Stabilization of an underactuated 1+2 linear hyperbolic system with a proper control

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Abstract— **In this paper we consider the stabilization of an underactuated system of linear first-order hyperbolic Partial Differential Equations consisting of one rightward-convecting transport PDE and two leftward-convecting equations. The system is underactuated in the sense that only one of the two leftward-convecting equations is actuated at the boundary. This is a fundamental difference compared to existing results in the literature, as it is usually assumed that all of the equations convecting in at least one direction are independently actuated at the boundary. The proposed approach combines a backstepping transformation and successive state transformations which allow the reformulation of the original system as a neutral system with distributed state delays and distributed input delays. Assuming that the actuator dynamics are asymptotically stable, the stabilization of the system is reduced to that of a simpler neutral system with only distributed state delays and a delayed control. Designing a predictor, it becomes possible to stabilize this simpler system. The proposed feedback law is then combined with a low pass-filter to guarantee the existence of delay-robustness margins for the closed-loop system.**

I. Introduction

In all real-life systems, signals, energy, matter and other quantities cannot be immediately transported across space. When modeling systems where propagation phenomena are *fast enough* with respect to other dynamics, these transport delays are usually neglected. Nevertheless, as performance requirements for physical systems grow more stringent and large-scale interconnected systems become more prevalent (sometimes spanning hundreds of kilometers in length, as is the case for electrical grids and pipelines), detailed models that include propagation mechanisms are required to adequately represent their dynamic behavior [1], [2], [3], [4], [5]. One natural mathematical representation of these transport phenomena is through hyperbolic Partial Differential Equations (PDEs).

The control of coupled hyperbolic PDEs is an active research topic [6], [7], [8], [9]. Unlike results for linear Ordinary Differential Equations (ODEs), constructive control designs, even for linear hyperbolic PDEs, are harder to find and often require specific controllability results [10], [11], or many independent actuators to be available [12]. From an application standpoint, reducing the number of required actuators provides clear advantages in terms of cost, weight, and overall system design complexity.

In this paper, we explore the effect of removing full actuation from one boundary of the backstepping design in [13]. We consider a system consisting of one rightward-convecting transport PDE and two leftward-convecting equations, with only one of the leftward-convecting equations actuated at the boundary. To construct a stabilizing control law for this system, we rely on backstepping techniques inspired by [14] in order to equivalently reformulate the three-PDE system stabilization problem as that of stabilizing a difference equation with pointwise and distributed delays in both the state and the control. Using a transform inspired by [15], the (direct) control action on one of the PDEs is recast as an (indirect) action on the two remaining PDEs with a pointwise-plus-distributed delay structure. If the resulting *actuator transfer function* does not have any poles on the complex right-half plane, a dynamic inversion procedure can be applied to stabilize the whole system using predictor-based techniques [16], [17], [18]. The stability requirement for this transfer function is the main limitation in the proposed approach as one could conceivably still stabilize the system as long as there are no unstable modes in the system corresponding to transmission zeros of the control operator. Nevertheless, this more restrictive assumption allows for a simpler explicit construction of a dynamic control operator. As a final step, it will be shown that an adequate low-pass filter can be added to the control design that renders the final control operator strictly proper. This property is desirable from a robustness perspective (with respect to high-frequency noise or delays in the control loop) as has been shown in [19], [20], [21].

The paper is organized as follows: in Section II, we present the problem under consideration, before reformulating it as a time-delay difference equation in Section III. Then, Section IV focuses on the design of a control law stabilizing this difference equation, which is then modified via a low-pass filter in Section V to guarantee delay-robust stabilization. Finally, the merits of our design are illustrated in numerical simulations in Section VI, before drawing conclusions.

Notations

We denote the functional space where the PDE states will be defined as $\chi = (L^2([0,1];\mathbb{R}))^3$ with the associated norm

$$||w||_\chi = \left(\int_0^1 w^T(\nu)w(\nu)d\nu\right)^{\frac{1}{2}}. \tag{1}$$

We denote $D = L^2([-\tau,0],\mathbb{R}^2)$ the Banach space of L^2 functions mapping the interval $[-\tau,0]$ into $\mathbb{R}^2$. For a function $\phi : [-\tau,\infty) \mapsto \mathbb{R}$, its partial trajectory $\phi_{[t]} \in D$ is defined by $\phi_{[t]}(\theta) = \phi(t+\theta)$, $-\tau \leq \theta \leq 0$. The associated norm is given by $||\phi_{[t]}||_D = \left(\int_{-\tau}^0 \phi^T(t+\theta)\phi(t+\theta)d\theta\right)^{\frac{1}{2}}$. We define for every $r > 0$, $||\phi_{[t]}||_r = \left(\int_{-r}^0 \phi^T(t+\theta)\phi(t+\theta)d\theta\right)^{\frac{1}{2}}$. We denote Id the identity matrix of dimension 2. We define

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the right closed half-plane as $\mathbb{C}^+ = \{s \in \mathbb{C} \mid \Re(s) \geq 0\}$. Finally, for any proper and stable transfer matrix $G(s)$, we denote $\bar{\sigma}(G(s))$ the largest singular value of $G(s)$ ($s \in \mathbb{C}$) and $\underline{\sigma}(G(s))$ its smallest one.

II. System under consideration

We consider a class of underactuated linear balance laws composed of one rightward-convecting transport PDE and two leftward-convecting equations. These equations can model (among others) a Drif-Flux Model (DFM) as described in [22]. For this example, it models the flow of liquid (oil, water and drilling fluid being considered as one liquid phase) and gas along a drillstring. More precisely, the equations correspond to two mass and one momentum conservation laws. This class of systems has been studied in the literature [13] in the case of full actuation at one-boundary. Here, we consider that only one of the two leftward-convecting equations can be actuated. More precisely, the class of systems under consideration in this paper is

$$\partial_t w(t,x) + \Lambda \partial_x w(t,x) = \Sigma(x) w(t,x), \tag{2}$$

where $w(t,x) = (u(t,x), v_1(t,x), v_2(t,x))^T$ is the state of the system, the different arguments evolving in $\{(t,x) \text{ s.t. } t > 0, x \in [0,1]\}$. We consider the boundary conditions

$$u(t,0) = q_1 v_1(t,0) + q_2 v_2(t,0), \tag{3}$$

$$v_1(t,1) = \rho_1 u(t,1) + V(t), \;\; v_2(t,1) = \rho_2 u(t,1), \tag{4}$$

The diagonal matrix Λ is given by $\Lambda = \text{diag}(\lambda, -\mu_1, -\mu_2)$, where the different velocities λ, μ_1, μ_2 are assumed to be constant and positive. The boundary couplings q_1, q_2, ρ_1 and ρ_2 are assumed to be constant. The in-domain coupling matrix is given by

$$\Sigma(x) = \begin{pmatrix} 0 & \sigma_1^+(x) & \sigma_2^+(x) \\ \sigma_1^{-+}(x) & 0 & \sigma_1^{--}(x) \\ \sigma_2^{-+}(x) & \sigma_2^{--}(x) & 0 \end{pmatrix},$$

where the functions σ_i^+ and σ_i^{--} and σ_i^{-+} are continuous. Note that the matrix Σ does not have any diagonal term, as these terms can be easily removed using a simple exponential change of coordinates (see [23] for details). The initial conditions of the state $w = (u, v_1, v_2)^T$ is denoted $w^0 = (u^0, v_1^0, v_2^0)^T$. It is assumed to belong to χ. The control input $V(t)$ takes values in $\mathbb{R}$. We denote τ_1 and τ_2 the characteristic transport times $\tau_1 = \frac{1}{\lambda} + \frac{1}{\mu_1}$, $\tau_2 = \frac{1}{\lambda} + \frac{1}{\mu_2}$, and $\tau = \max(\tau_1, \tau_2)$ the largest transport time. The system (2)-(4) is well-posed [23, Theorem A.6, page 254]. This class of system (2)-(4) is fundamentally different to the one considered in [13] as only one of the two leftward-convecting equations is actuated. Thus, the techniques developed in[13] cannot be straightforwardly adjusted to stabilize (2)-(4). In addition, we require the two following assumptions.

Assumption 1: The boundary parameters are such that $|\rho_1 q_1| + |\rho_2 q_2| < 1$.

Assumption 2: The boundary couplings of the system (2)-(4) verify $\rho_2 q_1 \neq 0$.

To these two (somehow natural) assumptions, we will add, later in the paper, a third assumption that corresponds to an asymptotic stability condition on the actuation chain.

Assumption 1 is equivalent (in the case where τ_1 and τ_2 are rationally independent) to the fact that the difference system

$$Z(t) = \begin{pmatrix} \rho_1 q_1 & 0 \\ \rho_2 q_1 & 0 \end{pmatrix} Z(t-\tau_1) + \begin{pmatrix} 0 & \rho_1 q_2 \\ 0 & \rho_2 q_2 \end{pmatrix} Z(t-\tau_2), \tag{5}$$

is exponentially stable (where $Z \in D$). More precisely, it has been shown in [24] and [14] that a necessary condition to guarantee the existence of delay-robustness margins for (2)-(4) is that the open-loop system without in-domain couplings must be exponentially stable. This is equivalent to the exponential stability of (5) (see [14] for details). Therefore, Assumption 1 constitutes a reasonable assumption as it is necessary for the existence of robustness margins for the closed-loop system. Second, Assumption 2 guarantees the stabilizability of the second leftward-convecting PDE through its boundary. Note that, however, it may be conservative as (under some conditions) it should be possible to stabilize this unactuated PDE through its in-domain coupling terms. This situation, which considerably complexifies the control design, may be a direction of future work but is out of the scope of the paper.

III. Time-delay formulation of the PDE system

In this section, we show that the original system (2)-(4) can be rewritten as a time-delay system of neutral type (difference system) with distributed delay terms. This is a straightforward application of [14]. More precisely, we have the following theorem.

Theorem 1: There exist $L^\infty([0,\tau],\mathbb{R})$-functions $G_{1,1}$, $G_{1,2}$, $G_{2,1}$ and $G_{2,2}$ which only depend on the system parameters such that the system (2)-(4) has stability properties equivalent to those of the difference system defined by

$$z_1(t) = \rho_1 q_1 z_1(t-\tau_1) + \rho_1 q_2 z_2(t-\tau_2) + V(t) + \int_0^\tau [G_{1,1}(\nu) z_1(t-\nu) + G_{1,2}(\nu) z_2(t-\nu)] \, d\nu \tag{6}$$

$$z_2(t) = \rho_2 q_1 z_1(t-\tau_1) + \rho_2 q_2 z_2(t-\tau_2) + \int_0^\tau [G_{2,1}(\nu) z_1(t-\nu) + G_{2,2}(\nu) z_2(t-\nu)] \, d\nu, \tag{7}$$

i.e., there exist two constants $C_1 > 0$ and $C_2 > 0$ and a constant $r > 0$ such that for all $t > \tau$,

$$C_1 ||(z_1,z_2)||_r \leq ||w||_\chi \leq C_2 ||(z_1,z_2)||_D. \tag{8}$$

Moreover, there exists a transformation $\mathcal{F}$ such that for all $t > \tau$, $(z_1(t), z_2(t)) = \mathcal{F}(w(t,\cdot))$.

Proof: The proof of this Theorem can be found in[14]. It relies on successive backstepping transformations. ■

This theorem means that we can consider the system (6)-(7) for the design of the control law. The resulting feedback law could then be expressed as a function of w using the operator $\mathcal{F}$. System (6)-(7) consists of two difference equations, only one being actuated. To simplify the problem, a natural choice is to define an alternative control law by

$$\bar{V}(t) = V(t) + \rho_1 q_1 z_1(t-\tau_1) + \rho_1 q_2 z_2(t-\tau_2) + \int_0^\tau [G_{1,1}(\nu) z_1(t-\nu) + G_{1,2}(\nu) z_2(t-\nu)] \, d\nu. \tag{9}$$

Then the system (6)-(7) rewrites as

$$z_1(t) = \bar{V}(t) \tag{10}$$

$$z_2(t) = \rho_2 q_1 \bar{V}(t-\tau_1) + \rho_2 q_2 z_2(t-\tau_2) + \int_0^\tau \left[G_{2,1}(\nu) \bar{V}(t-\nu) + G_{2,2}(\nu) z_2(t-\nu)\right] d\nu. \tag{11}$$

Choosing $\bar{V}$ as a feedback in z_2 and such that the solution of equations (11) converges to zero will imply the stabilization of z_1 (and consequently of the original state w using Theorem 1). Consequently, we can consider only equation (11) for the design of the control law $\bar{V}$. This is the approach we pursue in the next two sections.

However, it is worth noticing that the control law $V(t)$ resulting from such an approach requires the cancelation of the reflection terms $\rho_1 q_1 z_1(t-\tau_1)$ and $\rho_1 q_2 z_2(t-\tau_2)$. As shown in [19], this may have major consequences regarding the robustness margins of the closed-loop system since the corresponding feedback law is not strictly proper. To avoid this problem and make the control law strictly proper, we choose to combine it with a well-tuned low pass filter, the design of which is detailed in Section V.

IV. Stabilization of the difference equation (11)

In this section, we design a control law $\bar{V}$ that stabilizes the difference equation (11). The resulting feedback law can then be used for the stabilization of (2).

Note that the actuation in (11) appears both through the pointwise delay term $\rho_2 q_1 \bar{V}(t-\tau_1)$ and the distributed delay term $\int_0^\tau G_{2,1}(\nu)V(t-\nu)d\nu$. This distributed actuation term has been seldom studied in the literature [25], [26], is uncommon when considering difference equation and is a major difference compared to the existing results given in [14]. An additional difficulty is induced by the fact that the delay inside the integral lies between 0 and τ, whereas the pointwise delay is equal to τ_2. Thus, we cannot directly cancel the integral term using the pointwise delayed control term as the resulting control law would not be causal. The approach we have chosen to design our stabilizing control law is described as follows

1) We first consider successive state transformations that make the delays in the distributed part of the actuation larger that the pointwise delay (Subsection IV-A).
2) Then, it becomes possible to use the pointwise delayed actuation term to cancel the integral part, while preserving causality. Under an additional condition on the dynamics, the compensation of this integral actuation term does not create unstable loops and the resulting system rewrites as a difference equation with only a pointwise delayed actuation. (Subsection IV-B).
3) For such a system, we design a state-predictor. We finally show that the corresponding proposed control law guarantees the stabilization of the original system (Subsection IV-C).

A. Successive state transformations

In this section, we perform successive state transformations that will make the delays in the distributed part of the actuation larger that the pointwise delay. This is crucial to be able to causally cancel this integral term. Inspired by [15], let us consider the following family of state transformations defined for $k \in \mathbb{N}$ by $y_0(t) = z_2(t)$ and

$$y_{k+1}(t) = y_k(t) + \int_{k\tau_2}^{(k+1)\tau_2} R_k(\xi)\bar{V}(t-\xi)d\xi. \qquad (12)$$

where the functions R_k are $L^\infty([k\tau_2,(k+1)\tau_2],\mathbb{R})$-functions defined by the integral equation

$$R_k(x) = -\bar{G}_k(x) + \int_{k\tau_2}^{x} G_{2,2}(x-\xi)R_k(\xi)d\xi. \qquad (13)$$

Note that for a given $\bar{G}_k$, this equation always admit a solution as it is a Volterra equation [27]. The functions $\bar{G}_k$ are $L^\infty([k\tau_2, k\tau_2+\tau],\mathbb{R})$ defined for all $k \in \mathbb{N}^\star$ by $\bar{G}_0(x) = G_{2,1}(x)$ and by (14) if $(k+2)\tau_2 \geq k\tau_2+\tau$ and by (15) if $(k+2)\tau_2 \leq k\tau_2+\tau$.

With these definitions, we have the following lemma

Lemma 1: For every $k \in \mathbb{N}$, the state y_k is solution of the following difference equation

$$y_k = \rho_2 q_2 y_k(t-\tau_2) + \int_0^\tau G_{2,2}(\nu)y_k(t-\nu)d\nu + \rho_2 q_1 \bar{V}(t-\tau_1) + \int_{k\tau_2}^{\tau+k\tau_2} \bar{G}_k(\nu)\bar{V}(t-\nu)d\nu. \qquad (16)$$

Proof: The proof is based on an induction argument. The case $k=0$ is a consequence of the definition of y_0 and $\bar{G}_0$. Let us assume that the property holds at rank k. Using (12), we have

$$\begin{aligned} & y_{k+1} - \rho_2 q_2 y_{k+1}(t-\tau_2) - \int_0^\tau G_{2,2}(\nu)y_{k+1}(t-\nu)d\nu \\ = \;& \rho_2 q_1 \bar{V}(t-\tau_1) + \int_{k\tau_2}^{\tau+k\tau_2} \bar{G}_k(\nu)\bar{V}(t-\nu)d\nu + \int_{k\tau_2}^{(k+1)\tau_2} \\ & R_k(\nu)\bar{V}(t-\nu)d\nu - \rho_2 q_2 \int_{(k+1)\tau_2}^{(k+2)\tau_2} R_k(\nu-\tau_2)\bar{V}(t-\nu)d\nu \\ & - \int_0^\tau \int_{k\tau_2}^{(k+1)\tau_2} G_{2,2}(\nu)R_k(\xi)\bar{V}(t-\xi-\nu)d\xi d\nu. \qquad (17) \end{aligned}$$

Using the change of variables $\eta = \xi+\nu$, the last integral rewrites $I_k = \int_{k\tau_2}^{(k+1)\tau_2}(\int_\xi^{\tau+\xi} G_{2,2}(\eta-\xi)R_k(\xi)\bar{V}(t-\eta)d\eta)d\xi$. Using Fubini's theorem and using the definitions of $\bar{G}_k$ and R_k concludes the proof. ∎

Let us now define $N = \min_k\{k \in \mathbb{N} \mid k\tau_2 \geq \tau_1\}$. For this N, we have

$$y_N(t) = \rho_2 q_2 y_N(t-\tau_2) + \int_0^\tau G_{2,2}(\nu)y_N(t-\nu)d\nu \qquad (18)$$
$$+ \rho_2 q_1 \bar{V}(t-\tau_1) + \int_{N\tau_2-\tau_1}^{N\tau_2+\tau-\tau_1} \bar{G}_N(\nu+\tau_1)\bar{V}(t-\tau_1-\nu)d\nu.$$

Thanks to the proposed changes of variables, we have managed to make the distributed delay on the actuation larger than the actuation pointwise delay.

B. Reformulation as a system with only one actuation term

We can now use the pointwise-delayed actuation to cancel the distributed delayed actuation term. However, this direct cancellation may create some unstable loops. To avoid this situation, we consider that the actuation dynamics is asymptotically stable by formulating the following assumption [1].

Assumption 3: The holomorphic function

$$C(s) = \rho_2 q_1 + \int_{N\tau_2-\tau_1}^{\tau+N\tau_2-\tau_1} \bar{G}_N(\nu+\tau_1)e^{-\nu s}d\nu, \qquad (19)$$

[1] This assumption corresponds to some specific conditions on the different parameters of the system. The analysis of such a condition is out of the scope of this paper. However, one must be aware that this assumption is conservative. More precisely, see [28], the only necessary condition for the stabilizability of (18) is that the state operator ($O(s) = 1 - \rho_2 q_2 e^{-\tau_2 s} - \int_0^\tau G_{2,2}(\nu)e^{-\nu s}d\nu$) and $C(s)$ do not share any common zeros on the closed right half-plane. Unfortunately, we have not been able to design a control law only based on this general condition. This will be the purpose of further investigations

$$\bar{G}_{k+1}(x) = \begin{cases} \bar{G}_k(x) - \rho_2 q_2 R_k(x-\tau_2) - \int_{k\tau_2}^{(k+1)\tau_2} G_{2,2}(x-\xi)R_k(\xi)d\xi & \text{if } (k+1)\tau_2 \le x \le k\tau_2 + \tau \\ -\rho_2 q_2 R_k(x-\tau_2) - \int_{x-\tau}^{(k+1)\tau_2} G_{2,2}(x-\xi)R_k(\xi)d\xi & \text{if } k\tau_2 + \tau \le x \le (k+2)\tau_2 \\ -\int_{x-\tau}^{(k+1)\tau_2} G_{2,2}(x-\xi)R_k(\xi)d\xi & \text{if } (k+2)\tau_2 \le x \le (k+1)\tau_2 + \tau \end{cases} \quad (14)$$

$$\bar{G}_{k+1}(x) = \begin{cases} \bar{G}_k(x) - \rho_2 q_2 R_k(x-\tau_2) - \int_{k\tau_2}^{(k+1)\tau_2} G_{2,2}(x-\xi)R_k(\xi)d\xi & \text{if } (k+1)\tau_2 \le x \le (k+2)\tau_2 \\ \bar{G}_k(x) - \int_{k\tau_2}^{(k+1)\tau_2} G_{2,2}(x-\xi)R_k(\xi)d\xi & \text{if } (k+2)\tau_2 + \tau \le x \le k\tau_2 + \tau \\ -\int_{x-\tau}^{(k+1)\tau_2} G_{2,2}(x-\xi)R_k(\xi)d\xi & \text{if } k\tau_2 + \tau \le x \le (k+1)\tau_2 + \tau \end{cases} \quad (15)$$

does not have any roots on the closed right half-plane.

Lemma 2: A feedback law stabilizing the difference equation

$$y_N(t) = \rho_2 q_2 y_N(t-\tau_2) + \int_0^\tau G_{2,2}(\nu) y_N(t-\nu)d\nu + \tilde{V}(t-\tau_1) \quad (20)$$

results in a control law V stabilizing the system (10)–(11), defined through the following integral equation

$$\tilde{V}(t) = \rho_2 q_1 \bar{V}(t) + \int_{N\tau_2-\tau_1}^{\tau+N\tau_2-\tau_1} \bar{G}_N(\nu+\tau_1)\bar{V}(t-\nu)d\nu. \quad (21)$$

Proof: Define a new control input $\tilde{V}$ as in (21) with which (18) rewrites as (20). Consequently, if there exists a feedback law $\tilde{V}$ stabilizing (20), then the convergence of $\tilde{V}$ to zero implies the convergence of $\bar{V}$ to zero due to Assumption 3 (the control chain is asymptotically stable). This, in turn, implies the convergence of both z_1 and z_2 to zero due to (12). ∎

C. Predictor design

In this section, in virtue of Lemma 2, we choose a control law that stabilizes (20) as

$$\tilde{V}(t) = -\int_0^\tau G_{2,2}(\nu)P(t, t-\nu)d\nu \quad (22)$$

in which, for $t \ge 0$ and $s \in [t-\tau_1-\tau, t]$, $P(t,s)$ is the state prediction (see [16], [17])

$$P(t,s) = \begin{cases} y_N(s+\tau_1) & \text{if } s \in [t-\tau-\tau_1, t-\tau_1] \\ \rho_2 q_2 P(t, s-\tau_2) + \int_0^\tau G_{2,2}(\nu)P(t,s-\nu)d\nu + \tilde{V}(s) & \text{otherwise} \end{cases} \quad (23)$$

Observe that the function[2] $P(t,\cdot)$ is a τ_1 units of time ahead prediction of the function $y_{N,t} : s \in [-\tau, 0] \mapsto y_N(t+s)$.

Theorem 2: Under Assumptions 1-3, the closed-loop system consisting of the plant (20) and the control law (22) is exponentially stable.

Proof: The proof follows straightforwardly from the fact that $P(t,s) = y_N(s+\tau_1)$ for any $s \in [t-\tau_1-\tau, t]$ which implies that $\tilde{V}(t) = -\int_0^\tau G_{2,2}(\nu)y_N(t+\tau_1-\nu)d\nu$ and the result follows by plugging this control law back into (20) which implies $y_N(t) = \rho_2 q_2 y_N(t-\tau_2)$ with $|\rho_2 q_2| < 1$ by Assumption 1. ∎

[2] We write P as a function of two arguments to emphasize the fact that the prediction should be computed by incorporating measured delayed states available at time t, to improve its robustness in practice.

Note that we voluntarily choose not to cancel the pointwise delay term $\rho_2 q_2 y_N(t-\tau_2)$ in the closed-loop dynamics, to guarantee that the transfer function relating $\tilde{V}$ to y_N is strictly proper, a characteristic which is necessary in the following section.

V. Robustness aspects

Although the control law $V(t)$ designed in the previous section guarantees the stabilization of the system (6)-(7), it cancels the reflection terms $\rho_1 q_1 z_1(t-\tau_1)$ and $\rho_1 q_2 z_2(t-\tau_2)$ and is consequently non strictly proper. As mentioned above, this may raise important issues with respect to the existence of robustness margins at high frequencies (where the integral terms vanish) as we may, in presence of a small delay, add instabilities instead of canceling them [19]. In this section, we combine the control law $V(t)$ with a low-pass filter to make it strictly proper, while guaranteeing the nominal stabilization. The analysis we propose will be done in the Laplace domain. We will denote $Z(s) = (z_1(s), z_2(s))^T$ the Laplace transform of the state $(z_1(t), z_2(t))^T$. Let us denote $F(s)$ the holomorphic function defined by

$$F = \begin{pmatrix} \rho_1 q_1 \mathrm{e}^{-\tau_1 s} & \rho_1 q_2 \mathrm{e}^{-\tau_2 s} \\ \rho_2 q_1 \mathrm{e}^{-\tau_1 s} & \rho_2 q_2 \mathrm{e}^{-\tau_2 s} \end{pmatrix}. \quad (24)$$

It corresponds to the pointwise delay part of (6)-(7). Let us also denote $H(s)$, the transfer matrix corresponding to the integral part of (6)-(7)

$$H(s) = \begin{pmatrix} \int_0^\tau G_{1,1}(\nu)\mathrm{e}^{-\nu s}d\nu & \int_0^\tau G_{1,2}(\nu)\mathrm{e}^{-\nu s}d\nu \\ \int_0^\tau G_{2,1}(\nu)\mathrm{e}^{-\nu s}d\nu & \int_0^\tau G_{2,2}(\nu)\mathrm{e}^{-\nu s}d\nu \end{pmatrix}. \quad (25)$$

The function H is strictly proper due to Riemann-Lebesgues' lemma. Following the analysis done in the previous section, he control law $\bar{V}$ can be rewritten in the Laplace domain as $\bar{V}(s) = \bar{K}(s)Z(s)$, where $\bar{K}(s)$ is a line-vector whose components are holomorphic functions that are strictly proper. We do not give their explicit expression due to the space restrictions but they can be obtained using (12), (22) and an alternative expression of (23) based on the transition matrix of (20) (see [29]). Using (9), the control law $V(s)$ can then be rewritten as

$$V(s) = \bar{V} - (F_{11}(s) + H_{11}(s))Z_1(s) - (F_{12}(s) + H_{12}(s))Z_2(s),$$

which can be represented as

$$V(s) = (\bar{K}(s) - \begin{pmatrix}1 & 0\end{pmatrix}(F(s) + H(s)))Z(s) = K(s)Z(s).$$

The transfer function $K(s)$ is composed of strictly proper terms (namely the functions $H(s)$ and $\bar{K}(s)$) and of non-strictly proper terms (namely the term $F(s)$). More precisely, the functions $H(s)$ and $\bar{K}(s)$ corresponds to integral terms that vanish at high frequency, while the term $\begin{pmatrix}1 & 0\end{pmatrix} F(s)$ corresponds to pointwise delays terms and (as mentioned above) may be the source of robustness issues [19]. This is why we are going to low-pass filter this term. To distinguish the effects of the proper terms and of the non-proper terms in the control law, we will rewrite the function $K(s)$ as follows

$$K(s) = P(s) - (1 \quad 0)\, F(s), \tag{26}$$

where $P(s) = \bar{K}(s) - (1 \quad 0)\, H(s)$. The following lemma assesses several interesting properties on the different transfer functions.

Lemma 3: Consider F, H, K and P defined in (24)–(26). There exist $\epsilon_0 > 0$ and $0 < \epsilon_1 < 1$ such that, for any $s \in \mathbb{C}^+$,

$$\underline{\sigma}(Id - F(s) - H(s) - \begin{pmatrix}1\\0\end{pmatrix} K(s)) > \epsilon_0, \tag{27}$$

$$\bar{\sigma}(F(s)) < \epsilon_1 < 1. \tag{28}$$

Furthermore, there exists $M > 0$ such that for any $s \in \mathbb{C}^+$ with $|s| > M$ we have $\underline{\sigma}(Id - H(s) - \begin{pmatrix}1\\0\end{pmatrix} P(s)) > \epsilon_1$.

Proof: The stability of the nominal closed-loop system (6)-(7) assessed by its reformulation into (10)–(11) and Lemma 2 together with Theorem 2 implies that the characteristic equation of the closed-loop system is lower-bounded on the closed complex right half-plane [30]. This implies the first inequality (27) The second inequality (28) is a consequence of Assumption 1. Since P and H are strictly proper, $\underline{\sigma}(Id - H(s) - \begin{pmatrix}1\\0\end{pmatrix} P(s))$ converges to 1 when $|s| \to +\infty$ ($s \in \mathbb{C}^+$). This implies the last inequality. ■

We then have the following theorem

Theorem 3: Let $w(s)$ be any low-pass filter such that for all $s \in \mathbb{C}^+$

$$\begin{cases} |1 - w(s)| < \frac{\epsilon_0}{\epsilon_1} \text{ if } |s| \leq M, \\ |1 - w(s)| < \frac{1}{\epsilon_1}\underline{\sigma}(Id - H(s) - \begin{pmatrix}1\\0\end{pmatrix} P(s)) \text{ if } |s| > M, \end{cases} \tag{29}$$

where H and P are defined in (25)–(26) and M, ϵ_0 and ϵ_1 are defined in Lemma 3. Let us consider the control law defined in the Laplace domain by

$$V_f(s) = K_f(s) Z(s) \tag{30}$$

with $K_f(s) = P(s) - (w(s) \quad 0)\, F(s)$. Then, this control law delay-robustly stabilizes the system (6)-(7).

Proof: Definition of w: Note that the filter w is well defined since the second condition of (29) allows the convergence of w to zero for high frequencies. The first condition implies that w is close to one for sufficiently small $|s|$.

Stabilization: We first prove that the new control law V_f still guarantees the stabilization of the system (6)-(7). Plugging the control law inside (6)-(7), the characteristic equation of the closed-loop system now rewrites

$$\det(Id - F(s) - H(s) - \begin{pmatrix}1\\0\end{pmatrix} K_f(s)) = 0.$$

Using (30), this characteristic equation can be rewritten as

$$\det(Id - H(s) - \begin{pmatrix}1\\0\end{pmatrix} P(s) - \begin{pmatrix}1 - w(s) & 0\\ 0 & 1\end{pmatrix} F(s)) = 0.$$

To ease the notations, we will denote $Q(s) = Id - F(s) - H(s) - (1 \quad 0)^T K_f(s)$ so that the characteristic equation rewrites $\det(Q(s)) = 0$. To prove that the closed-loop system is exponentially stable, we need to show that this characteristic equation does not have any solution on $\mathbb{C}^+$. By contradiction, let us assume that there exists $s \in \mathbb{C}^+$ such that $\det(Q(s)) = 0$. If $|s| > M$, we have, using (29),

$$|1 - w(s)|\bar{\sigma}(F(s)) < \underline{\sigma}(Id - H(s) - \begin{pmatrix}1\\0\end{pmatrix} P(s)).$$

Thus, $\underline{\sigma}(Q(s)) > 0$ which leads to a contradiction. A similar contradiction can be obtained when $|s| \leq M$. Consequently, the characteristic equation cannot be satisfied on $\mathbb{C}^+$. This proves the stability of the closed-loop system.

Delay-robustness: The new control law is now strictly proper. This means that it is robust to small delays in the input and uncertainties on the parameters. The complete robustness proof follows the same ideas as that in [31]. ■

VI. Simulation results

The proposed control law has been tested in simulations using Matlab. The PDE system is simulated using a classical finite volume method based on a Godunov scheme [32]. We used 61 spatial discretization points (and a CFL number of 1). The algorithm we use to compute the different kernels (which are required to obtain the system (6)-(7)) is adapted from the one proposed in [31]. Using the method of characteristics, we write the integral equations associated to the kernel PDE-systems. These integral equations are solved using a fixed-point algorithm. The predictor is implemented using a backward Euler approximation of the integral involved in (23). The numerical values used are: $\lambda = 1$, $\mu_1 = 2$, $\mu_2 = 1$, $q_1 = q_2 = 0.6$ $\rho_1 = 0.9$, $\rho_2 = 0.4, \sigma_1^{+-} = \sigma_2^{+-} = 0.7, \sigma_1^{+} = 0, \sigma_2^{+} = 0.6, \sigma_1^{--} = 0.9, \sigma_2^{--} = 0$. These coefficients are chosen such that the PDEs system is unstable in open-loop. Assumption 1 and Assumption 2 are obviously satisfied. Finally, we have checked during the simulations that Assumption 3 was also satisfied. We show in Figure 1 the evolution of the χ-norm of the system for three different situations in presence of a delay of 0.2 seconds. In the first case, we have considered the ideal situation of [13] where two actuations are available, one acting at the boundary condition (4) (that we denote V_1, i.e. $v_1(t, 1) = \rho_1 u(t, 1) + V_1(t)$), the other one acting at the boundary condition (4) (that we denote V_2, i.e. $v_2(t, 1) = \rho_2 u(t, 1) + V_2(t)$). To avoid robustness problems (see [14] for details), we use these two control laws to cancel only the integral terms in (6)-(7). As expected, the resulting control system exponentially converges to zero. In the second case, we consider the framework of Section II in which V_2 is not available anymore ($V_2 \equiv 0$). We consider a naive approach, where we would only use the control law $V(t) \equiv V_1(t)$ to simply cancel the terms in (6). This implies $\bar{V} = 0$ and that equation (7) remains unactuated. This strategy is not convincing since the resulting closed-loop system diverges, as one could expect due to the unstable nature of the open loop dynamics. Finally, in the third case, we consider the filtered control law V_f defined in (30) based on the new original methodology developed in this paper. Such a control law robustly stabilizes the system (2)-(4). The control efforts for the first and the last strategies have been plotted in Figure 2.

VII. Concluding remarks

In this paper, we have developed a new and original strategy to stabilize an underactuated PDEs system consisting

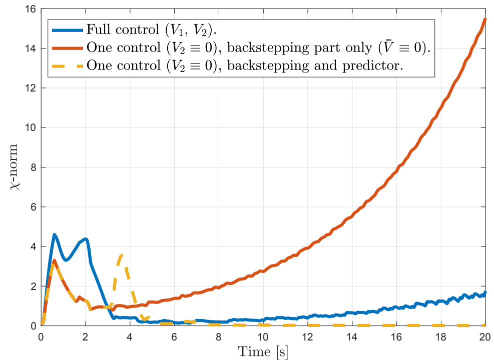

Fig. 1. Evolution of the χ-norm of the closed-loop system with an input delay of $0.2s$ for three different strategies: a) Two actuators available, backstepping strategy [14], b) One actuator, naive backstepping strategy, c) One actuator, filtered control law (30).

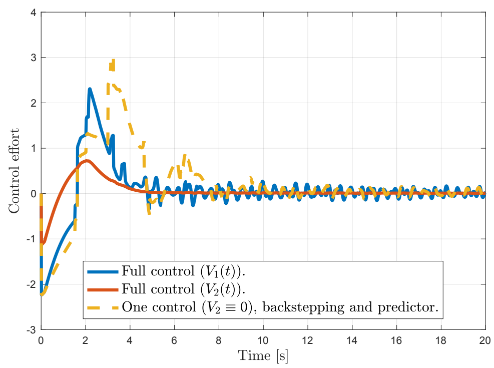

Fig. 2. Evolution of the control effort for two different strategies a) Two actuators available b) One actuator, filtered control law (30)

of one rightward-convecting transport PDE and two leftward-convecting equations. A current limitation of the proposed approach is that it requires a conservative assumption (Assumption 3) that guarantees the asymptotic stability of the control chain. How to overcome such an assumption and how to generalize the proposed approach to systems composed of more than three equations are crucial questions which will be the purpose of our next contributions.

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