Remark 1 *Note that we cannot simply adjust the backstepping approach derived in [4], where a robust stabilizing output feedback control law was designed for an underactuated cascade network of n systems of two heterodirectional linear first-order hyperbolic PDEs interconnected through their boundaries. Indeed, contrary to [4], we are dealing here with non-scalar subsystems. Thus, as it will appear in the sequel, while performing backstepping transformations, some in-domain coupling terms may remain in the target system .*

3 A preliminary result: stabilization of a delayed $n+m$ PDE system

The first step of our control strategy is to find a suitable function $V_{\mathrm{virt}}^{\mathrm{ref}}(t)$ such that the tracking of $Q_{dp}v^p(t+\frac{1}{\mu_1^p},0)$ to $V_{\mathrm{virt}}^{\mathrm{ref}}(t)$ guarantees the stabilization of the distal subsystem. Let us define the virtual actuation as follows:

$$V_{\mathrm{virt}}(t) = Q_{dp}v^p\left(t+\frac{1}{\mu_1^p},0\right). \tag{22}$$

Again, we do not choose $V_{\mathrm{virt}}(t) = Q_{dp}v^p(t,0)$ to guarantee the causality of the final control law ($\frac{1}{\mu_1^p}$ is the largest transport time between the real actuation and the distal subsystem). The distal subsystem now rewrites

$$\begin{aligned} \partial_t u^d(t,x) + \Lambda_d^+\partial_x u^d(t,x) &= \Sigma_d^{++}(x)u^d(t,x) \\ &\quad + \Sigma_d^{+-}(x)v^d(t,x), \\ \partial_t v^d(t,x) - \Lambda_d^-\partial_x v^d(t,x) &= \Sigma_d^{-+}(x)u^d(t,x) \end{aligned} \tag{23}$$

$$+\Sigma_d^{--}(x)v^d(t,x), \quad (24)$$

with the boundary conditions

$$u^d(t,-1) = R_d v^d(t,-1), \quad (25)$$

$$v^d(t,0) = Q_{dd}u^d(t,0) + V_{\mathrm{virt}}(t-\frac{1}{\mu_1^p}). \quad (26)$$

3.1 Time-delay formulation of the PDE system

The system (23)-(26) can be rewritten as a time-delay system of neutral type (difference system) with distributed delay terms. This is a straightforward application of [8]. More precisely, we have the following theorem.

Theorem 5 *There exist $L^\infty([0,\tau_d],\mathrm{R})$-functions G_{ij} (with $i \in \{1,...,n_d\}$ for $j \in \{1,...,m_d\}$) which only depend on the system parameters such that the stability properties of the system (23)-(26) are equivalent to those of the difference system defined for all $1 \leq i \leq m_d$ by*

$$(z_d)_i(t) = \sum_{k=1}^{n_d}\sum_{l=1}^{m_d}(Q_{dd})_{ik}(R_d)_{kl}(z_d)_l\left(t-\frac{1}{\lambda_k^d}-\frac{1}{\mu_l^d}\right)+ \sum_{l=1}^{m_d}\int_0^{\tau_d}G_{il}(\nu)(z_d)_l(t-\nu)d\nu + (V_{\mathrm{virt}})_i\left(t-\frac{1}{\mu_1^p}\right), \quad (27)$$

i.e., there exist two constants $C_1 > 0$ and $C_2 > 0$ and a constant $r > 0$ such that for all $t > \tau$,

$$C_1||z_{d[t]}||_r \leq ||(u_d,v_d)||_{L^2} \leq C_2||z_{d[t]}||. \quad (28)$$

Moreover, for all $t > \tau$, the state $(z_d(t))$ can be expressed as a function of $u^d(t,\cdot), v^d(t,\cdot)$, that is, there exists a linear operator $\mathcal{F}_d$ such that for all $t > \tau$, $(z_d(t)) = \mathcal{F}_d(u^d(t,\cdot),v^d(t,\cdot))$.

PROOF. The proof of this theorem can be found in [8]. It relies on successive backstepping transformations. □

It is important to mention that the difference system (27) and the original PDE system (23)-(26) have equivalent stability properties in the sense of (28). However, they are not strictly equivalent since (without any additional assumptions) it may be impossible to reconstruct part of the PDE states (initial condition for instance) from the state z_d . In that sense, the system (27) can be seen as a **comparison system** for the design of a stabilizing control law for the original PDE system (see, e.g., [36] and the references therein for some discussions for delay systems). Indeed, using (28), the exponential stability of the state z_d implies the one of the state (u_d, v_d) (since the initial condition of z_d can be expressed as a function of the initial condition of (u_d, v_d)). Consequently, we now consider the system (27) for the design of the control law V_{virt}. The resulting feedback law will then be expressed as a function of (u^d, v^d) using the operator $\mathcal{F}_d$.

Remark 6 *The time-delay equation (27) corresponds to a typical flatness based parametrization of the distal PDE system (23)-(26). In that sense, z_d can be seen as a particular flat output of the system. Similar parametrizations have been used in the literature for flatness-based open-loop design, controllability analysis, and closed-loop design (see, e.g., [44] for the case of a scalar second-order hyperbolic PDE, [37] for the control of a heavy chain). However, most of the existing literature results have been developed for PDEs with a low number of states.*

3.2 Predictor design

We now design a control law that stabilizes (27). Let us define $V_{\mathrm{virt}}^{\mathrm{ref}}$ as follows:

$$(V_{\mathrm{virt}}^{\mathrm{ref}})_i(t) = -\sum_{l=1}^{m_d}\int_0^{\tau_d}G_{il}(\nu)P_l(t,t-\nu)d\nu, \quad (29)$$

in which, for all $1 \leq i \leq m_d$, for $t \geq 0$ and $s \in [t-\frac{1}{\mu_1^p}-\tau_d, t]$, $P(t,s)$ is the state prediction (see [10,12])

$$P_i(t,s) = \quad (30)$$
$$\begin{cases} (z_d)_i(s+\frac{1}{\mu_1^p}) & \text{if } s \in \left[t-\tau_d-\frac{1}{\mu_1^p}, t-\frac{1}{\mu_1^p}\right] \\ \sum_{k=1}^{n_d}\sum_{l=1}^{m_d}(R_d)_{ik}(Q_{dd})_{kl}(P)_l(t,s-\frac{1}{\lambda_k^d}-\frac{1}{\mu_l^d}) & \\ \quad +\sum_{l=1}^{m_d}\int_0^{\tau_d}G_{il}(\nu)(P)_l(t,s-\nu)d\nu + \tilde{V}(s) & \\ & \text{otherwise.}\end{cases}$$

Observe that the function [4] $P(t,\cdot)$ is a $\frac{1}{\mu_1^p}$ units of time ahead prediction of the function $z_d : s \in [-\tau_d, 0] \mapsto z_d(t+s)$. Though its definition is implicit, through an integral relation of Volterra type, it is well-defined and unique, as the solution to the difference equation (27).

Theorem 7 *The closed-loop system consisting of the plant (27) and the control law $V_{\mathrm{virt}} = V_{\mathrm{virt}}^{\mathrm{ref}}$ (where $V_{\mathrm{virt}}^{\mathrm{ref}}$ is defined by (29)) is exponentially stable.*

---

[4] We write P as a function of two arguments to emphasize the fact that the prediction should be computed by incorporating measured delayed states available at time t, to improve its robustness in practice.

PROOF. The proof follows straightforwardly from the fact that $P(t,s) = z_d(s + \frac{1}{\mu_1^p})$ for any $s \in [t - \frac{1}{\mu_1^p} - \tau_d, t]$ which implies that $(V_{\text{virt}}^{\text{ref}})_i(t) = -\sum_{l=1}^{m_d} \int_0^{\tau_d} G_{il}(\nu)(z_d)_l(t+ \frac{1}{\mu_1^p} - \nu)d\nu$. By plugging this control law back into (27), we obtain

$$(z_d)_i(t) = \sum_{k=1}^{n_d}\sum_{l=1}^{m_d}(R_d)_{ik}(Q_{dd})_{kl}(z_d)_l(t - \frac{1}{\lambda_k^d} - \frac{1}{\mu_l^d}). \tag{31}$$

Due to Assumption 1, equation (31) is exponentially stable. More precisely, equation (31) corresponds to equation (19) in which all the terms that come from the proximal subsystem are set to zero. Since (19) is exponentially stable, equation (31) is exponentially stable as well due to the variations of constants formula [26] (considering the proximal terms in (19) as an input that goes to zero). This concludes the proof. □

The control law (29) (and the predictor (30)) can be expressed as a function of the original distal state (u_d, v_d) using the transformation $\mathcal{F}_d$. This is a necessary step in view of a practical implementation. However, to avoid long computations (that can be found in [8]) we choose not to give it. Note that we voluntarily choose not to cancel the pointwise delay term in the closed-loop dynamics to guarantee that the transfer function relating $\tilde{V}$ to z_d is strictly proper, a characteristic which reveals necessary for robustness purposes.

4 Tracking of the virtual input $V_{\text{virt}}^{\text{ref}}(\mathbf{t})$

The objective of this section is to design a control law $V(t)$ such that the virtual input $Q_{dp}v^p(t + \frac{1}{\mu_1^p}, 0) = V_{\text{virt}}$ converges to $V_{\text{virt}}^{\text{ref}}$. This will guarantee the stabilization of the distal subsystem. To do so, we use a backstepping transformation that moves most of the in-domain coupling terms located in equation (6) to the actuated boundary (only some non-local coupling terms remain in the target system). Canceling these terms will guarantee a clear actuation path and make the tracking of the virtual input easier.

4.1 Backstepping transformation

Let us consider the backstepping transformation defined by

$$\begin{aligned}\beta^p(t,x) &= v^p(t,x) - \int_0^x K_p^u(x,\xi)u^p(t,\xi)d\xi \\ &- \int_0^x K_p^v(x,\xi)v^p(t,\xi)d\xi - \int_{-1}^0 K_d^u(x,\xi)u^d(t,\xi)d\xi \\ &- \int_{-1}^0 K_d^v(x,\xi)v^d(t,\xi)d\xi,\end{aligned} \tag{32}$$

where the kernels K_p^u and K_p^v are $L^\infty(\mathcal{T}_b)$-functions while the kernels K_d^u and K_d^v are $L^\infty(\mathcal{U})$-functions (the spaces $\mathcal{T}_b$ and $\mathcal{U}$ being defined in equations (1) and (2)). They satisfy the following set of PDEs on their respective domains of definition

$$\Lambda_p^-\partial_x K_p^u(x,\xi) - \partial_\xi K_p^u(x,\xi)\Lambda_p^+ = K_p^u(x,\xi)\Sigma_p^{++}(\xi) + K_p^v(x,\xi)\Sigma_p^{-+}(\xi), \tag{33}$$

$$\Lambda_p^-\partial_x K_p^v(x,\xi) + \partial_\xi K_p^v(x,\xi)\Lambda_p^- = K_p^u(x,\xi)\Sigma_p^{+-}(\xi) + K_p^v(x,\xi)\Sigma_p^{--}(\xi), \tag{34}$$

$$\Lambda_p^-\partial_x K_d^u(x,\xi) - \partial_\xi K_d^u(x,\xi)\Lambda_d^+ = K_d^u(x,\xi)\Sigma_d^{++}(\xi) + K_d^v(x,\xi)\Sigma_d^{-+}(\xi), \tag{35}$$

$$\Lambda_p^-\partial_x K_d^v(x,\xi) + \partial_\xi K_d^v(x,\xi)\Lambda_d^- = K_d^u(x,\xi)\Sigma_d^{+-}(\xi) + K_d^v(x,\xi)\Sigma_d^{--}(\xi), \tag{36}$$

along with the following set of boundary conditions

$$K_d^u(x,0)\Lambda_d^+ = K_d^v(x,0)\Lambda_d^- Q_{dd} + K_p^u(x,0)\Lambda_p^+ Q_{pd}, \tag{37}$$

$$K_d^v(x,-1)\Lambda_d^- = K_d^u(x,-1)\Lambda_d^+ R_d, \tag{38}$$

$$K_d^u(0,\xi) = 0, \quad K_d^v(0,\xi) = 0, \tag{39}$$

$$\Lambda_p^- K_p^u(x,x) + K_p^u(x,x)\Lambda_p^+ = -\Sigma_p^{-+}(x), \tag{40}$$

$$\Lambda_p^- K_p^v(x,x) - K_p^v(x,x)\Lambda_p^- = -\Sigma_p^{--}(x), \tag{41}$$

$$(K_p^v)_{ij}(x,0)\Lambda_p^- = (K_p^u(x,0)\Lambda_p^+ Q_{pp})_{ij} + (K_d^v(x,0)\Lambda_d^- Q_{dp})_{ij}, \text{ if } i \geq j. \tag{42}$$

To ensure well-posedness of the equations, we add artificial boundary conditions for $(K_p^v)_{ij}(i < j)$. We have the following lemma that assesses the well-posedness of the kernel equations (33)-(42)

Lemma 8 *Consider the system* (33)-(42)*. There exists a unique solution K_p^u and K_p^v in $L^\infty(\mathcal{T}_b)$ and K_d^u and K_d^v in $L^\infty(\mathcal{U})$.*

PROOF. Let us define the sequence x_n $(n \in \mathrm{N})$ by $x_n = \min\{n\frac{\mu_1^p}{\lambda_{n_d}^d}, 1\}$, where we recall that $\lambda_{n_d}^d$ is the largest positive velocity for the distal subsystem, while μ_1^p is the smallest negative velocity for the proximal subsystem. They are defined in equation (9). The sequence x_n converges in a finite number of iteration to 1. Let us consider the rectangular domain $\mathcal{R}_n = \{(x,\xi) \in [0,1] \times [-1,0],\ x_n \leq x \leq x_{n+1}\}$. Note the union of $\mathcal{R}_n$ corresponds to $\mathcal{U}$. We will show that the kernels equation admits a unique solution using an induction argument. Let us consider the case $n = 0$. Let us cut the square domain $\mathcal{R}_0$ in two triangular domains $\mathcal{R}_u$ and $\mathcal{R}_l$ defined by

$$\mathcal{R}_u = \{(x,\xi) \in [0,1] \times [-1,0],\ x \geq -x_1\xi\}, \quad (43)$$
$$\mathcal{R}_b = \{(x,\xi) \in [0,1] \times [-1,0],\ x \leq -x_1\xi\}. \quad (44)$$

The union of $\mathcal{R}_u$ and $\mathcal{R}_b$ corresponds to $\mathcal{R}_0$. On the triangular domain $\mathcal{R}_b$, using the kernel PDEs (35)-(36) and the boundary conditions (38) and (39), we can apply [22, Theorem 3.2] to prove the existence of the L^∞-kernels K_d^u and K_d^v. This means that the kernels K_d^u and K_d^v are perfectly defined on the boundary $x = -x_1\xi$ of $\mathcal{R}_u$. We now perform a change of variables in order to express the kernels K_p^u and K_p^v on the domain $\mathcal{R}_u$. Let us define the kernels $\bar{K}_u^p$ and $\bar{K}_b^p$ on $\mathcal{R}_u$ by

$$\bar{K}_u^p(x,\xi) = \bar{K}_u^p(x, -\frac{\xi}{x_1}),\ \bar{K}_v^p(x,\xi) = \bar{K}_v^p(x, -\frac{\xi}{x_1}).$$

We can rewrite the kernel PDEs (33)-(34) for these new kernels. The boundary conditions (37) and (42) remain unchanged. Using these boundary conditions, and the value of all the kernels on the line $x = -x_1\xi$, it becomes possible to apply [22, Theorem 3.2] on this new triangular domain. This guarantees the well-posedness of the kernels K_d^u, K_d^v, $\bar{K}_p^u$ and $\bar{K}_p^v$ on $\mathcal{R}_u$. This implies the existence of L^∞-functions K_d^u and K_d^v on the domain $\mathcal{R}_0$ and of L^∞ functions K_p^u and K_p^v on $\{(x,\xi) \in \mathcal{T}_b,\ x \leq x_1\}$. Repeating the procedure on each $\mathcal{R}_n$ leads to the expected result. □

The transformation (32) is a Volterra transformation to which an affine term that depends on the distal state is added. Consequently, it is invertible [45] and there exist L_p^u and L_p^v in $L^\infty(\mathcal{T}_b)$ and L_d^u and L_d^v in $L^\infty(\mathcal{U})$ such that

$$\begin{aligned} v^p(t,x) &= \beta^p(t,x) - \int_0^x L_p^u(x,\xi)u^p(t,\xi)d\xi \\ &- \int_0^x L_p^v(x,\xi)\beta^p(t,\xi)d\xi - \int_{-1}^0 L_d^v(x,\xi)u^d(t,\xi)d\xi \\ &- \int_{-1}^0 L_d^v(x,\xi)v^d(t,\xi)d\xi. \end{aligned} \quad (45)$$

Differentiating (32) with respect to time and space and integrating by parts, the original system (5)-(13) is mapped to the following target system, schematically pictured in Figure 3,

$$\begin{aligned} \partial_t u^p(t,x) + \Lambda_p^+ \partial_x u^p(t,x) &= \Sigma_p^{++}(x)u^p(t,x) \\ &+ \Sigma_p^{+-}(x)\beta^p(t,x) - I_1(x,u^p,\beta^p,u^d,v^d), \quad (46) \\ \partial_t \beta^p(t,x) - \Lambda_p^- \partial_x \beta^p(t,x) &= \Omega(x)\beta^p(t,0), \quad (47) \\ \partial_t u^d(t,x) + \Lambda_d^+ \partial_x u^d(t,x) &= \Sigma_d^{++}(x)u^d(t,x) \\ &+ \Sigma_d^{+-}(x)v^d(t,x), \quad (48) \\ \partial_t v^d(t,x) - \Lambda_d^- \partial_x v^d(t,x) &= \Sigma_d^{-+}(x)u^d(t,x) \\ &+ \Sigma_d^{--}(x)v^d(t,x), \quad (49) \end{aligned}$$

with the boundary conditions

$$u^d(t,-1) = R_d v^d(t,-1), \quad (50)$$
$$v^d(t,0) = Q_{dp}\beta^p(t,0) + Q_{dd}u^d(t,0), \quad (51)$$
$$u^p(t,0) = Q_{pp}\beta^p(t,0) + Q_{pd}u^d(t,0), \quad (52)$$
$$\beta^p(t,1) = R_p u^p(t,1) + V(t) - I_2(u^p,v^p,u^d,v^d), \quad (53)$$

where the function $\Omega(x)$ is defined by

$$\begin{aligned} \Omega(x) = K_p^v(x,0)\Lambda_p^- &- K_d^v(x,0)\Lambda_d^- Q_{dp} \\ &- K_p^u(x,0)\Lambda_p^+ Q_{pp}. \end{aligned} \quad (54)$$

Note that this matrix is **upper-triangular** due to (42). The integral terms I_1 and I_2 are defined by

$$\begin{aligned} I_1(x,u^p,\beta^p,u^d,v^d) &= \Sigma_p^{+-}(x)\Big(\int_0^x L_p^u(x,\xi)u^p(\cdot,\xi)d\xi) \\ &+ \int_0^x L_p^v(x,\xi)\beta^p(\cdot,\xi)d\xi + \int_{-1}^0 L_d^v(x,\xi)u^d(\cdot,\xi)d\xi \\ &+ \int_{-1}^0 L_d^v(x,\xi)v^d(\cdot,\xi)d\xi), \end{aligned} \quad (55)$$

$$\begin{aligned} I_2(u^p,v^p,u^d,v^d) &= \int_0^1 K_p^u(1,\xi)u^p(\cdot,\xi)d\xi \\ &+ \int_0^1 K_p^v(1,\xi)v^p(\cdot,,\xi)d\xi \\ &+ \int_{-1}^0 \left[K_d^v(1,\xi)u^d(\cdot,,\xi) + K_d^v(1,\xi)v^d(\cdot,,\xi)\right]d\xi. \end{aligned} \quad (56)$$

The local terms originally present in equation (6) have been replaced by a non-local term that depends on $\beta^p(t,0)$. This structure will help us to track the desired virtual input. In what follows we decide to cancel the right-hand part of the boundary condition (53) using the actuation. More precisely, we choose $V(t) = \bar{V}(t) - R_p u^p(t,1) + I_2(u^p,v^p,u^d,v^d)$. where $\bar{V}$ is our new control input that will be used to track the virtual input previously defined. With this new control input, the boundary condition (53) rewrites $\beta^p(t,1) = \bar{V}(t)$. Although, this choice of control law considerably simplifies the analysis, it is worth noticing that such an approach requires the cancelation of the reflection term $R_p u^p(t,1)$. As shown in [5], this may have major consequences regarding the robustness margins of the closed-loop system. More precisely, the corresponding feedback law is not proper which may lead to vanishing delay margins. To avoid this problem and make the control law strictly proper, we choose to combine it with a well-tuned low pass filter, the design of which is detailed in Section 6.

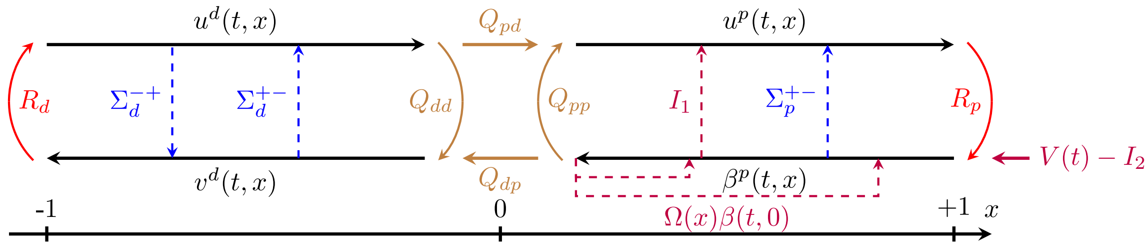

Fig. 3. Schematic representation of the target system (46)-(53).

4.2 Tracking control design

The backstepping transformation (32) has allowed us to map the original system (5)-(13) to the simpler target system (46)-(53). A closer analysis to this target system shows that once the distal subsystem has been stabilized, then the proximal subsystem also converges to zero. Indeed, having $u^d \equiv 0$ and $v^d \equiv 0$ in (46)-(47), the states u^p and v^p converge to zero due to the cascade structure of (46)-(47) (the matrix Ω is upper-triangular) [28]. Thus, our objective is to guarantee the exponential stabilization of the distal subsystem. In other words, we want the function $Q_{dp}\beta^p(t+\frac{1}{\mu_1^p},0)$ to track $V_{\text{virt}}^{\text{ref}}$ (29). In this section, we propose a flatness-based feedforward tracking control design for $\bar{V}$. More precisely, inspired by [28], we show in the following lemma that the the boundary value $\beta^p(t,0)$ corresponds to a flat output [35,30], which we later use for trajectory planning similarly to [34].

Lemma 9 *Let us consider the control law $\bar{V}(t)$ defined for all $1 \leq i \leq m_p$ by*

$$\bar{V}_i(t) = \zeta_i(t+\frac{1}{\mu_i^p}) - \sum_{j=i+1}^{m_p}\int_0^{\frac{1}{\mu_i^p}} \Omega_{i,j}(\mu_i^p\nu)\zeta_j(t+\frac{1}{\mu_i^p}-\nu)d\nu, \quad (57)$$

where ζ is an arbitrary known function. Then, for any $t \geq \sum_{j=1}^{m_p}\frac{1}{\mu_j^p}$, we have $\beta^p(t,0) \equiv \zeta(t)$.

PROOF. The proof is inspired by [28, Theorem 5.1]. Applying the method of characteristics to equation (47), we obtain for all $1 \leq i \leq m_p$ and for all $t > \frac{1}{\mu_i^p}$

$$\beta_i^p(t,0) = \bar{V}_{m_p}(t-\frac{1}{\mu_i^p}) + \sum_{k=i+1}^{m_d}\int_0^{\frac{1}{\mu_i^p}} \Omega_{ik}(\mu_i\nu)\beta_k^p(t-\nu,0)d\nu. \quad (58)$$

Let us consider the case $i = m_p$. For $t \geq \frac{1}{\mu_{m_p}^p}$ we have $\beta_{m_p}^p(t,0) = \bar{V}_{m_p}\left(t - \frac{1}{\mu_{m_p}^p}\right)$. Choosing $\bar{V}_{m_p}(t) = \zeta_{m_p}(t+\frac{1}{\mu_{m_p}^p})$ guarantees $\beta_{m_p}^p(t,0) = \zeta_{m_p}(t)$ for $t \geq \frac{1}{\mu_{m_p}^p}$. Let us now consider the case $i = m_p - 1$. For $t \geq \frac{1}{\mu_{(m_p-1)}^p}$, we have, using equation (58),

$$\bar{V}_{m_p-1}(t) = \beta_{m_p-1}^p\left(t+\frac{1}{\mu_{(m_p-1)}^p},0\right) - \int_0^{\frac{1}{\mu_i^p}} \Omega_{m_p-1,m_p}(\mu_{m_p-1}^p\nu)\beta_{m_p}^p\left(t+\frac{1}{\mu_{(m_p-1)}^p}-\nu,0\right)d\nu. \quad (59)$$

When $t \geq \frac{1}{\mu_{m_p}^p}$ we have $\beta_{m_p}^p(t,0) = \zeta_{m_p}(t)$. Thus, choosing

$$\bar{V}_{m_p-1}(t) = \zeta_{m_p-1}\left(t+\frac{1}{\mu_{(m_p-1)}^p}\right) - \int_0^{\frac{1}{\mu_i^p}} \Omega_{m_p-1,m_p}(\mu_{m_p-1}^p\nu)\zeta_{m_p}\left(t+\frac{1}{\mu_{(m_p-1)}^p}-\nu\right)d\nu, \quad (60)$$

we obtain $\beta_{m_p-1}^p(t,0) = \zeta_{m_p-1}(t)$ for $t \geq \frac{1}{\mu_{(m_p-1)}^p} + \frac{1}{\mu_{m_p}^p}$ (as we have to wait an extra $\frac{1}{\mu_{(m_p-1)}^p}$ for the control input to propagate). Iterating the procedures gives the expected result. $\square$

Consequently, any arbitrary function ζ can be tracked using the input $\bar{V}$. However, this requires knowing future values of the function ζ, which can lead to a non-causal control law. This explains why we have defined $V_{\text{virt}}(t) = Q_{dp}v^p(t+\frac{1}{\mu_1^p},0)$ instead of simply choosing $Q_{dp}v^p(t,0)$ as the virtual input. We can then design a feedforward controller for the finite-time stable cascade with the new input $\bar{V}$. Note that the tracking error resulting from incompatible initial conditions decays in finite time with such a control law.

5 Stabilizing control law

We are now able to design a stabilizing control law for the system (5)-(13). Indeed, it is sufficient to choose the function ζ as the reference trajectory for the flat output

$\beta^p(t,0)$ to be tracked. This reference trajectory should stabilize the distal subsystem.

Theorem 10 *Let us consider the function I_2 defined by* (56) *and the control law $V_{\text{virt}}^{\text{ref}}$, defined in* (29)*. Let us consider the function ζ defined by*

$$\zeta(t) = Q_{dp}^T(Q_{dp}Q_{dp}^T)^{-1}V_{\text{virt}}(t - \frac{1}{\mu_1^p}). \tag{61}$$

Then, the control law $V(t)$ defined for all $1 \leq i \leq n_p$ by

$$V_i(t) = -(R_p u^p(t,1))_i + (I_2(u^p, v^p, u^d, v^d))_i + \zeta_i(t + \frac{1}{\mu_i^p}) - \sum_{j=i+1}^{m_p} \int_0^{\frac{1}{\mu_i^p}} \Omega_{i,j}(\mu_i^p \nu)\zeta_j(t + \frac{1}{\mu_i^p} - \nu)d\nu, \tag{62}$$

stabilizes the system (5)-(13) *in the sense of the L^2-norm.*

PROOF. The matrix $Q_{dp}^T(Q_{dp}Q_{dp}^T)^{-1}$ is well-defined due to Assumption 2. Note that the control law $V(t)$ defined by equation (62) is causal. Using the backstepping transformation (32), we can map the original system (5)-(8) to the target system (46)-(53). Using Lemma 9, the control law (62) ensures the convergence of $v^p(t,0)$ to the function $\zeta(t)$ (defined in (61)) in finite time. Then, it implies that $Q_{dp}v^p(t,0) = V_{\text{virt}}^{\text{ref}}\left(t - \frac{1}{\mu_1^p}\right)$. Thus, applying Theorem 7, we have the L^2-exponential convergence of the state u^d and v^d to zero. This implies the convergence of ζ and of $v^p(t,0)$ to zero. Consequently, due to the transport structure of (47), the state $\beta^p(t,\cdot)$ exponentially converges to zero. Using the boundary condition (52), the state $u^p(t,0)$ exponentially converges to zero. Thus, the state $u^p(t,x)$ now converges to zero, since all the terms that do not depend on u^p in equation (46) converge to zero and since the homogeneous equation is exponentially stable [28]. Using the invertibility of the backstepping transformation (32), this implies the exponential convergence of the state $v^p(t,\cdot)$ to zero. Consequently, the state (u^p, v^p, u^d, v^d) exponentially converges to zero in the sense of the L^2-norm. This concludes the proof. □

Though stabilizing, the control law (62) presents the drawback of canceling (among others) the proximal reflection term $R_p u^p(t,1)$. In that sense, it is not a strictly proper feedback law. It has been shown in [5] that such a cancelation may raise important issues concerning the existence of robustness margins at high frequencies. To overcome this issue, a first solution could be to cancel only a part of the reflection terms in (62). This could be achieved by modifying the tracking design with a convolutional procedure as performed e.g. in [6,44]. Although somehow more standard, this approach presents the drawback of not distinguishing the effects of high and low frequencies in terms of stability and robustness. In addition, such an approach can be difficult to implement when considering chains with a higher number of subsystems. Instead, we now propose an approach based on the filtering of the control.

6 Robustness aspects

In this section, we combine the control law V(t) with a low-pass filter to make it strictly proper, while guaranteeing the nominal stabilization. We then show that the resulting closed-loop system is robust with respect to delays in the loop and uncertainties on the parameters. The analysis we propose will be done in the Laplace domain.

6.1 Neutral formulation and Laplace transform

Regarding robustness aspects, it may be easier to work with the system's neutral formulation rather than the PDE formulation. Then, we rewrite the interconnected system (16)-(18) as a neutral system. We have the following lemma, which is a direct application of [8].

Lemma 11 *There exist $L^\infty([0,\tau],\mathrm{R})$-functions H_{ij} (with $i \in \{1,...,n_p+m_d\}$ and $j \in \{1,...,m_p+n_d\}$) which only depend on the parameters of the system* (16)-(18) *such that the stability properties of the system* (16)-(18) *are equivalent to those of the difference system defined for all $1 \leq i \leq m_p + n_d$ by*

$$z_i(t) = \sum_{k=1}^{n_p+m_d} \sum_{l=1}^{m_p+n_d} R_{ik}Q_{kl}(z_d)_l(t - \frac{1}{\lambda_k} - \frac{1}{\mu_l}) + \sum_{l=1}^{m_p+n_d} \int_0^\tau H_{il}(\nu)z_l(t-\nu)d\nu + A_iV_i(t), \tag{63}$$

i.e., there exist two constants $D_1 > 0$ and $D_2 > 0$ and a constant $r_0 > 0$ such that for all $t > \tau$,

$$D_1||z_{[t]}||_{r_0} \leq ||(u,v)||_{L^2} \leq D_2||z_{[t]}||. \tag{64}$$

There exists a bounded transformation $\mathcal{F}_0$ such that for all $t > \tau$, $z(t) = \mathcal{F}_0(u(t,\cdot), v(t,\cdot))$. Moreover, for all $x \in [0,1]$, there exist $L^\infty([0,\tau],\mathrm{R})$-functions H_{il}^u and H_{jl}^v such that for all $t \geq \tau$, for $1 \leq j \leq m_p + n_d$, for all $1 \leq i \leq m_d + n_p$, we have

$$u_i(t,x) = \sum_{k=1}^{m_p+n_d} Q_{ik}z_k(t - \frac{x}{\lambda_i} - \frac{1}{\mu_k}) + \sum_{l=1}^{m_p+n_d} \int_0^\tau H_{il}^u(\nu,x)z_l(t-\nu)d\nu. \tag{65}$$

$$v_j(t,x) = z_j(t - \frac{1-x}{\mu_j})$$

$$+\sum_{l=1}^{m_p+n_d}\int_0^\tau H^v_{jl}(\nu,x)z_l(t-\nu)d\nu, \qquad (66)$$

PROOF. The proof of this lemma can be found following [8]. It relies on successive backstepping transformations and on the method of characteristics. □

We can now rewrite the system (63) in the Laplace domain. We have the following lemma

Lemma 12 *There exist two holomorphic matrix-functions $F(s)$ and $H(s)$ such that the Laplace transform of the state z verifies*

$$\hat{z}(s) = F(s)\hat{z}(s) + H_1(s)\hat{z}(s) + A\hat{V}(s). \qquad (67)$$

in which the function H_1 is strictly proper on C^+, that is for every $s \in \mathrm{C}^+$, $\lim_{|s|\to+\infty} H_1(s) = 0$.

PROOF. Using equation (63), we immediately obtain

$$F_{ij}(s) = \sum_{k=1} R_{ik}Q_{kj}\mathrm{e}^{-(\frac{1}{\lambda_k}+\frac{1}{\mu_j})s}, \qquad (68)$$

$$(H_1)_{ij}(s) = \int_0^\tau H_{ij}(\nu)\mathrm{e}^{-\nu s}d\nu, \qquad (69)$$

Using Riemann-Lebesgue's lemma, we can show that $(H_1)_{ij}(s)$ goes to zero when $|s| \to +\infty$ as long as $\mathrm{Re}(s) \geq 0$. This proves that H_1 is strictly proper. □

We now express the states $u(t,x)$ and $v(t,x)$ in the Laplace domain as functions of $\hat{z}(s)$.

Lemma 13 *For all $x \in [0,1]$, there exist holomorphic functions $F_u(s,x)$, $F_v(s,x)$, $H^u_1(s,x)$ and $H^v_1(s,x)$, H^u_1 and H^v_1 being strictly proper on C^+, such that the Laplace transforms of the states $u(t,x)$ and $v(t,x)$ verify*

$$\hat{u}(s,x) = F^u(s,x)\hat{z}(s) + H^u_1(s,x)\hat{z}(s) \qquad (70)$$
$$\hat{v}(s,x) = F^v(s,x)\hat{z}(s) + H^v_1(s,x)\hat{z}(s) \qquad (71)$$

Moreover, $RF^u(s,1) = F(s)$.

PROOF. Taking the Laplace transform of (66), we obtain for $1 \leq i \leq m_p + n_d$

$$\hat{v}_i(s,x) = \mathrm{e}^{-\frac{1-x}{\mu_i}s}\hat{z}_i(s) + \sum_{l=1}^{m_p+n_d}(\int_0^\tau H^v_{il}(\nu,x)\mathrm{e}^{-\nu s}d\nu)\hat{z}_l(s).$$

This implies that for all $1 \leq i,j \leq m_p+n_d$ $(F^v)_{ij}(s,x) = \mathrm{e}^{-\frac{1-x}{\mu_i}s}$ if $i=j$ and $(F^v)_{ij}(s,x) = 0$ otherwise. In the mean time $(H^v_1)_{ij}(s,x) = (\int_0^\tau H^v_{ij}(\nu,x)\mathrm{e}^{-\nu s}d\nu)$. Using Riemann-Lebesgue's lemma, we have that H^v_1 is strictly proper. Similar computations can be done to obtain equation (70). The equality $RF^u(s,1) = F(s)$ can be obtained by direct computations. □

We now rewrite the stabilizing control law $V(t)$ defined in equation (62) in terms of the state z.

Lemma 14 *There exists a strictly proper holomorphic function $P(s)$ such that the Laplace transform of the control law $V(t)$ defined by equation (62) rewrites*

$$\hat{V}(s) = (P(s) - A^T F(s))\hat{z}(s), \qquad (72)$$

PROOF. Let us consider the different terms that appear in equation (62) and compute their Laplace transform. We have $R_p u^p(t,1) = A^T Ru(t,1)$ Thus, taking the Laplace transform of (6.1) and using (70), we obtain

$$R_p\hat{u}^p(s,1) = A^T(F(s) + RH^u_1(s,1))\hat{z}(s). \qquad (73)$$

Using (70), we directly obtain

$$\int_0^1 K^u(1,\xi)u(t,\xi)d\xi = \int_0^1 K^u(1,\xi)(F^u(s,\xi) + H^u_1(s,\xi))d\xi\hat{z}(s).$$

Again, due to Riemann-Lebesgues' lemma, the function $\int_0^1 K^u(1,\xi)(F^u(s,\xi)+H^u_1(s,\xi))d\xi$ is strictly proper. Repeating the procedure for all the terms that form I_2 (defined by (56)), we can obtain $\hat{I}_2(s) = H_I(s)\hat{z}(s)$, where H_I is strictly proper. The function $V^{\mathrm{ref}}_{\mathrm{virt}}$ is strictly proper (since it only cancels integral terms). Thus, there exists a strictly proper function H_V such that $\hat{V}^{\mathrm{ref}}_{\mathrm{virt}}(s) = H_V(s)\hat{z}_d(s)$. Using the operator $\mathcal{F}_d$ defined in Theorem 5 and Lemma 13, we obtain that $\hat{V}^{\mathrm{ref}}_{\mathrm{virt}}(s) = \hat{H}_V(s)\hat{z}(s)$ with H_V being strictly proper. Using the fact that the transfer function between ζ and $V^{\mathrm{ref}}_{\mathrm{virt}}$ is bounded, we deduce that the transfer function between $\hat{z}$ and ζ is strictly proper. Combining the different transfer functions leads to the definition of P and to the expected result. □

6.2 Low-pass filtering of the actuation

We now show that there exists a low-pass filter $\hat{w}(s)$ such that the filtered control law $\hat{w}(s)\hat{V}(s)$ still guarantees the stabilization of the system (63) (and consequently of (16)-(18)). The objective behind this filtering is to make the control law strictly proper. We first state an useful lemma

Lemma 15 *Consider F, H_1 and P defined in* (67) *and* (72). *There exist $\epsilon_0 > 0$ and $1 > \epsilon_1 > 0$ such that, for any $s \in \mathrm{C}^+$*

$$\bar{\sigma}(F(s)) < \epsilon_1 < 1. \tag{74}$$
$$\underline{\sigma}(\mathrm{Id} - F - H_1 - A(P - A^T F)) > \epsilon_0, \tag{75}$$

Furthermore, there exists $M > 0$ such that for any $s \in \mathrm{C}^+$ with $|s| > M$ we have

$$\bar{\sigma}(P(s)) < 1 - \epsilon_1 \tag{76}$$
$$\underline{\sigma}(\mathrm{Id} - H_1(s) - AP(s)) > \epsilon_1 + \bar{\sigma}(P(s)) \tag{77}$$

PROOF. The first inequality is a direct consequence of Assumption 1. Due to Theorem 10, the system (16)-(18) with the control law (62) is exponentially stable. This in turns implies (using Lemma 11) the exponential stability of the state $z(t)$, solution of (63). This means that the characteristic equation of the closed-loop system is lower-bounded on the complex right-half plane [26]. Injecting (72) in (67), we obtain the closed-loop system

$$\hat{z}(s) = (F(s) + H_1(s) + A(P(s) - A^T F(s)))\hat{z}(s), \tag{78}$$

whose characteristic equation is given by

$$\det(\mathrm{Id} - F(s) - H_1(s) - A(P(s) - A^T F(s))) = 0.$$

This implies the second inequality. Since H_1 and P are strictly proper, we have

$$\underline{\sigma}(\mathrm{Id} - H_1(s) - AP(s)) \underset{|s|\to+\infty}{\overset{s\in\mathrm{C}^+}{\longrightarrow}} \underline{\sigma}(\mathrm{Id}) = 1. \tag{79}$$

Since P is strictly proper, there exists $M_0 > 0$ such that for any $s \in \mathrm{C}^+$ with $|s| > M_0$ we have

$$\epsilon_1 + \bar{\sigma}(P(s)) < 1. \tag{80}$$

Combining this inequality and (79) implies the last inequality. □

Theorem 16 *Let $w(s)$ be any low-pass filter such that for all $s \in \mathrm{C}^+$*

$$\begin{cases} |1 - w(s)| < \min(1, \frac{\epsilon_0}{\epsilon_1 + \bar{\sigma}(P(s))}) \text{ if } |s| \le M, \\ |1 - w(s)| < 1 \text{ if } |s| > M, \end{cases} \tag{81}$$

where H_1 and P are defined in (69) *and* (72) *and M, ϵ_0 and ϵ_1 are defined in Lemma 15. Let us consider the control law defined in the Laplace domain by*

$$\hat{V}_f(s) = w(s)(P(s) - A^T F(s))\hat{z}(s) \tag{82}$$

Then, this control law delay-robustly stabilizes the system (63).

PROOF. Definition of w: Note that the filter w is well defined since the second condition of (81) allows the convergence of w to zero for large $|s|$. The first condition of (81) implies that w is close to one for sufficiently small $|s|$.

Stabilization: Let us prove that the new control law V_f still guarantees the stabilization of the system (63). Plugging the control law inside (27), the characteristic equation of the closed-loop system now rewrites

$$\det(\mathrm{Id} - F(s) - H_1(s) - w(s)A(P(s) - A^T F(s))) = 0.$$

To ease the notations, we will denote $P_1(s) = \mathrm{Id} - F(s) - H_1(s) - w(s)A(P(s) - A^T F(s))$ so that the characteristic equation rewrites $\det(P_1(s)) = 0$. To prove that the closed-loop system is exponentially stable, we need to show that this characteristic equation does not have any solution on C^+. By contradiction, let us assume that there exists $s \in \mathrm{C}^+$ such that $\det(P_1(s)) = 0$.

If $|s| > M$, we have $P_1(s) = \mathrm{Id} - H_1(s) - AP(s) - (\mathrm{Id} - w(s)AA^T)F(s) + (1 - w(s))AP(s)$. We have

$$\bar{\sigma}(\mathrm{Id} - w(s)AA^T) \le 1, \tag{83}$$
$$\bar{\sigma}((1 - w(s))AP(s)) \le \bar{\sigma}(P(s)). \tag{84}$$

This implies,

$$\begin{aligned} &\bar{\sigma}((\mathrm{Id} - w(s)AA^T)F(s) + (1 - w(s))AP(s)) \\ &\quad \le \epsilon_1 + \bar{\sigma}(P(s)) < \underline{\sigma}(\mathrm{Id} - H_1(s) - AP(s)), \end{aligned} \tag{85}$$

In the mean time, since $\det(P_1(s)) = 0$, we have $\underline{\sigma}(P_1(s)) = 0$. Thus, we must have $\underline{\sigma}(\mathrm{Id} - H_1(s) - AP(s)) \le \bar{\sigma}((\mathrm{Id} - w(s)AA^T)F(s) + (1 - w(s))AP(s))$. This is a contradiction with equation (85).

If $|s| \le M$. We have

$$\begin{aligned} P_1(s) = \mathrm{Id} &- F(s) - H_1(s) - A(P(s) - A^T F(s)) \\ &+ (1 - w(s))A(P(s) - A^T F(s)). \end{aligned}$$

Using (81), we have

$$\begin{aligned} \bar{\sigma}((1 - w(s))A(P(s) - A^T F(s))) &\le |1 - w(s)|(\bar{\sigma}(F(s)) \\ &+ \bar{\sigma}(P(s))) < \epsilon_0. \end{aligned} \tag{86}$$

Since $\det(P_1(s)) = 0$ (and consequently $\underline{\sigma}(P_1(s)) = 0$), we obtain $\underline{\sigma}(\mathrm{Id} - F(s) - H_1(s) - A(P(s) - A^T F(s))) \le \bar{\sigma}((1 - w(s))A(P(s) - A^T F(s))) < \epsilon_0$. This is a contradiction with (75). Consequently, the characteristic equation cannot be satisfied on C^+. This proves the stability of the closed-loop system [26], since the asymptotic vertical chain of zeros of this characteristic equation cannot be the imaginary axis due to Assumption 1. □

The following lemma gives an explicit method to design a suitable low-pass filter.

Lemma 17 *There exists* $\nu_0 > 0$ *such that the low-pass filter defined for all* $s \in \mathrm{C}^+$ *by* $w(s) = \frac{1}{1+\nu_0 s}$ *satisfies equation* (81)*.*

PROOF. The filter $w(s)$ verifies $|1 - w(s)| < 1$ on C^+. The set $\mathcal{S} = \{s \in \mathrm{C}^+, |s| \leq \bar{M}\}$ is a compact set. Then, we can define $\tilde{M}$ as

$$\tilde{M} = \inf_{s \in \mathcal{S}} \left(\frac{\epsilon_0}{\epsilon_1 + \bar{\sigma}(P(s))}\right).$$

The constant $\tilde{M}$ is positive by definition. It is then sufficient to choose $\nu_0 < \frac{\tilde{M}}{M}$. Indeed, on the set $\mathcal{S}$, we obtain

$$|1 - w(s)| = |\frac{\nu_0 s}{1 + \nu_0 s}| \leq |\nu_0 s| \leq |\nu_0| M < \tilde{M}.$$

By definition of $\tilde{M}$, condition (81) is then always satisfied.

This lemma and its proof give a constructive procedure to design the low-pass filter introduced in Theorem 16. Note that this design requires the estimation of the parameters ϵ_0 and ϵ_1. This can be obtained numerically off-line from their definitions in (74)–(75).

6.3 Delay-robustness and w-stability

We are now able to assess the robustness properties of our system.

Theorem 18 *The control law* (82) *delay-robustly stabilizes the system* (5)-(13)*. The resulting closed-loop system is w-stable.*

PROOF. Note that the last sentence of the theorem is a consequence of the fact that the conditions for delay-robustness and w-stability are the same when the system is input-output stable which is the case here due to the boundedness of the different operators. We already know that the control law $V_f(t)$, whose Laplace transform is defined in (82), stabilizes (63). Consequently, it stabilizes (5)-(13) since the two systems have equivalent stability properties. Let us now prove that it is robust to small delays in the loop. Let us consider a delay vector δ whose components $\delta_1, \delta_2, .., \delta_{m_p}$ are positive delays. Let us consider that each component $(V_f(t))_j$ is delayed by δ_j. The characteristic equation of the closed-loop system now rewrites

$$\det(\mathrm{Id} - F(s) - H_1(s) - w(s)A\Delta(s)(P(s) - A^T F(s))) = 0, \quad (87)$$

where $\Delta(s) = \mathrm{diag}(\mathrm{e}^{-\delta_1 s}, ..., \mathrm{e}^{-\delta_{m_p} s})$. Since $\underline{\sigma}(\mathrm{Id} - F(s)) > 0$, since $H_1(s) - w(s)A\Delta(s)(P(s) - A^T F(s))$ is strictly proper and since $\bar{\sigma}(\Delta(s)) \leq 1$, there exists $N > 0$ such that for all $\delta_j > 0$, equation (87) does not have any solution if $|s| > N$ ($s \in \mathrm{C}^+$). On the compact set $\{s \in \mathrm{C}^+, |s| \leq N\}$, due to the continuity of the functions and to the fact that the nominal system is exponentially stable (and that consequently its characteristic equation do not vanish on C^+), there exists $\delta^\star > 0$ such that for any $\delta_j < \delta^\star$, $\det(\mathrm{Id} - F(s) - H_1(s) - w(s)A\Delta(s)(P(s) - A^T F(s))) \neq 0$. Thus, for $\delta_j < \delta^\star$, the characteristic equation of the system does not have any root on C^+. This concludes the proof [26], since again the asymptotic vertical chain of zeros of this characteristic equation cannot be the imaginary axis due to Assumption 1.

It is worth mentioning that the value ν_0 chosen in Lemma 17 may influence the largest admissible delay $\delta^\star$.

7 Simulation results

The proposed control law has been tested in simulations using Matlab. The PDE system is simulated using a classical finite volume method based on a Godunov scheme. We used 61 spatial discretization points (and a CFL number of 1). The algorithm we use to compute the different kernels is the following. Using the method of characteristics, we write the integral equations associated to the kernel PDE-systems. These integral equations are solved using a fixed-point algorithm. The predictor is implemented using a backward Euler approximation of the integral involved in (30). The numerical values used are given below:

$$\Lambda_p^+ = 1,\ \Lambda_d^+ = 2,\ \Lambda_p^- = \begin{pmatrix} 1.3 & 0 \\ 0 & 1.7 \end{pmatrix},\ \Lambda_d^- = \begin{pmatrix} 0.8 & 0 \\ 0 & 1.5 \end{pmatrix},$$

$$\Sigma_p^{++} = \Sigma_d^{++} = 0,\ \Sigma_p^{+-} = \Sigma_d^{+-} = \begin{pmatrix} 0.5 & 0.25 \end{pmatrix},$$

$$\Sigma_p^{-+} = \begin{pmatrix} 0 & 0.5 \end{pmatrix},\ \Sigma_p^{--} = \begin{pmatrix} 0 & -0.1 \\ 0.2 & 0 \end{pmatrix},$$

$$\Sigma_d^{-+} = \begin{pmatrix} -0.2 & 0.2 \end{pmatrix},\ \Sigma_d^{--} = \begin{pmatrix} 0 & 0 \\ 0.45 & 0 \end{pmatrix},$$

$$Q_{pd} = 0.3,\ Q_{pp} = \begin{pmatrix} 0.3 & 0.6 \end{pmatrix},\ R_d = \begin{pmatrix} 0.8 & 0.6 \end{pmatrix},$$

$$Q_{dp} = \begin{pmatrix} 0.4 & 0.24 \\ 0 & 0.4 \end{pmatrix},\ Q_{dd} = \begin{pmatrix} 0.6 \\ 0.6 \end{pmatrix},\ R_p = \begin{pmatrix} 0.4. \\ 0.3 \end{pmatrix}$$

These coefficients are chosen such that the distal and the proximal PDEs subsystems are independently unstable in open-loop and such that the resulting interconnected system remains unstable. Assumption 2 is obviously satisfied, while we can check numerically that Assumption 1

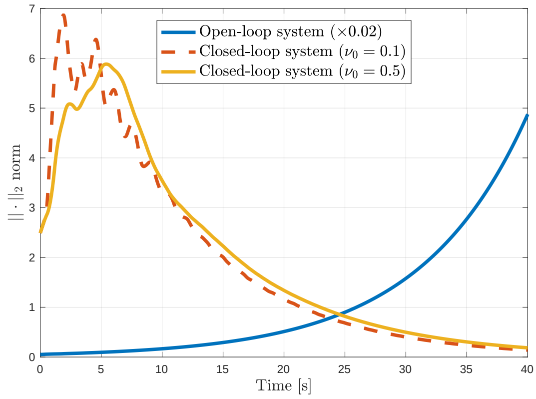

Fig. 4. Evolution of the $|| \cdot ||_2$-norm of the open-loop system and of the closed-loop system (for two different filters) with an input delay of $0.2s$. The control law is defined by (82). The filters have been designed with $\nu_0 = 0.1$ and $\nu_0 = 0.5$. The L^2-norm of the open-loop system has been divided by 50 for readability reasons.

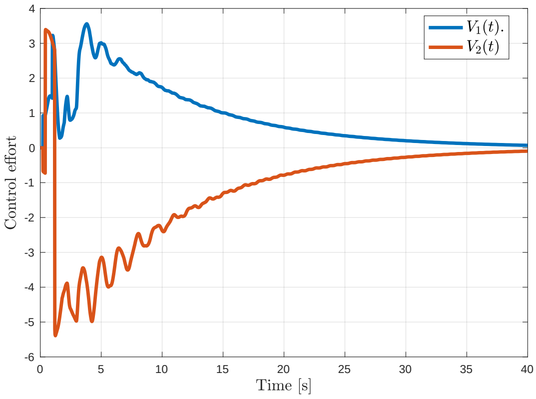

Fig. 5. Evolution of the control effort $V_1(t)$ and $V_2(t)$ ($\nu_0 = 0.1$).

is also verified. Using Lemma 17, we define the low-pass filter $w(s) = \frac{1}{1+\nu_0 s}$. We choose $\nu_0 = 0.1$ and $\nu_0 = 0.5$ for the simulations. Note that a complete analysis of the quantitative effects of the low-pass filter on the closed-loop system is out of the scope of this paper. We have pictured in Figure 4 the evolution of the L_2-norm of the open-loop system and of the closed-loop system (using the control law (82)) in presence of a delay of 0.2 seconds. As expected, the resulting system exponentially converges to zero. The corresponding control effort has been plotted in Figure 5.

8 Concluding remarks

In this paper, we have designed an explicit stabilizing state feedback control law for an underactuated cascade network of two hyperbolic PDEs systems connected through their boundaries, the control law being located at one boundary of the network. The proposed approach combines backstepping transformations that simplify the network structure with a predictive tracking controller that stabilizes the distal subsystem. The closed-loop system's robustness properties are guaranteed by combining the stabilizing control law with a well-tuned low-pass filter. In future contributions, we will focus on the design of a state-observer, which is necessary to obtain an output-stabilizing feedback law. We will also have a closer look at the quantitative effects of the low-pass filter This will be the purpose of our next contributions.

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