Prediction-based controller for linear systems with stochastic input delay

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Abstract

We study the stabilization problem of a linear time-invariant system with an unknown stochastic input delay. We propose to robustly compensate for the stochastic delay with a constant time horizon prediction-based controller. We prove the mean-square exponential stabilization of the closed-loop system under a sufficient condition, which requires the range of the delay values to be sufficiently narrow and the constant delay used in the prediction-based controller to be chosen in this range. Numerical simulations illustrate the relevance of this condition and the merits of our control design.

Key words: linear systems; delay systems; random delay; backstepping control of distributed parameter systems; prediction-based control; delay systems.

1 Introduction

Time delays are ubiquitous in engineering systems. Especially, the development of communication technology led to the large spread of sophisticated network control systems [39, 44]. Yet, information transmitted through these networks often suffers from lag [40], data reordering, packets dropouts [13], data corruption or quantization [7]. Such phenomena play a crucial role in the dynamic of vehicular traffic [16]. Indeed, in addition to the driver reaction time, wireless vehicle-to-vehicle communication [46] used to monitor vehicles ahead when beyond the line of sight often introduces substantial communication delays and packet losses [4,32] while transmitting the remote vehicles information. These phenomena can be accounted for by a stochastic delay model (see [14, 15]).

When a delay affects the input of a dynamical system, prediction-based laws [1, 41] are the state-of-the-art control strategy [35]. It was first applied to linear systems with constant input delays (see [27, 30]), then extended to handle time-varying delays (see [2, 34]), uncertain input delays or disturbances (see [28, 33]), and nonlinear systems (see [3,20]). The main ingredient of this class of control law requires calculating a state prediction over a time window of the length of the delay or future value of the delay in the case of a time-varying delay. However, this strategy difficultly translates to the case of unknown future delay variations and even more to stochastic delays.

While a vast number of works [17, 18, 22, 31, 42] have investigated the stability or stabilization of Stochastic Delay Differential Equations (SDDEs) in various contexts, only a few works have considered the case where the delay itself is a stochastic variable. Indeed, prediction-based control laws have been applied to linear SDDEs in [6], but the delay itself is assumed to be constant. Up to our knowledge, ones of the few studies to consider the delay as a stochastic variable are [23,24,29,43]. While [43] studies a piecewise constant process and [29] analyzes a deterministic delay term multiplied by a random variable, [23, 24] consider stochastic state delays modeled as a Markov process with a finite number of states. The authors then consider each delay value separately, following the so-called technique of probabilistic delay averaging. This constant delay reasoning inspired the core of our analysis methodology.

In this paper, we consider for the first time the problem of prediction-based control of dynamical systems subject to stochastic input delays. We consider linear dynamics and model the delay as a Markov process with a finite number of states. We propose to use a prediction-based controller aiming at compensating for this delay. Yet, due to its stochastic nature and the fact that the current delay value is unlikely to be measured, we design a constant-horizon prediction.

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This control strategy builds on the delay-robustness results obtained for prediction-based controllers in the deterministic case. Indeed, delay-robust compensation can be achieved not only for a constant time-delay [33], but also for a time-varying one provided that the delay rate remains sufficiently small [3] or using a small-gain approach [19]. In this paper, we extend these results to the stochastic delay case and establish that a constant horizon prediction-based controller guarantees mean-square exponential stabilization of the system provided that the horizon prediction is sufficiently close to the delay values. This is the main contribution of the paper.

This paper is organized as follows. In section 2, we begin with the problem statement and state our main theorem. In section 3, we propose a backstepping transformation to reformulate the system, which establishes the preliminaries to analyze the stability of the closed-loop system in section 4. Simulation results are given in section 5 to illustrate the relevance of the stabilization conditions we formulated.

Notations.
In the following sections, for a signal $v:(x,t)\in[0,1]\times\mathbb{R}\rightarrow v(x,t)\in\mathbb{R}$, we denote $\|v(t)\|$ its spatial $\mathscr{L}_2$-norm

$$\|v(t)\|=\sqrt{\int_0^1 v(x,t)^2dx} \tag{1}$$

For a square matrix A, $\lambda(A)$ denotes its spectrum. For a symmetric square matrix A, min($\lambda(A)$) and max($\lambda(A)$) are its minimum and maximum eigenvalues respectively.

Additionally, $|A|$ denotes its Euclidean norm

$$|A|=\sqrt{max(\lambda(A^TA))} \tag{2}$$

in which A^T denotes the transpose of A.

$\mathbb{E}(x)$ denotes the expectation of a random variable x. For a random signal $x(t)$ ($t\in\mathscr{T}\subset\mathbb{R}$), the conditional expectation of $x(t)$ at the instant t knowing that $x(s)=x_0$ at the instant $s\leq t$ is denoted $\mathbb{E}_{[s,x_0]}(x(t))$.

Finally, $e_i\in\mathbb{R}^r$ ($r\in\mathbb{N}_+$ and $i\in\{1,...,r\}$) denotes the i^{th} unit vector, that is, $e_1=\begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}^T$, $e_2=\begin{pmatrix}0 & 1 & \cdots & 0\end{pmatrix}^T$, ..., $e_r=\begin{pmatrix}0 & 0 & \cdots & 1\end{pmatrix}$.

2 Problem Statement and Main Result

We consider the following controllable linear dynamics

$$\dot{X}(t)=AX(t)+BU(t-D(t)) \tag{3}$$

in which the $\mathbb{R}^n$-valued random variable X and $U\in\mathbb{R}$ are the state and control input, respectively. The stochastic delay D is a Markov process with the following properties:

(1) $D(t)\in\{D_i,i\in\{1,...,r\}\}$, $r\in\mathbb{N}$ with $0<\underline{D}\leq D_1<D_2<...<D_r\leq\overline{D}$.
(2) The transition probabilities $P_{ij}(t_1,t_2)$, which quantify the probability to switch from D_i at time t_1 to D_j at time t_2 ($(i,j)\in\{1,...,r\}^2$, $t_2\geq t_1\geq 0$), are differentiable functions $P_{ij}:\mathbb{R}^2\rightarrow[0,1]$ satisfying

$$\sum_{j=1}^{r}P_{ij}(t_1,t_2)=1,\quad (0\leq t_1\leq t_2) \tag{4}$$

(3) The realizations of $t\mapsto D(t)$ are continuous from the right.

We consider the following constant time horizon prediction-based control law

$$U(t)=K\left[e^{AD_0}X(t)+\int_{t-D_0}^{t}e^{A(t-s)}BU(s)ds\right] \tag{5}$$

in which K is a feedback gain such that $A+BK$ is Hurwitz, and $D_0\in[\underline{D},\overline{D}]$ is constant.

Exact compensation of the non-constant delay in (3) requires the function $\phi:t\rightarrow t-D(t)$ to be invertible in order to define the feedback law $U(t)=KX(\phi^{-1}(t))$. However, in the case of a stochastic delay, the function has no reason to be invertible. In addition, even if it were, the computation of $\phi^{-1}(t)$ would require to know future realizations of the delay which is impossible in practice.

For these reasons, we therefore propose to use the constant horizon prediction-based controller (5). Note that, if the time lag D was constant and equal to D_0, the control law would then correspond to the exact prediction of the state X over a time window of D_0 units. Consistently, we now formulate the main result of the paper which states that robust compensation is achieved if the time delay remains sufficiently close to D_0.

Theorem 1 *Consider the closed-loop system consisting of the system (3) and the control law (5). There exists a positive constant $\varepsilon^{\star}(K)$ such that, if*

$$|D_0-D_j|\leq\varepsilon^{\star}(K),\quad j\in\{1,...,r\} \tag{6}$$

there exist positive constants R and γ such that

$$\mathbb{E}_{[0,(\Upsilon(0),D(0))]}(\Upsilon(t))\leq R\Upsilon(0)e^{-\gamma t} \tag{7}$$

with

$$\Upsilon(t)=|X(t)|^2+\int_{t-\overline{D}-D_0}^{t}U(s)^2ds \tag{8}$$

The prediction-based control law (5) would exactly compensate for the input delay with a constant time lag equal to D_0. In other words, if $D(t) = D_0$, applying the variation of constant formula, it would follow that $U(t) = KX(t + D_0)$ and that, after D_0 units of time, the corresponding closed-loop dynamics would be $\dot{X} = (A + BK)X$ which is exponentially stable. Condition (6) guarantees that the prediction performed in (5) remains sufficiently accurate in the case of a stochastic delay. In details, (6) requires the sequence of the random delay $\mathbf{D} = \begin{pmatrix} D_1 & \cdots & D_i & \cdots & D_r \end{pmatrix}^T_{r\in\mathbb{N}}$ to be limited in a vicinity $\varepsilon^\star$ of the constant D_0.

Note that this result is consistent with the delay-robustness results obtained in the deterministic delay case. Indeed, [3, 25] provide a similar robust compensation result for a time-differentiable delay function, under the assumptions that both the range of variation of the delay and its variation rate is sufficiently limited. A similar result was obtained in [19] but through a small-gain approach enabling to avoid restricting the delay rate. Finally, as the delay process under consideration only takes a finite number of values, it is worth noticing that similar robustness properties were also obtained in a discrete-time context in [8] where the prediction is approximated to respect causality. Hence, this theorem falls within this framework and extends it to the stochastic context.

Finally, note that the limit $\varepsilon^\star$ depends on the feedback gain K, the choice of which is likely to play a crucial role in practice. However, capturing this dependence is a complex task from a Lyapunov stability point of view and would require additional studies.

We now provide the proof of this theorem in the following sections.

3 PDE Representation of the Delay and Backstepping Transformation

First, to represent the control input which is subject to a stochastic delay, we define a distributed actuator vector as, for $x \in [0,1]$, $\mathbf{v}(x,t) = \begin{pmatrix} v_1(x,t) & \cdots & v_k(x,t) & \cdots & v_r(x,t) \end{pmatrix}^T$ with $v_k(x,t) = U(t + D_k(x-1))$. This enables to rewrite (3) as

$$\begin{cases} \dot{X}(t) = AX(t) + B\delta(t)^T \mathbf{v}(0,t) \\ \Lambda_D \mathbf{v}_t(x,t) = \mathbf{v}_x(x,t) \\ \mathbf{v}(1,t) = \mathbf{1}U(t) \end{cases} \tag{9}$$

in which $\Lambda_D = diag(D_1,...,D_r)$, $\mathbf{1}$ is a r-by-1 all-ones vector and $\delta(t) \in \mathbb{R}^r$ is such that, if $D(t) = D_j$,

$$\delta_i(t) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

Hence, $\delta(t)$ is a Markov process with the same transition probabilities as the process $D(t)$, but with the finite number of states (e_i) instead of (D_i). In the sequel, $\delta(t)$ and $D(t)$ will thus be equivalently used.

Now, we introduce $\hat{v}(x,t)$ to represent the control input $U(t)$ within the interval $[t - D_0, t]$, and the corresponding input estimation error $\tilde{\mathbf{v}}(x,t)$ defined as

$$\begin{cases} \hat{v}(x,t) = U(t + D_0(x-1)) \\ \tilde{\mathbf{v}}(x,t) = \mathbf{v}(x,t) - \mathbf{1}\hat{v}(x,t) \end{cases} \tag{11}$$

Then, the extended state $(X(t), \hat{v}(x,t), \tilde{\mathbf{v}}(x,t))$ satisfies

$$\begin{cases} \dot{X}(t) = AX(t) + B\hat{v}(0,t) + B\delta(t)^T \tilde{\mathbf{v}}(0,t) \\ D_0 \hat{v}_t(x,t) = \hat{v}_x(x,t) \\ \hat{v}(1,t) = U(t) \\ \Lambda_D \tilde{\mathbf{v}}_t(x,t) = \tilde{\mathbf{v}}_x(x,t) - \Sigma_D \hat{v}_x(x,t) \\ \tilde{\mathbf{v}}(1,t) = \mathbf{0} \end{cases} \tag{12}$$

in which $\Sigma_D = (\frac{D_1 - D_0}{D_0}, ..., \frac{D_r - D_0}{D_0})^T$ and $\mathbf{0}$ is a r-by-1 all-zeros vector.

Besides, to ease the stability analysis, we also introduce another actuator $\mu(x,t) = U\left(t - D_0 + \overline{D}(x-1)\right)$ to describe the history of the input on a longer time window $[t - \overline{D} - D_0, t - D_0]$. Correspondingly, we extend the dynamic to $(X(t), \hat{v}(x,t), \tilde{\mathbf{v}}(x,t), \mu(x,t))$ satisfying

$$\begin{cases} \dot{X}(t) = AX(t) + B\hat{v}(0,t) + B\delta(t)^T \tilde{\mathbf{v}}(0,t) \\ D_0 \hat{v}_t(x,t) = \hat{v}_x(x,t) \\ \hat{v}(1,t) = U(t) \\ \Lambda_D \tilde{\mathbf{v}}_t(x,t) = \tilde{\mathbf{v}}_x(x,t) - \Sigma_D \hat{v}_x(x,t) \\ \tilde{\mathbf{v}}(1,t) = \mathbf{0} \\ \overline{D}\mu_t(x,t) = \mu_x(x,t) \\ \mu(1,t) = \hat{v}(0,t) \end{cases} \tag{13}$$

In other words, $\hat{v}$ now cascades into the transport PDE satisfied by μ. Finally, in view of stability analysis, we introduce the backstepping transformation (see [26])

$$\begin{aligned} w(x,t) =& \hat{v}(x,t) - Ke^{AD_0 x}X(t) \\ & - D_0 \int_0^x Ke^{AD_0(x-y)} B\hat{v}(y,t) dy \end{aligned} \tag{14}$$

Lemma 2 *The backstepping transformation (14), jointly with the control law (5), transform the plant (13) into the* *target system* $(X(t), w(x,t), \tilde{\mathbf{v}}(x,t), \mu(x,t))$

$$\begin{cases} \dot{X}(t) = (A+BK)X(t) + B\delta(t)^T\tilde{\mathbf{v}}(0,t) + Bw(0,t) \\ D_0 w_t(x,t) = w_x(x,t) - D_0 K e^{AD_0 x} B\delta(t)^T \tilde{\mathbf{v}}(0,t) \\ w(1,t) = 0 \\ \Lambda_D \tilde{\mathbf{v}}_t(x,t) = \tilde{\mathbf{v}}_x(x,t) - \Sigma_D h(t + D_0(x-1)) \\ \tilde{\mathbf{v}}(1,t) = \mathbf{0} \\ \overline{D}\mu_t(x,t) = \mu_x(x,t) \\ \mu(1,t) = KX(t) + w(0,t) \end{cases} \quad (15)$$

in which, h is defined for $t \geq 0$ *as*

$$\begin{aligned} h(t) = D_0 K \big[&(A+BK)e^{AD_0}X(t) + e^{AD_0}B\delta(t)^T\tilde{\mathbf{v}}(0,t) \quad (16) \\ &+ e^{AD_0}Bw(0,t) + D_0(A+BK)\int_0^1 e^{AD_0(1-x)}B\big(w(x,t) \\ &+ Ke^{(A+BK)D_0 x}X(t) + \int_0^x KD_0 e^{(A+BK)D_0(x-y)}Bw(y,t)dy\big)dx \big] \end{aligned}$$

PROOF. The space-derivative of the backstepping transformation (14) can be written as

$$\begin{aligned} w_x(x,t) =& \hat{v}_x(x,t) - KAD_0 e^{AD_0 x}X(t) - D_0 KB\hat{v}(x,t) \\ &- D_0 \int_0^x KAD_0 e^{AD_0(x-y)}B\hat{v}(y,t)dy \end{aligned} \quad (17)$$

Besides, the time-derivative of (14) reads

$$\begin{aligned} w_t(x,t) =& \hat{v}_t(x,t) - D_0 \int_0^x Ke^{AD_0(x-y)}B\hat{v}_t(y,t)dy \\ &- Ke^{AD_0 x}AX(t) - Ke^{AD_0 x}B\hat{v}(0,t) \\ &- Ke^{AD_0 x}B\delta(t)^T\tilde{\mathbf{v}}(0,t) \end{aligned} \quad (18)$$

From (14), (17) and (18) with an integration by parts, we obtain the two equations with respect to w in (15). Finally, from the definition of $\tilde{\mathbf{v}}$ and $\hat{v}$ in (11), one can observe that $h(t + D_0(x-1)) = \hat{v}_x(x,t) = D_0\dot{U}(t + D_0(x-1))$ which gives the desired expression of $h(t)$ for $t \geq 0$, taking a time-derivative of (5) and using the inverse backstepping transformation of (14), which is

$$\begin{aligned} \hat{v}(x,t) =& w(x,t) + Ke^{(A+BK)D_0 x}X(t) \\ &+ \int_0^x KD_0 e^{(A+BK)D_0(x-y)}Bw(y,t)dy \end{aligned} \quad (19)$$

We are now ready to carry out the stability analysis.

4 Lyapunov Stability Analysis

4.1 Preliminaries

Let us define the state of the target system (15) as $\Psi = (X, w, \tilde{\mathbf{v}}, \mu) \in \mathbb{R}^n \times \mathscr{L}_2([0,1],\mathbb{R}) \times \mathscr{L}_2([0,1],\mathbb{R}^r) \times \mathscr{L}_2([0,1],\mathbb{R}) \triangleq \mathscr{D}_\Psi$. Note that (15) was reformulated as a dynamical system involving a random parameter, as studied in [21] or [10]. However, the results presented in [10] on the existence of solutions consider a more complex stochastic framework. This is why, for the sake of self-containedness, we now formulate a well-posedness result.

Following [21], by a weak solution to the closed-loop system (3) and (5), we refer to a $\mathbb{R}^n \times \mathscr{L}_2([-\overline{D},0],\mathbb{R}) \times \mathbb{R}$-valued random variable $(X(X_0,t), U_t(U_0,\cdot), D(t))$, the realizations of which satisfy an integral form of (3) and (5), that is,

$$X(t) = X(0) + \int_0^t (AX(s) + BU(s - D(s)))\,ds \quad (20)$$

and (5) for $t \geq 0$.

Similarly, by a weak solution to (15), we refer to a $\mathscr{D}_\Psi \times \mathbb{R}$-valued random variable $(X(X_0,t), w(w_0,\cdot,t), \tilde{\mathbf{v}}(\tilde{\mathbf{v}}_0,\cdot,t), \mu(\mu_0,\cdot,t), D(t))$, the realizations of which are a weak solution of (15), that is, in the PDEs standard sense [11, Definition 3.1.4] of weak solutions for the transport PDEs and under an integral form for the ODE.

Lemma 3 *For every initial condition* $(X_0, U_0) \in \mathbb{R}^n \times \mathscr{L}_2([-\overline{D},0],\mathbb{R})$*, the closed-loop system consisting of* (3) *and the control law* (5) *has a unique weak solution such that*

$$X(t) = e^{At}X(0) + \int_0^t e^{A(t-s)}BU(s - D(s))ds \quad (21)$$

Consequently, for each initial condition in $\mathscr{D}_\Psi$*, the target system* (15) *also has a unique weak solution.*

PROOF. We first focus on the existence of a solution. Notice that, for X defined in (21), performing an integration by parts,

$$\begin{aligned} &X(0) + \int_0^t (AX(s) + BU(s - D(s)))\,ds \\ =& X(0) + \int_0^t BU(s - D(s))ds + \int_0^t Ae^{As}X(0)ds \\ &+ \int_0^t \left(\int_0^s Ae^{A(s-\xi)}ds \right) BU(\xi - D(\xi))d\xi \\ =& e^{At}X(0) + \int_0^t e^{A(t-\xi)}BU(\xi - D(\xi))d\xi = X(t) \end{aligned} \quad (22)$$

which corresponds to an integral form of (3). Observe that $U(t - D(t)) = \delta(t)\,(U(t - D_1)\ \dots\ U(t - D_r))^T$, in which $\delta(t)$ is a bounded almost-everywhere continuous function (due to the assumption that the realizations of D and thus δ are right-continuous [9, Exercise 4 p.7]) and thus integrable and also square-integrable. As $U_0 \in \mathscr{L}_2([-\overline{D},0],\mathbb{R})$, it follows from Cauchy-Schwarz's inequality, that $t \to U(t-D(t))$ is integrable on the interval $[0,\underline{D}]$. Consequently, the integral in (21) is well-defined for $t \in [0,\underline{D}]$, and X is bounded on the interval $[0,\underline{D}]$.

Then, as $U_0 \in \mathscr{L}_2([-\overline{D},0],\mathbb{R})$, U defined in (5) remains bounded on the interval $[0,\underline{D}]$ from the corresponding inverse Volterra integral equation [5,45]. Consequently, by a straightforward iterative argument on time intervals of length $\underline{D}$, one can prove that both X and U, as defined through (21) and (5) remain bounded for positive times, and that, consequently, the integral in (21) is well-defined. Hence, (21) and (5) define a weak solution to the closed-loop system.

Secondly, we prove the uniqueness of this solution. Suppose that there exist two different solutions (X_1,U_1) and (X_2,U_2) for a given initial condition. It then holds

$$\begin{cases} (\dot{X}_1-\dot{X}_2)(t) = A(X_1-X_2) \\ \qquad\qquad +B(U_1(t-D(t))-U_2(t-D(t))) \\ (X_1-X_2)(0)=0 \end{cases} \tag{23}$$

with $U_1=U_2$ for $t<0$, and for $t\geq 0$

$$\begin{cases} U_1(t)=K\left(e^{AD_0}X_1(t)+\int_{t-D_0}^{t} e^{A(t-s)}BU_1(s)ds\right) \\ U_2(t)=K\left(e^{AD_0}X_2(t)+\int_{t-D_0}^{t} e^{A(t-s)}BU_2(s)ds\right) \end{cases} \tag{24}$$

For any delay realization, it thus holds that $U_1(t-D(t)) = U_2(t-D(t))$ for $t\in[0,\underline{D}]$, which, in turns, gives $X_1(t) = X_2(t)$ for $t\in[0,\underline{D}]$. Iterating on intervals of length $\underline{D}$, we then obtain $X_1=X_2$ and $U_1=U_2$ for $t\in\mathbb{R}_+$.

Let us now observe that the well-posedness of the closed-loop system (3) and (5) implies the one of (9) and (5) by equivalence. The one of (13) and (5) and thus of (15) by backstepping transformation then follow (see [38, Theorem 3.1]).

From Lemma 3, (Ψ,δ) thus defines a continuous-time Markov process and we can therefore introduce the following elements for stability analysis.

In the sequel, we consider the following Lyapunov functional candidate

$$\begin{aligned} V(\Psi) =& X^TPX + bD_0\int_0^1 (1+x)w(x)^2dx \\ &+c\sum_{l=1}^{r}\int_0^1(1+x)(e_l\cdot\mathbf{D})^T\tilde{\mathbf{v}}(x)^2dx \\ &+d\overline{D}\int_0^1(1+x)\mu(x)^2dx \end{aligned} \tag{25}$$

with $b,c,d>0$, P the symmetric positive definite solution of the equation $P(A+BK)+(A+BK)^TP=-Q$, for a given symmetric positive definite matrix Q, and $\mathbf{D}=\begin{pmatrix}D_1 & \cdots & D_i & \cdots & D_r\end{pmatrix}^T_{r\in\mathbb{N}}$ and where $\cdot$ denotes the Hadamard multiplication and the square in $\tilde{\mathbf{v}}(x)^2$ should be understood component-wise.

As the functional V is not differentiable with respect to time t when evaluated at $\Psi(t)$ and $\delta(t)$, we introduce the infinitesimal generator L (see [24] and [22]) as

$$\begin{aligned} &LV(\Psi,\delta) \\ &=\limsup_{\Delta t\to 0^+}\frac{1}{\Delta t}\left(\mathbb{E}_{[t,(\Psi,\delta)]}(V(\Psi(t+\Delta t),\delta(t+\Delta t)))-V(\Psi,\delta)\right) \end{aligned} \tag{26}$$

We also define L_j, the infinitesimal generator of the Markov process (Ψ,δ) for the target system obtained from (15) by fixing $\delta(t)=e_j$, as

$$L_jV(\Psi)=\frac{dV}{d\Psi}(\Psi,e_j)f_j(\Psi) \tag{27}$$

in which f_j denotes the operator corresponding to the dynamics of the target system (15) with the fixed value $\delta(t)=e_j$, that is, for $\Psi=(X,w,\tilde{\mathbf{v}},\mu)$,

$$f_j(\Psi)(x)=\begin{pmatrix}(A+BK)X+Be_j^T\tilde{\mathbf{v}}(0)+Bw(0) \\ \frac{1}{D_0}\left[w_x(x)-D_0Ke^{AD_0x}Be_j^T\tilde{\mathbf{v}}(0)\right] \\ \Lambda_D^{-1}\left[\tilde{\mathbf{v}}_x(x)-\Sigma_Dh(\cdot+D_0(x-1))\right] \\ \frac{1}{\overline{D}}\mu_x(x)\end{pmatrix} \tag{28}$$

For the sake of conciseness, in the sequel, we denote $V(t)$, $LV(t)$ and $L_jV(t)$, for short, instead of $V(\Psi(t),\delta(t))$, $LV(\Psi(t),\delta(t))$ and $LV(\Psi(t))$ respectively.

It is worth noticing that, due to the fact that V does not depend explicitly on δ and (4), the infinitesimal generators are related as follows

$$\begin{aligned} LV(t) &= \sum_{j=1}^{r}P_{ij}(0,t)\frac{dV}{d\Psi}(\Psi(t))f_j(\Psi(t))+\sum_{j=1}^{r}\frac{\partial P_{ij}}{\partial t}(0,t)V(t) \\ &=\sum_{j=1}^{r}P_{ij}(0,t)L_jV(t) \end{aligned} \tag{29}$$

Therefore, in view of stability analysis, as a first step, one can focus on the derivative of the Lyapunov functional evaluated for a dynamic with a fixed delay, that is, L_jV. This is the approach we follow in the sequel.

4.2 Lyapunov analysis

Lemma 4 *Assume there exists a positive constant ε such that*

$$|D_0 - D_j| \leq \varepsilon, \quad j \in \{1,...,r\} \tag{30}$$

Then, there exist $(b,c,d,\eta) \in (\mathbb{R}_+^*)^4$ which are independent of ε such that the Lyapunov functional V defined in (25) satisfies

$$LV(t) \leq -(\eta - g(\varepsilon))V(t), \quad t \geq \overline{D} \tag{31}$$

with the function $g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\lim_{\varepsilon\to 0} g(\varepsilon) = 0$.

PROOF. Taking a derivative of (25) and applying integrations by parts and Young's inequality, we obtain

$$\begin{aligned}
&\frac{dV}{d\Psi}(\Psi)f_j(\Psi) \quad (32)\\
&= -X(t)^TQX(t) + 2X(t)^TPB(w(0,t)+\tilde{v}_j(0,t))\\
&\quad + 2b\int_0^1 (1+x)w(x,t)\big(w_x(x,t) - D_0Ke^{AD_0x}\\
&\quad \times B\tilde{v}_j(0,t)\big)dx + 2c\sum_{l=1}^r \int_0^1 (1+x)\tilde{v}_l(x,t)\Bigg(\tilde{v}_{lx}(x,t)\\
&\quad + \left(1-\frac{D_l}{D_0}\right)h(t+D_0(x-1))\Bigg)dx\\
&\quad + 2d\int_0^1 (1+x)\mu(x,t)\mu_x(x,t)dx\\
&\leq -\left(\frac{\min(\lambda(Q))}{2} - 4d|K|^2\right)|X(t)|^2 - d\|\mu(t)\|^2\\
&\quad - b(1-2D_0|K||B|e^{|A|D_0}\gamma_1)\|w(t)\|^2\\
&\quad - c\sum_{l=1}^r\left(1-\frac{2}{D_0}|D_0-D_l|\gamma_2\right)\|\tilde{v}_l(t)\|^2\\
&\quad - \left(b - 4d - \frac{4|PB|^2}{\min(\lambda(Q))}\right)w(0,t)^2 - \Big(c - \frac{4|PB|^2}{\min(\lambda(Q))}\\
&\quad - 2bD_0|K||B|e^{|A|D_0}\frac{1}{\gamma_1}\Big)\tilde{v}_j(0,t)^2 - c\sum_{l\neq j}\tilde{v}_l(0,t)^2 - d\mu(0,t)^2\\
&\quad + \frac{2c}{D_0}\sum_{l=1}^r |D_0-D_l|\frac{1}{\gamma_2}\|h(t+D_0(x-1))\|^2
\end{aligned}$$

for any $\gamma_1, \gamma_2 \geq 0$.

Therefore, applying (30) and Lemma 6 given in Appendix, one gets for $t \geq \overline{D}$,

$$\begin{aligned}
&LV(t) = \sum_{j=1}^r P_{ij}(0,t)\frac{dV}{d\Psi}(\Psi)f_j(\Psi) \quad (33)\\
&\leq -\left(\frac{\min(\lambda(Q))}{2} - 4d|K|^2\right)|X(t)|^2\\
&\quad - b\left(1-2D_0|K||B|e^{|A|D_0}\gamma_1\right)\|w(t)\|^2\\
&\quad - c\sum_{l=1}^r\left(1-\frac{2}{D_0}|D_0-D_l|\gamma_2\right)\|\tilde{v}_l(t)\|^2 - d\|\mu(t)\|^2\\
&\quad - \left(b-4d-\frac{4|PB|^2}{\min(\lambda(Q))}\right)w(0,t)^2 - \Big(c - \frac{4|PB|^2}{\min(\lambda(Q))}\\
&\quad - 2bD_0|K||B|e^{|A|D_0}\frac{1}{\gamma_1}\Big)\sum_{j=1}^r P_{ij}(0,t)\tilde{v}_j(0,t)^2\\
&\quad - c\sum_{j=1}^r P_{ij}(0,t)\sum_{l\neq j}\tilde{v}_l(0,t)^2 - d\mu(0,t)^2\\
&\quad + \frac{2cr\varepsilon}{D_0}\frac{1}{\gamma_2}MV(t)
\end{aligned}$$

in which the positive constant M does not depend on ε and is defined in Lemma 6.

Observing that $D_0 \in [\underline{D}, \overline{D}]$, let us choose $(b,c,d,\gamma_1,\gamma_2) \in (\mathbb{R}_+^*)^5$ as follows

(a) $$d < \frac{\min(\lambda(Q))}{8|K|^2} \tag{34}$$

(b) $$b \geq 4d + \frac{4|PB|^2}{min(\lambda(Q))} \tag{35}$$

(c) $$\gamma_1 < \frac{1}{2\overline{D}|K|e^{|A|\overline{D}}|B|} \tag{36}$$

(d) $$\gamma_2 < \frac{1}{4}\min\left\{\left(1-\frac{D_1}{\overline{D}}\right)^{-1}, \left(\frac{D_r}{\underline{D}}-1\right)^{-1}\right\} \tag{37}$$

(e) $$c \geq \frac{4|PB|^2}{\min(\lambda(Q))} + 2b\overline{D}|K||B|e^{|A|\overline{D}}\frac{1}{\gamma_1} \tag{38}$$

From (33), one then obtains (31) with the well-defined constant $\eta = \min\left\{\frac{\min(\lambda(Q))-8d|K|^2}{2\max(\lambda(P))}, \frac{1-2\overline{D}|K||B|e^{|A|\overline{D}}\gamma_1}{2\overline{D}}, \frac{1}{4D_r}, \frac{1}{2\overline{D}}\right\}$, and the function

$$g(\varepsilon) = \frac{2cr}{D_0}\frac{1}{\gamma_2}M\varepsilon \tag{39}$$

which satisfies $\lim_{\varepsilon\to 0} g(\varepsilon) = 0$.

4.3 Proof of Theorem 1

Firstly, as $\lim_{\varepsilon\to 0} g(\varepsilon) = 0$, there exists $\varepsilon^* > 0$ such that $\eta - g(\varepsilon) = \gamma > 0$ for $\varepsilon < \varepsilon^*$. Therefore, according to Dynkin's formula [12, Theorem 5.1, p. 133], from (31), one obtains for $\varepsilon < \varepsilon^*$

$$\begin{aligned}&\mathbb{E}_{[\overline{D},(\Psi,D)(\overline{D})]}(e^{\gamma t}V(t)) - e^{\gamma\overline{D}}V(\overline{D}) \\ &= \mathbb{E}_{[\overline{D},(\Psi,D)(\overline{D})]}\left(\int_{\overline{D}}^{t}[\gamma e^{\gamma s}V(s) + e^{\gamma s}LV(s)]\,ds\right) \leq 0\end{aligned} \tag{40}$$

from which, using standard conditional expectation properties, one deduces

$$\mathbb{E}_{[0,(\Psi,D)(0)]}(e^{\gamma t}V(t)) \leq \mathbb{E}_{[0,(\Psi,D)(0)]}(e^{\gamma\overline{D}}V(\overline{D})) \tag{41}$$

Hence, it follows that

$$\mathbb{E}_{[0,(\Psi,D)(0)]}(V(t)) \leq \mathbb{E}_{[0,(\Psi,D)(0)]}(V(\overline{D}))e^{-\gamma(t-\overline{D})} \tag{42}$$

Noticing that V and Υ are equivalent, that is, that there exist positive constants q_1 and q_2 such that for $\forall t \geq 0$, $q_1V(t) \leq \Upsilon(t) \leq q_2V(t)$ (see [25]), it thus follows that $\mathbb{E}_{[0,(\Upsilon(0),D(0))]}(\Upsilon(t)) \leq \frac{q_2}{q_1}\mathbb{E}_{[0,(\Upsilon(0),D(0))]}(\Upsilon(\overline{D}))e^{-\gamma(t-\overline{D})}$.

From the definition of Υ in (8), one deduces that there exists a constant R_0 such that

$$\Upsilon(t) \leq R_0\Upsilon(0), \quad t \in [0,\overline{D}] \tag{43}$$

(see Lemma 5 in the appendix for a proof of this property). Finally, the function Υ thus satisfies $\mathbb{E}_{[0,(\Upsilon(0),D(0))]}(\Upsilon(t)) \leq \frac{q_2}{q_1}R_0\Upsilon(0)e^{-\gamma(t-\overline{D})}$, Theorem 1 is then proved with $R = \frac{q_2}{q_1}R_0e^{\gamma\overline{D}}$.

5 Simulations

To illustrate Theorem 1 and in particular the role played by the condition (6), we consider the following toy example

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} X(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(t - D(t)) \tag{44}$$

The control law (5) is applied with the feedback gain $K = -\begin{bmatrix} 1 & 2 \end{bmatrix}$ resulting in conjugate closed-loop eigenvalues $\lambda(A+BK) = \{-0.5000 + 1.3229i, -0.5000 - 1.3229i\}$. The initial conditions are chosen as $X(0) = [1 \quad 0]^T$ and $U(t) = 0$, for $t \leq 0$. The integral in (5) is discretized using a trapezoïdal scheme. Finally, the simulations are carried out with a discrete-time solver in Matlab-Simulink and a sampling time $\Delta t = 0.01$ s.

We consider 5 different delay values $(D_1, D_2, D_3, D_4, D_5) = (0.5, 0.75, 1, 1.25, 1.5)$. Besides, the initial transition probabilities are taken as [1] $P_{ij}(0,0^+) = 0.4985$ ($i \in \{1,...,5\}$, $j = \{3,4\}$) and $P_{ij}(0,0^+) = 0.001$ ($i \in \{1,...,5\}$, $j = \{1,2,5\}$), which means that the delay values are initially concentrated in D_3 and D_4.

In addition, we introduce the following Kolmogorov equation [23, 36, 37] to describe the time-evolution of the transition probabilities

$$\begin{aligned}\frac{\partial P_{ij}(s,t)}{\partial t} &= -c_j(t)P_{ij}(s,t) + \sum_{k=1}^{r} P_{ik}(s,t)\tau_{kj}(t), \quad s < t \\ P_{ii}(s,s) &= 1 \text{ and } P_{ij}(s,s) = 0 \text{ for } i \neq j\end{aligned} \tag{45}$$

in which τ_{ij} and $c_j = \sum_{k=1}^{r}\tau_{jk}$ are positive-valued functions such that $\tau_{ii}(t) = 0$.

In details, $\tau_{ij}\Delta t$ is approximately the probability of transition from D_i to D_j on the interval $[t, t+\Delta t)$. Similarly, $1 - c_j(t)\Delta t$ represents somehow the probability of staying at D_j during the time interval $[t, t+\Delta t)$.

Here, we choose for simulation the transition rates τ_{ij} as

$$\begin{aligned}\tau(t) &= \left(\{\tau_{ij}(t)\}\right)_{1\leq i,j\leq r} \\ &= 0.03\begin{pmatrix} 0 & e^{-10t} & e^{-10t} & e^{-10t} & e^{-10t} \\ e^{-10t} & 0 & \frac{1}{2}-e^{-10t} & \frac{1}{2}-e^{-10t} & e^{-10t} \\ e^{-10t} & 1-3e^{-10t} & 0 & e^{-10t} & e^{-10t} \\ e^{-10t} & 1-3e^{-10t} & e^{-10t} & 0 & e^{-10t} \\ e^{-10t} & e^{-10t} & e^{-10t} & e^{-10t} & 0 \end{pmatrix}\end{aligned} \tag{46}$$

The delay values will therefore gradually evolve towards a uniform distribution among the delay values D_2, D_3 and D_4.

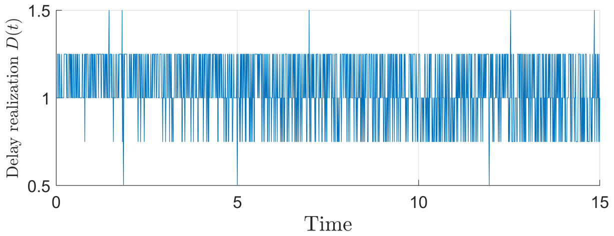
Firstly, we pick $D_0 = 1$. This results in a value [2] $\varepsilon = \max_{j=1,...,r}|D_0 - D_j| = 0.5$. Fig. 1 represents the results obtained for Monte-Carlo simulations of 100 trials. One can observe that the resulting mean value of the state, which approximates the expectation of Theorem 1, indeed converges to the origin.

On the other hand, the choice of a larger prediction horizon $D_0 = 1.25$ (corresponding to the larger value $\varepsilon = 0.75$) results into a diverging behaviour pictured in Fig. 2. This

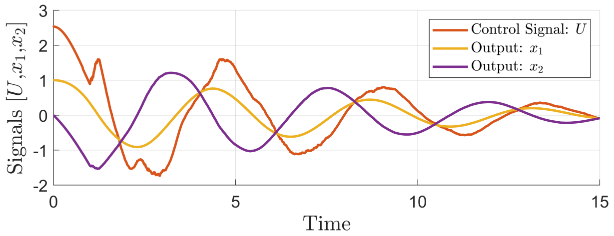
[1] To avoid a conflict between the initial condition in (45) and their discretized version used in simulation, we denote their initial conditions as $P_{ij}(0,0^+)$.

[2] The previous proof guarantees the existence of $\varepsilon^\star$, but can difficulty be used to provide an interesting estimate of it. Indeed, picking $Q = I_2$ in the Lyapunov equation and the intermediate constants according to (34)–(38) results into a value of $\varepsilon^\star \approx 3e-20$, which is of course very conservative and cannot be used in practice. This originates mainly from the value of $M \approx 1e15$ obtained in Lemma 6 and could in all likelihood be decreased by avoiding the use of equivalence constants in the definitions of both η and M.

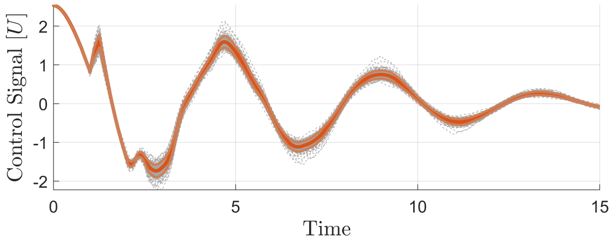
confirms that the choice of prediction horizon D_0 should be restricted in a sufficiently small range to guarantee the stability of the dynamics, on average.



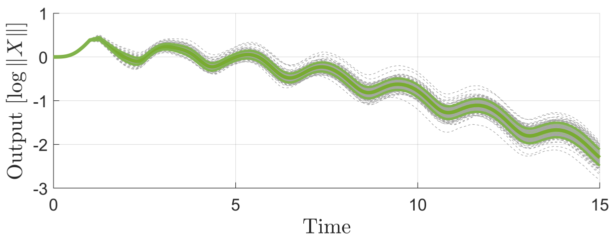
(a) Example of a realization of the stochastic delay D



(b) Realization of the signals U, x_1 and x_2 corresponding to the delay pictured in (a)



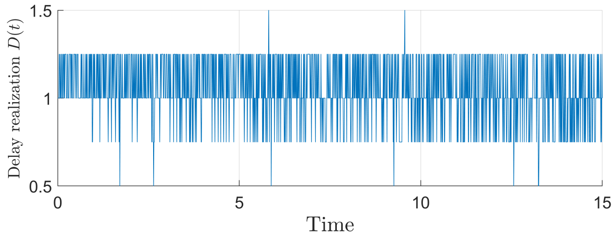
(c) Monte Carlo simulation of the closed-loop input U (100 trials)



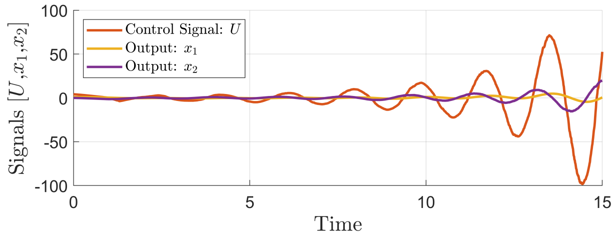
(d) Monte Carlo simulation of $log\|x\|$ (100 trials)

Fig. 1. Simulation results of the closed-loop system (44) and (5) for $\mathbf{D} = (0.5, 0.75, 1.0, 1.25, 1.5)^T$, $X(0) = [1 \quad 0]^T$ and $U(t) = 0$ for $t \leq 0$. The prediction horizon is $D_0 = 1.0$. The transition probabilities follow the dynamics (45)-(46). (a) and (b) picture results corresponding to one delay realization. (c) and (d) present the results of 100 trials, in which the means and the standard deviations are highlighted by the coloured lines.

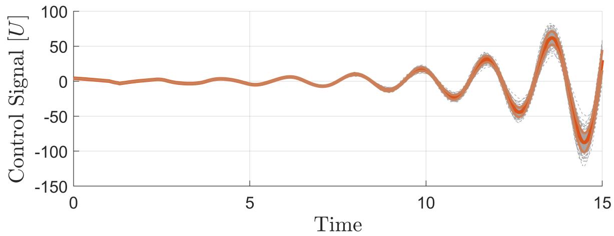
It can be seen from the graph of the time lag change (Fig.1.a) that the delay, which originally takes mainly its value among D_3 and D_4, becomes gradually evenly distributed between D_2, D_3 and D_4. This change in transition probabilities explains why the choice of $D_0 = D_3$ yields closed-loop stability on average, while the one of $D_0 = D_4$ does not. The prediction horizon should thus reflect on this distribution evolution to improve the compensation capabilities of the controller. Future works should focus on this aspect.



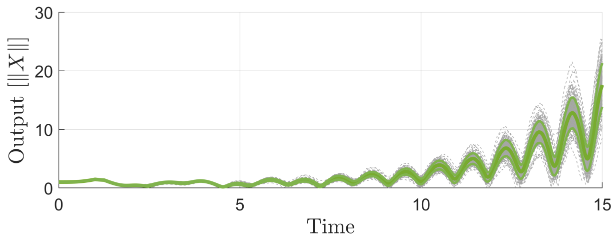
(a) Example of a realization of the stochastic delay D



(b) Realization of the signals U, x_1 and x_2 corresponding to the delay pictured in (a)



(c) Monte Carlo simulation of the closed-loop input U (100 trials)



(d) Monte Carlo simulation of $\|x\|$ (100 trials)

Fig. 2. Simulation results of the closed-loop system (44) and (5) for $\mathbf{D} = (0.5, 0.75, 1.0, 1.25, 1.5)^T$, $X(0) = [1 \quad 0]^T$ and $U(t) = 0$ for $t \leq 0$. The prediction horizon is $D_0 = 1.25$. The transition probabilities follow the dynamics (45)-(46). (a) and (b) picture results corresponding to one delay realization. (c) and (d) present the results of 100 trials, in which the means and the standard deviations are highlighted by the coloured lines.

6 Conclusion

In this paper, we proposed a constant horizon prediction-based controller to compensate for a stochastic input delay modelled as a Markov process with a finite number of values. We proved the exponential mean-square stability of the closed-loop control system provided that the delay values are limited and in the vicinity of the chosen prediction horizon. Simulations on a toy example illustrated the relevance of this condition and the interest of this prediction-based control law.

Simulation results emphasize the crucial role played by the delay distribution and its evolution. However, the robustness analysis provided in this paper does not distinguish between the probability distributions. It is built on the worst-case scenario of a uniform probability of taking any delay value. Hence, the sufficient condition we obtain is likely to be somehow conservative. Relaxing this condition by including the delay distribution into the stability analysis is an important direction of future work. Adapting the prediction-horizon to the current delay distribution could also be an interesting design feature to explore, as it is likely to increase the closed-loop delay-robustness. Similarly, capturing the dependence on the feedback gain on this robustness margin is an important practical question for control tuning, which should be explored.

Finally, the delay process considered in this work has a finite number of states. Extending this analysis to the case where the delay can take values in a given continuum is another challenging theoretical issue worth exploring in the future.

A Technical Lemmas

Lemma 5 *Consider* (3). *There exists a constant* R_0 *such that the function* Υ *defined in* (8) *satisfies*

$$\Upsilon(t) \leq R_0 \Upsilon(0), \quad t \in [0, \overline{D}] \tag{A.1}$$

PROOF. Using (21) in Lemma 3, for $t \in [0, \overline{D}]$, and defining $N_1 = 2e^{2|A|\overline{D}} \max\{1, |B|^2 \overline{D}\}$, it holds

$$\begin{aligned} &|X(t)|^2 \\ \leq& N_1 \left(|X(0)|^2 + \int_{-\overline{D}}^{t-\underline{D}} U(s)^2 ds \right) \\ =& N_1 \left(|X(0)|^2 + \int_{-\overline{D}}^{\min\{t-\underline{D},0\}} U_0(s)^2 ds + \int_{\min\{t-\underline{D},0\}}^{t-\underline{D}} U(s)^2 ds \right) \end{aligned} \tag{A.2}$$

with $U(t) = U_0(t)$ for $t \leq 0$. Thus, by using Theorem 2 in [5], the prediction-based control law can be also written as

$$\begin{aligned} &U(t) \\ &= K_D \Big[X(t) + \int_0^t \Phi_D(t,s) X(s) ds + \int_{-D_0}^0 \Phi_0(t,s) U_0(s) ds \Big] \end{aligned} \tag{A.3}$$

with $K_D = Ke^{AD_0}$, and Φ_D and Φ_0 two continuous functions defined in [5]. Replacing (A.3) into (A.2), one obtains

$$\begin{aligned} |X(t)|^2 \leq& N_1 \Big(|X(0)|^2 + \int_{-\overline{D}}^{\min\{t-\underline{D},0\}} U_0(s)^2 ds \\ &+ 3K_D \int_{\min\{t-\underline{D},0\}}^{t-\underline{D}} |X(s)|^2 ds \\ &+ 3K_D \int_{\min\{t-\underline{D},0\}}^{t-\underline{D}} \int_{-D_0}^0 \Phi_0(s,\xi)^2 U_0(\xi)^2 d\xi ds \\ &+ 3K_D \int_{\min\{t-\underline{D},0\}}^{t-\underline{D}} \int_0^s \Phi_D(s,\xi)^2 |X(\xi)|^2 d\xi ds \Big) \end{aligned} \tag{A.4}$$

Thus, using again (A.2) and (A.3) and by a straightforward iteration on time intervals of length $\underline{D}$, we can get that there exist $N_2 > 0$ and a continuous function $\tilde{\Phi}_0$ such that

$$|X(t)|^2 \leq N_2 |X(0)|^2 + \int_{-\overline{D}}^0 \tilde{\Phi}_0(t,s) U_0(s)^2 ds \tag{A.5}$$

Similarly, there exist a constant $N_3 > 0$ and a continuous function $\tilde{\tilde{\Phi}}_0$ such that

$$U(t)^2 \leq N_3 |X(0)|^2 + \int_{-\overline{D}}^0 \tilde{\tilde{\Phi}}_0(t,s) U_0(s)^2 ds \tag{A.6}$$

Therefore, as from (8), it holds

$$\Upsilon(t) = |X(t)|^2 + \int_{t-\overline{D}-D_0}^0 U_0(s)^2 ds + \int_0^t U(s)^2 ds \tag{A.7}$$

the conclusion follows from (A.5) and (A.6).

Lemma 6 *Consider the function h defined in (16). There exists* $M > 0$ *such that*

$$\|h(t + D_0(\cdot - 1))\|^2 \leq M V(t), \quad t \geq D_0 \tag{A.8}$$

PROOF. First, from (16), h can be expressed as

$$\begin{aligned} h(t) =& D_0 K \Big[e^{AD_0} A X(t) + e^{AD_0} B U(t - D(t)) \\ &+ BU(t) - e^{AD_0} BU(t - D_0) + A \int_{t-D_0}^t e^{A(t-s)} BU(s) ds \Big] \end{aligned} \tag{A.9}$$

Then, (A.9) gives

$$\begin{aligned} &\|h(t + D_0(\cdot - 1))\|^2 \\ &= \int_0^1 h(t + D_0(x-1))^2 dx \\ &\leq 5|K|^2 D_0^2 \int_0^1 \Big[M_1 |X(t + D_0(x-1))|^2 \\ &+ M_2 |U(t + D_0(x-1) - D(t + D_0(x-1)))|^2 \\ &+ M_3 |U(t + D_0(x-1))|^2 + M_4 |U(t + D_0(x-2))|^2 \\ &+ |A|^2 \int_{t+D_0(x-2)}^{t+D_0(x-1)} e^{2|A|(t+D_0(x-1)-s)} |B|^2 |U(s)|^2 ds \Big] dx \end{aligned} \tag{A.10}$$

with

$$\begin{cases} M_1 = e^{2|A|\overline{D}}|A|^2 \\ M_2 = e^{2|A|\overline{D}}|B|^2 \\ M_3 = |B|^2 \\ M_4 = e^{2|A|\overline{D}}|B|^2 \end{cases} \tag{A.11}$$

From the definition of the dynamics (3), it holds

$$\begin{aligned} &|X(t+D_0(x-1))| \qquad\qquad (A.12)\\ =&\left|e^{AD_0(x-1)}\left(X(t)-\int_{t+D_0(x-1)}^{t}e^{A(t-s)}BU(s-D(s))ds\right)\right|\\ \le& e^{|A|D_0}\left(|X(t)|+\int_{t+D_0(x-1)}^{t}e^{|A|(t-s)}|B|\sum_{j=1}^{r}|U(s-D_j)|ds\right)\\ \le& e^{|A|D_0}|X(t)|+e^{2|A|D_0}|B|\sum_{j=1}^{r}\int_{t+D_0(x-1)}^{t}|U(s-D_j)|ds \end{aligned}$$

Then, with (19), the equation (A.10) gives

$$\begin{aligned} &\|h(t+D_0(\bullet-1))\|^2 \qquad\qquad (A.13)\\ &\le 5|K|^2D_0^2\Big[2M_1e^{2|A|D_0}|X(t)|^2\\ &+2M_1re^{4|A|D_0}|B|^2(\|\mu(t)\|^2+M_6|X(t)|^2+M_6\|w(t)\|^2)\\ &+M_2r(\|\mu(t)\|^2+M_6|X(t)|^2+M_6\|w(t)\|^2)\\ &+M_3M_6|X(t)|^2+M_3M_6\|w(t)\|^2+M_4\|\mu(t)\|^2\\ &+M_5(\|\mu(t)\|^2+M_6|X(t)|^2+M_6\|w(t)\|^2)\Big]\\ &\le 5|K|^2\overline{D}^2[M_X|X(t)|^2+M_w\|w(t)\|^2+M_\mu\|\mu(t)\|^2] \end{aligned}$$

in which $M_5 = |A|^2e^{2|A|D_0}|B|^2$, $M_6 = 3(1+|K|^2e^{2|A+BK|D_0}$ $\max\{1, D_0^2|B|^2\})$ and the positive constants (M_X, M_w, M_μ) are defined as follows

$$\begin{cases} M_X = 2M_1e^{2|A|\overline{D}}+M_6(2M_1re^{4|A|\overline{D}}|B|^2+M_2r+M_3+M_5) \\ M_w = M_6(2M_1re^{4|A|\overline{D}}|B|^2+M_2r+M_3+M_5) \\ M_\mu = 2M_1re^{4|A|\overline{D}}|B|^2+M_2r+M_4+M_5 \end{cases} \tag{A.14}$$

Consequently, from the definition of Lyapunov functional $V(t)$ in (25)

$$\begin{aligned} &\|h(t+D_0(\bullet-1))\|^2\\ \le& 5|K|^2\overline{D}^2\max\left\{\frac{M_X}{\min(\lambda(P))},\frac{M_w}{b\underline{D}},\frac{M_\mu}{d\overline{D}}\right\}V(t) \end{aligned} \tag{A.15}$$

The lemma is then proved with the positive constant $M = 5|K|^2\overline{D}^2\max\{\frac{M_X}{\min(\lambda(P))},\frac{M_w}{b\underline{D}},\frac{M_\mu}{d\overline{D}}\}$.

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