Constant time horizon prediction-based control for linear systems with time-varying input delay

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Abstract:
We introduce a constant time horizon prediction-based controller to compensate for a time-varying input delay in a linear control system. We establish that this controller guarantees closed-loop exponential stability, provided that the time-varying delay remains sufficiently close to its average value D_0 and its rate of variation sufficiently small. This conclution only has to hold in average, in a mathematical sense that we specify.

Keywords: Delay systems; time-varying delays; prediction-based controller; distributed parameter systems

1. INTRODUCTION

Time-delay is a common phenomenon that exists widely in engineering systems, and usually brings great difficulties to system analysis and controller design. As the internal operating mechanisms of a dynamic system may not be single-threaded, the delays could be time-varying. For instance, see [Witrant (2005), Simon et al. (2017)], for network systems, information traveling between two nodes can be transmitted through different channels, yielding data reordering and transmission lag. The latter can depend on the congestion of the channel, which itself depends on the routine algorithm, and can thus vary to a large extent. Besides, these variations are hard to quantify and the resulting time-delay is often unknown in practice.

Numerous control systems have been developed to reduce the influence of time-delays, as exposed in Gu and Niculescu (2003) and Richard (2003). When the delay affects the input of a system, prediction-based controllers are an interesting tool to consider as they aim at compensating for the delay and thus eliminating its effect in the closed-loop dynamics. However, state-of-the-art prediction-based techniques (see [Smith (1957), Artstein (1982), Kwon and Pearson (1980), Manitius and Olbrot (1979)]) only apply to the constant delay case. Extension in the works of Nihtila (1991), Bekiaris-Liberis and Krstic (2012) to time-varying delays reveals troublesome and, in any case, requires the knowledge of the delay variations.

In this paper, we propose to use a predictor with a constant prediction horizon, chosen as an a priori known average value of the time-varying delay. We establish that this controller guarantees closed-loop exponential stability, provided that the time-varying delay remains sufficiently close to its average value D_0 and its rate of variation sufficiently small. This conclusion only has to hold in average, in a mathematical sense that we specify. This is consistent with previous results reported in the study of Bekiaris-Liberis and Krstic (2013), which considers time-varying perturbations of a nominal constant delay and obtains similar conditions bearing on the delay perturbation. However, in this paper, contrary to Bekiaris-Liberis and Krstic (2013), we do not need to restrict the rate of delay with $|\dot{D}| < 1$. This is achieved by introducing a different transport PDE representation of the delay and a corresponding Lyapunov functional for stability analysis, inspired by our previous research in Bresch-Pietri et al. (2018). This is the main contribution of the paper.

We start the paper with Section 2 in which we describe the problem statement and give the main result. Section 3 contains the proof of this result, reformulating a transport PDE with a backstepping transformation, inspired by Krstic and Smyshlyaev (2008), to perform Lyapunov stability analysis. Finally, the conservativeness of the obtained conditions is discussed with a numerical example in Section 4, which highlights the interests of the proposed method.

Notations: In the following, for a given scalar function $x : \mathbb{R} \to \mathbb{R}$, we denote $x_t : s \in [-\overline{D}, 0] \to x(t+s)$ for $t \in \mathbb{R}$ and a given positive constant $\overline{D}$.

2. PROBLEM STATEMENT

We consider the following potentially unstable linear dynamic system

$$\dot{X} = AX + BU(t - D(t)) \tag{1}$$

in which $X \in \mathbb{R}^n$, U is scalar and $D(t) \in [\underline{D}, \overline{D}]$ $(\overline{D} > \underline{D} > 0)$ is a positive time-varying delay which is continuously time differentiable function for $t \geq 0$. We assume that this plant is controllable and choose a prediction-based controller as

$$U(t) = K\left[e^{AD_0}X(t) + \int_{t-D_0}^{t} e^{A(t-s)}BU(s)ds\right] \qquad (2)$$

in which D_0 is the average value or a positive time-invariant estimation of the delay, which is a priori known, and K is a stabilizing feedback gain, such that $A+BK$ is Hurwitz.

Theorem 1. Consider the closed-loop system which consists of system (1) and the control law (2). Define the Lyapunov-Krasovskii functional

$$\Upsilon(t) = |X(t)|^2 + \int_{t-\max\{D(t),D_0\}}^{t} U(s)^2 ds + \int_{t-D_0}^{t} \dot{U}(s)^2 ds \qquad (3)$$

Assume there exist $\delta > 0$ and $\Delta > 0$ such that

$$\frac{1}{\Delta}\int_{t}^{t+\Delta}\left(\frac{|D_0 - D(s)|}{D_0} + |\dot{D}(s)|\right)ds \leq \delta, \quad t \geq 0 \qquad (4)$$

Then, there exists $\delta^* > 0$ such that, if $\delta < \delta^*$, there exist two positive constants R and γ such that

$$\Upsilon(t) \leq R \max \Upsilon_0 e^{-\gamma t} \qquad (5)$$

The prediction-based controller (2) would give an exact prediction $X(t+D_0)$ in the case of a constant input delay $D(t) = D_0$. Intuitively, one could guess that, though exact delay compensation is not achieved with the controller (2) in the case of a time-varying delay, exponential stability would still hold if the time-varying delay can be reasonably approximated by a constant. This is the sense of condition (4), which requires the time-varying delay $D(t)$ to be approximately equal to the average value D_0. Note that this condition only holds in average on a time window of length Δ. This means that pointwise fluctuations can still occur as long as they are compensated the rest of the time.

We now provide the proof of this theorem.

3. PROOF OF THEOREM 1

3.1 Transport Equations and Backstepping Transformation

To represent the time-varying delay phenomemon, define a distributed actuator $u(x,t) = U(t + D(t)(x-1))$, for $x \in [0,1]$, which satisfies the following dynamics

$$\begin{cases} D(t)u_t(x,t) = \big(1 + \dot{D}(t)(x-1)\big)u_x \\ u(1,t) = U(t) \end{cases} \qquad (6)$$

To account for the control law (2) based on a D_0 units of time prediction horizon, introduce an estimate of the distributed input as $\hat{u}(x,t) = U(t + D_0(x-1))$, and the corresponding distributed input estimation error $\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t)$. The dynamics of the extended state $(X, \hat{u}, \tilde{u})$ can then be written as

$$\begin{cases} \dot{X} = AX + B\big(\hat{u}(0,t) + \tilde{u}(0,t)\big) \\ D_0\hat{u}_t = \hat{u}_x(x,t) \\ \hat{u}(1,t) = U(t) \\ D(t)\tilde{u}_t = \big(1 + \dot{D}(t)(x-1)\big)\tilde{u}_x(x,t) \\ \qquad + \left(\dfrac{D_0 - D(t)}{D_0} + \dot{D}(x-1)\right)\hat{u}_x \\ \tilde{u}(1,t) = 0 \end{cases} \qquad (7)$$

In view of the stability analysis, we reformulate this system by introducing the backstepping transformation (see [Krstic and Smyshlyaev (2008)])

$$\begin{aligned} w(x,t) =& \hat{u}(x,t) - Ke^{AD_0x}X(t) \\ & - D_0\int_0^x Ke^{AD_0(x-y)}B\hat{u}(y)dy \end{aligned} \qquad (8)$$

Lemma 2. The backstepping transformation (8), along with the control law (2), transform the plant (7) into the following target system

$$\begin{cases} \dot{X} = (A+BK)X + Bw(0,t) + B\tilde{u}(0,t) \\ D_0w_t(x,t) = w_x(x,t) - D_0Ke^{AD_0x}B\tilde{u}(0,t) \\ w(1,t) = 0 \\ D(t)\tilde{u}_t(x,t) = \left(\dfrac{D_0 - D(t)}{D_0} + \dot{D}(x-1)\right)f\big(X(t), w(\cdot,t)\big) \\ \qquad + \big(1 + \dot{D}(t)(x-1)\big)\tilde{u}_x(x,t) \\ \tilde{u}(1,t) = 0 \end{cases} \qquad (9)$$

in which

$$\begin{aligned} f(X,w) =& w_x(x,t) + D_0K(A+BK)e^{(A+BK)D_0x}X(t) \\ & + D_0KBw(x,t) + \int_0^x KD_0(A+BK)D_0 \\ & \times e^{(A+BK)D_0(x-y)}Bw(y,t)dy \end{aligned} \qquad (10)$$

In addition, the partial space-derivative of the backstepping variable satisfies

$$\begin{cases} D_0w_{xt}(x,t) = w_{xx}(x,t) - D_0^2KAe^{AD_0x}B\tilde{u}(0,t) \\ w_x(1,t) = D_0Ke^{AD_0}B\tilde{u}(0,t) \end{cases} \qquad (11)$$

Proof. Taking the space-derivation and the time-derivative of (8), one concludes that

$$\begin{cases} D_0w_t(x,t) = w_x(x,t) - D_0Ke^{AD_0x}B\tilde{u}(0,t) \\ w(1,t) = \hat{u}(1,t) - KX(t) = 0 \end{cases} \qquad (12)$$

Using the inverse backstepping transformation of (8) (see [Krstic and Smyshlyaev (2008)])

$$\begin{aligned} \hat{u}(x,t) =& w(x,t) + Ke^{(A+BK)D_0x}X(t) \\ & + \int_0^x KD_0e^{(A+BK)D_0(x-y)}Bw(y,t)dy \end{aligned} \qquad (13)$$

One can formulate the derivative of $\hat{u}$ with respect to x as

$$\begin{aligned} \hat{u}_x(x,t) &= f(X(t), w(\cdot,t)) \\ &= w_x(x,t) + D_0K(A+BK)e^{(A+BK)D_0x}X(t) \\ &\quad + D_0KBw(x,t) + \int_0^x KD_0(A+BK) \\ &\quad \times D_0e^{(A+BK)D_0(x-y)}Bw(y,t)dy \end{aligned} \qquad (14)$$

We take the spatial derivative of (12) and use the first equation in (12) for $x=1$ to obtain

$$\begin{cases} D_0w_{xt}(x,t) = w_{xx}(x,t) - D_0^2KAe^{AD_0x}B\tilde{u}(0,t) \\ w_x(1,t) = D_0w_t(1,t) + D_0Ke^{AD_0}B\tilde{u}(0,t) \\ \qquad = D_0Ke^{AD_0}B\tilde{u}(0,t) \end{cases} \qquad (15)$$

3.2 Lyapunov Analysis

Based on the work of Bresch-Pietri et al. (2018), we construct a positive Lyapunov functional as follows:

$$\begin{aligned} V(t) &= X(t)^T PX(t) + bD_0 \int_0^1 (1+x)w(x,t)^2 dx \\ &+ cD \int_0^1 (1+x)\tilde{u}(x,t)^2 dx \\ &+ dD_0 \int_0^1 (1+x)w_x(x,t)^2 dx \end{aligned} \tag{16}$$

with $b, c, d > 0$, and P is the symmetric positive definite solution of the equation $P(A+BK)+(A+BK)^T P = -Q$, for a given symmetric positive definite matrix Q.

Lemma 3. If $g : t \to \frac{|D(t)-D_0|}{D_0} + |\dot{D}|$ satisfies the condition (4) given in Theorem 1, there exist $(b, c, d, \delta^*) \in {\mathbb{R}_+^*}^4$ such that, if $\delta < \delta^*$, there exist positive constants R and γ such that

$$V(t) \le R \max V_0 e^{-\gamma t} \tag{17}$$

where V is defined in (16).

Proof. Firstly, using Lemma 2, we obtain the derivative of the Lyapunov functional $V(t)$ as follows

$$\begin{aligned} \dot{V}(t) &= -X(t)^T QX(t) + 2X(t)^T PB[w(0,t) + \tilde{u}(0,t)] \\ &+ 2b \int_0^1 (1+x)w(x,t)\Big(w_x(x,t) - D_0 K e^{AD_0 x} \\ &\times B\tilde{u}(0,t)\Big)dx + c\dot{D}(t) \int_0^1 (1+x)\tilde{u}(x,t)^2 dx \\ &+ 2c \int_0^1 (1+x)\tilde{u}(x,t)\Big[(1+\dot{D}(t)(x-1))\tilde{u}_x(x,t) \\ &+ \Big(\frac{D_0 - D(t)}{D_0} + \dot{D}(x-1)\Big) f(X(t), w(\cdot,t))\Big]dx \\ &+ 2d \int_0^1 (1+x)w_x(x,t)\Big(w_{xx}(x,t) \\ &- D_0^2 KAe^{AD_0 x}B\tilde{u}(0,t)\Big)dx \end{aligned} \tag{18}$$

Using Cauchy-Schwarz and Young's inequalities (see Appendix A), and classifying polynomials by grouping the like terms, we obtain the following inequality

$$\begin{aligned} \dot{V} &\le \Big(-\frac{min(\lambda(Q))}{2} + cM_3 g(t)\gamma_3\Big)|X(t)|^2 \\ &+ \Big(-b + bM_1\gamma_1 + cM_4 g(t)\gamma_4 + cM_5 g(t)\gamma_5\Big)\|w(t)\|^2 \\ &+ \Big(-c + 2c|\dot{D}| + \frac{4c}{\gamma_2} + \frac{cM_3}{\gamma_3} + \frac{cM_4}{\gamma_4} + \frac{cM_5}{\gamma_5}\Big)\|\tilde{u}(t)\|^2 \\ &+ \Big(-d + 4cg(t)\gamma_2 + dM_6\gamma_6\Big)\|w_x(t)\|^2 \\ &+ \Big(-b + \frac{4\|PB\|^2}{min(\lambda(Q))}\Big)w(0,t)^2 \\ &+ \Big(-c + \frac{4\|PB\|^2}{min(\lambda(Q))} + c|\dot{D}| + \frac{bM_1}{\gamma_1} + \frac{dM_6}{\gamma_6} \\ &\quad + dM_7\Big)\tilde{u}(0,t)^2 \\ &+ (-d)w_x(0,t)^2 \end{aligned} \tag{19}$$

in which (see Appendix B) M_1, M_3, M_4, M_5, M_6, M_7 are positive constants, and the positive constants γ_1, γ_2, γ_3, γ_4, γ_5, γ_6 are chosen as follows:

(1) $\gamma_1 \le \frac{1}{M_1}$, $\gamma_6 \le \frac{1}{M_6}$,

(2) $b \ge \frac{4\|PB\|^2}{min(\lambda(Q))}$, $d \ge 0$,

(3) $c \ge \frac{b}{\gamma_1^2} + \frac{d}{\gamma_6^2} + dM_7 + \frac{4\|PB\|^2}{min(\lambda(Q))}$

(4) $(\gamma_2, \gamma_3, \gamma_4, \gamma_5)$ such that $\frac{4}{\gamma_2} + \frac{M_3}{\gamma_3} + \frac{M_4}{\gamma_4} + \frac{M_5}{\gamma_5} \le 1$.

From (2),

$$U^2(t) \le M_0\big(|X(t)|^2 + \|\hat{u}(t)\|^2\big) \tag{20}$$

in which $M_0 = 2\|K\|^2 \max\{e^{2\|A\|\overline{D}}, e^{2\|A\|D_0}\|B\|^2\}$

Using the fact that $\tilde{u}(0,t) = U(t - D(t)) - U(t - D_0)$, it holds

$$\begin{aligned} &\tilde{u}^2(0,t) \\ &\le 2M_0 \max_{s\in[-\max(D(t),D_0),0]} \big(|X(t+s)|^2 + \|\hat{u}(t+s)\|^2\big) \\ &\le M \max_{s\in[-\overline{D},0]} V(t+s) \end{aligned} \tag{21}$$

in which $M = 2M_0 \max\{(r_1+1), r_2\}$ with r_1 and r_2 (see Appendix C) are positive constants.

Then, it holds

$$\dot{V}(t) \le -\eta_1 V(t) + \eta_2 g(t) \max_{s\in[-\overline{D},0]} V(t+s), \quad t \ge 0 \tag{22}$$

defining

$$\begin{cases} \eta_1 = \dfrac{1}{a} min\Big\{\dfrac{min(\lambda(Q))}{2}, b(1 - M_1\gamma_1), d(1 - M_6\gamma_6), \\ \quad c\Big(1 - \dfrac{4}{\gamma_2} - \dfrac{M_3}{\gamma_3} - \dfrac{M_4}{\gamma_4} - \dfrac{M_5}{\gamma_5}\Big)\Big\} > 0 \\ \eta_2 = \dfrac{c}{a} max\{M_3\gamma_3, (M_4\gamma_4 + M_5\gamma_5), 2, 4\gamma_2\} \\ \quad + \dfrac{4c}{a}\|K\|^2 max\{e^{2\|A\|\overline{D}}, e^{2\|A\|D_0}\|B\|^2\} \\ \quad max\{(r_1+1), r_2\} > 0 \end{cases} \tag{23}$$

where $a = min\{min(\lambda(P)), bD_0, c\underline{D}, dD_0\}$, λ represents the eigenvalues of the matrix.

Applying Lemma 5 (see Appendix D), one concludes that there exists δ^* such that, if $\delta \le \delta^*$, there exist two constants $R, \gamma > 0$ such that

$$V(t) \le R \max V_0 e^{-\gamma t} \tag{24}$$

thus, the exponential stability of the Lyapunov functional V holds.

3.3 Exponential stability in terms of Υ

Lemma 4. The two Lyapunov functionals $V(t)$ and $\Upsilon(t)$ are equivalent, that is, there exist $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 V(t) \le \Upsilon(t) \le \lambda_2 V(t)$ for $t \ge 0$.

Proof. See Appendix C.

The proof of Theorem 1 is then completed, as a straightforward consequence of Lemmas 3 and 4.

4. SIMULATIONS

To illustrate the merits of the proposed prediction-based control, we consider the following LTI system as an example

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} X(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(t - D(t)) \qquad (25)$$

The control law (2) is applied with the feedback gain $K = -[3.5 \ 4.5]$, which corresponds to closed-loop eigenvalues $\lambda_p(A + BK) = [-1.7500 + 1.1990i, -1.7500 - 1.1990i]$.

We propose to compute the delay margin of the corresponding constant delay dynamic system, to find a reasonable range of δ^* which is mentioned in Theorem 1. According to the study of Mondié et al. (2001), when D is constant, the dynamic system (1) with the control law (2) has the characteristic matrix

$$M_{char} = \begin{bmatrix} sI - A & -Be^{-Ds} \\ -Ke^{AD_0} & I_m - K(sI-A)^{-1}(I - e^{-D_0(sI-A)})B \end{bmatrix} \qquad (26)$$

which corresponds to the characteristic equation

$$\begin{aligned} det(M_{char}) =& det(sI - A - [I_m - e^{-D_0(sI-A)} \\ &+ e^{D_0A - DsI}]BK) = 0 \end{aligned} \qquad (27)$$

We solve this equation using the QPmR routine [Vyhlídal and Zítek (2014)] with $D = D_0 + \Delta D$ and for different values of ΔD. Corresponding roots are pictured in Fig.1. When $\Delta D = 0$, one recovers a finite spectrum as (27) simplifies into

$$det(M_{char}) = det(sI - A - BK) = 0 \qquad (28)$$

which is consistent with the finite spectrum assignment property of a prediction-based controller. As ΔD increases continuously, the system reaches instability when the real part of the caracteristic roots crosses the imaginary axis, which occurs for $\Delta D = 0.08$.

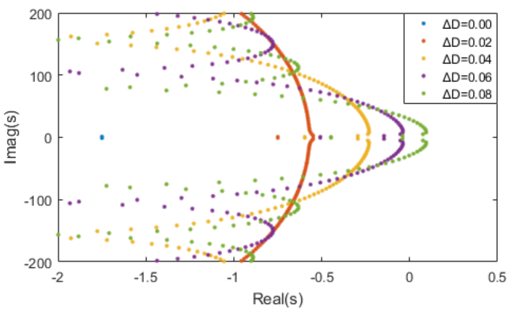
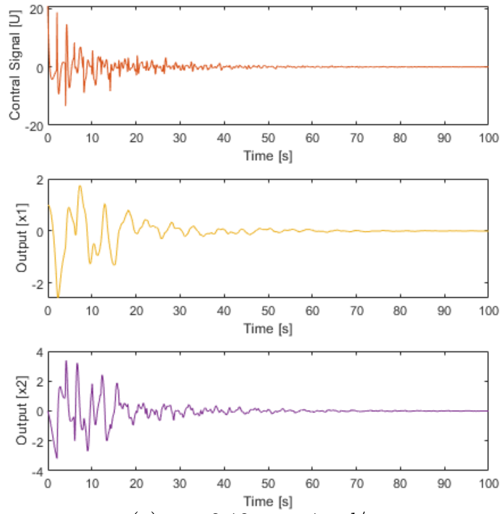
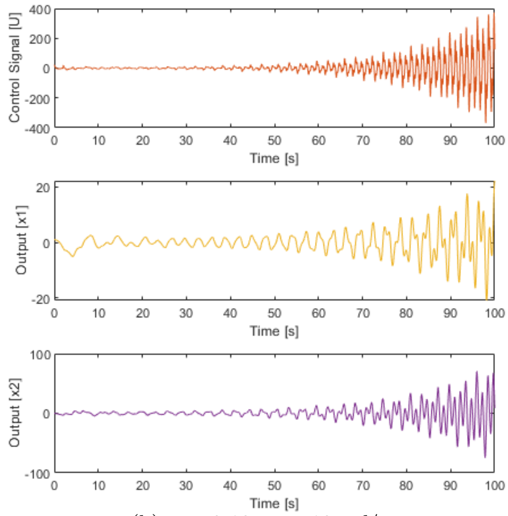


Fig. 1. Characteristic roots solutions to (27) for $D_0 = 2$, $D = D_0 + \Delta D$ and different values of ΔD.

Correspondingly, we now choose $D(t) = D_0 + asin(\omega t)$ with $a \leq \Delta D^*$. Firstly, we choose a set of parameters $(a, \omega) = (0.10, 1)$ that contributes to a stable system in Fig. 2(a), even though transient performances are deteriorated by a substantial delay variation rate. However, the selection of a higher pulsation $\omega_0 = 10rad/s$ results into an unstable closed-loop dynamics, in compliance with Theorem 1. The choice of a larger value of a ($a = 0.15$ for instance) would yield similar numerical results. This confirms the key role of both the delay magnitude and variation rate, which should be restricted in a sufficiently small range to guarantee closed-loop stability.



(a) $a = 0.10$, $\omega = 1rad/s$



(b) $a = 0.10$, $\omega = 10rad/s$

Fig. 2. (Delay perturbation $\Delta D = asin(\omega t)$) Simulation results with a feedback gain $K = -[3.5 \quad 4.5]$, initial conditions $X(0) = [1 \quad 0]^T$, estimated delay $D_0 = 2$.

5. CONCLUSION

In this paper, we formulated sufficient conditions to guarantee the exponential stability of a time-varying input delay system controlled with a constant horizon prediction-based controller. This result is of interest in the case where the current value of the delay is unknown and only statistical properties of the delay are available. Future works should focus on the extension of this technique in a stochastic framework which is an important concern of network systems.

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Appendix A. BOUNDS USED IN THE LYAPUNOV ANALYSIS

Using the Cauchy-Schwartz and Young's inequalites, each part of the derivative of Lyapunov functional V with respect to t can be bounded as follows:

$$2X(t)^T PB[w(0,t) + \tilde{u}(0,t)] \le \frac{min(\lambda(Q))}{2}|X(t)|^2 + \frac{4\|PB\|^2}{min(\lambda(Q))}(w(0,t)^2 + \tilde{u}(0,t)^2)$$

$$2b\int_0^1 (1+x)w(x,t)w_x(x,t)dx \le 2bD_0\|K\|e^{\|A\|D_0}\|B\|\left(\frac{1}{\gamma_1}\tilde{u}(0,t)^2 + \gamma_1\|w(t)\|^2\right)$$

$$c\dot{D}(t)\int_0^1 (1+x)\tilde{u}(x,t)^2 dx \le 2c|\dot{D}(t)|\|\tilde{u}(t)\|^2$$

$$2c\int_0^1 (1+x)\tilde{u}(x,t)[1+\dot{D}(t)(x-1)]\tilde{u}_x(x,t)dx \le c(|\dot{D}|-1)\tilde{u}(0,t)^2 + c(2|\dot{D}|-1)\|\tilde{u}(t)\|^2$$

$$2c\int_0^1 (1+x)\tilde{u}(x,t)\left(\frac{D_0 - D(t)}{D_0} + \dot{D}(x-1)\right)w_x(x,t)dx \le 2c\left(\frac{|D_0 - D|}{D_0} + \frac{|\dot{D}|}{2}\right)\left(\frac{1}{\gamma_2}\|\tilde{u}(t)\|^2 + \gamma_2\|w_x(t)\|^2\right)$$

$$2c\int_0^1 (1+x)\tilde{u}(x,t)\left(\frac{D_0 - D(t)}{D_0} + \dot{D}(x-1)\right) \times K(A+BK)D_0 e^{(A+BK)D_0 x}X(t)dx \le 2c\|K\|\|A+BK\|D_0 e^{\|A+BK\|D_0}\left(\frac{|D_0 - D|}{D_0} + \frac{|\dot{D}|}{2}\right) \times \left(\frac{1}{\gamma_3}\|\tilde{u}(t)\|^2 + \gamma_3|X(t)|^2\right)$$

$$2c\int_0^1 (1+x)\tilde{u}(x,t)\left(\frac{D_0 - D(t)}{D_0} + \dot{D}(x-1)\right)D_0 KB \times w(x,t)dx \le 2cD_0\|K\|\|B\|\left(\frac{|D_0 - D|}{D_0} + \frac{|\dot{D}|}{2}\right) \times \left(\frac{1}{\gamma_4}\|\tilde{u}(t)\|^2 + \gamma_4\|w(t)\|^2\right)$$

$$2c\int_0^1 (1+x)\tilde{u}(x,t)\left(\frac{D_0 - D(t)}{D_0} + \dot{D}(x-1)\right)D_0 \times \int_0^x K(A+BK)D_0 e^{(A+BK)D_0(x-y)}Bw(y,t)dydx \le 2cD_0^2\|K\|\|A+BK\|e^{\|A+BK\|D_0}\|B\| \times \left(\frac{|D_0 - D|}{D_0} + \frac{|\dot{D}|}{2}\right)(\frac{1}{\gamma_5}\|\tilde{u}(t)\|^2 + \gamma_5\|w(t)\|^2)$$

$$2d\int_0^1 (1+x)w_x(x,t)w_{xx}(x,t)dx \le 2dD_0^2(\|K\|e^{\|A\|D_0}\|B\|)^2\tilde{u}(0,t)^2 - dw_x(0,t)^2 - d\|w_x(t)\|^2$$

$$2d\int_0^1 (1+x)w_x(x,t)D_0^2 KAe^{AD_0 x}B\tilde{u}(0,t)dx \le 2dD_0^2\|K\|\|A\|e^{\|A\|D_0}\|B\|(\frac{1}{\gamma_6}\|w_x(t)\|^2 + \gamma_6\tilde{u}(0,t)^2)$$

Appendix B. CONSTANTS IN (19)

$(M_1, M_3, M_4, M_5, M_6, M_7)$ are positive constants that depend on the feedback gain and estimated delay of the system, and are given as

$$M_1 = 2D_0\|K\|e^{\|A\|D_0}\|B\| \tag{B.1}$$

$$M_3 = 2D_0\|K\|\|A+BK\|e^{\|A+BK\|D_0} \tag{B.2}$$

$$M_4 = 2D_0\|KB\| \tag{B.3}$$

$$M_5 = 2D_0^2\|K\|\|A+BK\|e^{\|A+BK\|D_0}\|B\| \tag{B.4}$$

$$M_6 = 2D_0^2\|KA\|e^{\|A\|D_0}\|B\| \tag{B.5}$$

$$M_7 = 2D_0^2(\|K\|e^{\|A\|D_0}\|B\|)^2 \tag{B.6}$$

Appendix C. PROOF OF LEMMA 4

Firstly, we define another Lyapunov functional $\Gamma(t)$ based on the state $(X, \hat{u}, \tilde{u})$

$$\Gamma(t) = |X(t)|^2 + \|u(t)\|^2 + \|\hat{u}(t)\|^2 + \|\hat{u}_x(t)\|^2 \quad \text{(C.1)}$$

From the definition of u and $\hat{u}$, one gets

$$\begin{aligned}\Gamma(t) &= |X(t)|^2 + \int_{t-D(t)}^{t} U(s)^2 ds + \int_{t-D_0}^{t} U(s)^2 ds \\ &\quad + \int_{t-D_0}^{t} \dot{U}(s)^2 ds \\ &= |X(t)|^2 + 2\int_{t-max\{D(t),D_0\}}^{t} U(s)^2 ds \\ &\quad - \int_{t-max\{D(t),D_0\}}^{t-min\{D(t),D_0\}} U(s)^2 ds + \int_{t-D_0}^{t} \dot{U}(s)^2 ds\end{aligned} \quad \text{(C.2)}$$

and hence

$$\Upsilon(t) \leq \Gamma(t) \leq 2\Upsilon(t) \quad \text{(C.3)}$$

From the backstepping transformation (8) and its inverse (13), there exist $r_i, s_i > 0$ such that

$$\|\hat{u}(t)\|^2 \leq r_1|X(t)|^2 + r_2\|w(t)\|^2 \quad \text{(C.4)}$$

$$\|\hat{u}_x(t)\|^2 \leq r_3|X(t)|^2 + r_4\|w(t)\|^2 + r_5\|w_x(t)\|^2 \quad \text{(C.5)}$$

$$\|\tilde{u}(t)\|^2 \leq 2\|u(t)\|^2 + 2\|\hat{u}(t)\|^2 \quad \text{(C.6)}$$

$$\|w(t)\|^2 \leq s_1|X(t)|^2 + s_2\|\hat{u}(t)\|^2 \quad \text{(C.7)}$$

$$\|w_x(t)\|^2 \leq s_3|X(t)|^2 + s_4\|\hat{u}(t)\|^2 + s_5\|\hat{u}_x(t)\|^2 \quad \text{(C.8)}$$

From which one can conclude that

$$\begin{aligned}\Gamma(t) &= |X(t)|^2 + \|u(t)\|^2 + \|\hat{u}(t)\|^2 + \|\hat{u}_x(t)\|^2 \\ &\leq |X(t)|^2 + 2\|\tilde{u}(t)\|^2 + 3\|\hat{u}(t)\|^2 + \|\hat{u}_x(t)\|^2 \\ &\leq \frac{max\{1+3r_1+r_3, 3r_2+r_4, r_5, 2\}}{min\{\underline{\lambda}(P), bD_0, c\underline{D}, dD_0\}} V(t)\end{aligned} \quad \text{(C.9)}$$

and that

$$\begin{aligned}V(t) &\leq \overline{\lambda}(P)|X(t)|^2 + 2bD_0\|w(t)\|^2 \\ &\quad + 2c\overline{D}\|\tilde{u}(t)\|^2 + 2dD_0\|w_x(t)\|^2 \\ &\leq (\overline{\lambda}(P) + 2bD_0s_1 + 2dD_0s_3)|X(t)|^2 \\ &\quad + (2bD_0s_2 + 4c\overline{D} + 2dD_0s_4)\|\hat{u}(t)\|^2 \\ &\quad + 4c\overline{D}\|u(t)\|^2 + 2dD_0s_5\|\hat{u}_x(t)\|^2 \\ &\leq max\{\overline{\lambda}(P) + 2bD_0s_1 + 2dD_0s_3, 4c\overline{D}, \\ &\quad 2bD_0s_2 + 4c\overline{D} + 2dD_0s_4, 2dD_0s_5\}\Gamma(t)\end{aligned} \quad \text{(C.10)}$$

The desired result follows defining

$$\begin{cases}\lambda_1 = \Big(2max\{\overline{\lambda}(P) + 2bD_0s_1 + 2dD_0s_3, 4c\overline{D}, \\ \quad + 4c\overline{D} + 2dD_0s_4, 2dD_0s_5\}\Big)^{-1} \\ \lambda_2 = \dfrac{max\{1+3r_1+r_3, 3r_2+r_4, r_5, 2\}}{min\{\underline{\lambda}(P), bD_0, c\underline{D}, dD_0\}}\end{cases} \quad \text{(C.11)}$$

Appendix D. TIME-VARYING HALANAY INEQUALITY

We detail here a new version of a time-varying Halanay inequality, inspired by the work of Bresch-Pietri et al. (2018).

Lemma 5. Consider a nonnegative differentiable function x such that for $t \geq 0$

$$\begin{cases}\dot{x}(t) \leq -\eta_1 x(t) + \eta_2 g(t) \max_{s\in[-\overline{D},0]} x(t+s) \\ x_0 = \psi \quad \in \mathcal{C}([-\overline{D}, 0], \mathbb{R})\end{cases} \quad \text{(D.1)}$$

in which $\overline{D} > 0$, $(\eta_1, \eta_2) \in {\mathbb{R}_+^*}^2$ and g is a nonnegative continuous function which satisfies, for certain $\Delta > 0$ and $\delta > 0$,

$$\frac{1}{\Delta}\int_t^{t+\Delta} g(s)ds \leq \delta, \quad t \geq 0 \quad \text{(D.2)}$$

Then, there exists $\delta^* > 0$, such that, if $\delta < \delta^*$,

$$x(t) \leq R \max\psi e^{-\gamma t} \quad \text{(D.3)}$$

in which R and γ are two positive constants.

Proof. First, we prove that there exists $R > 0$ such that

$$\max_{s\in[-\Delta-\overline{D},0]} x(\Delta+s) \leq R\max\psi \quad \text{(D.4)}$$

Let $k > \max\psi$ and define

$$y(t) = \begin{cases}k, & t \in [-\overline{D}, 0] \\ k\exp\big(\int_0^t \eta_2 g(s)ds\big), & t \in [0, \Delta]\end{cases} \quad \text{(D.5)}$$

which is a non-decreasing function and thus

$$\dot{y}(t) = \eta_2 g(t) \max_{s\in[-\overline{D},0]} y(t+s) > 0 \quad \text{(D.6)}$$

Consider $z = y - x$ which is a continuous function. Then, from (D.1) and (D.6), one obtains

$$\begin{aligned}\dot{z}(t) &= \eta_2 g(t) \max_{s\in[-\overline{D},0]} y(t+s) \\ &\quad - \big(-\eta_1 x(t) + \eta_2 g(t)\max_{s\in[-\overline{D},0]} x(t+s)\big) \\ &> \eta_2 g(t)\big(\max_{s\in[-\overline{D},0]} y(t+s) - \max_{s\in[-\overline{D},0]} x(t+s)\big) \\ &\quad + \eta_1 x(t) > 0\end{aligned} \quad \text{(D.7)}$$

which gives, as $k > \max\psi$

$$x(t) \leq \exp\big(\eta_2\int_0^t g(s)ds\big)\max\psi, \quad t \in [0, T] \quad \text{(D.8)}$$

from which (D.4) follows.

Second, $t \geq \Delta$, integrating (D.1) on $[t-\Delta, t]$, one gets

$$\begin{aligned}x(t) &\leq \Big(e^{-\eta_1\Delta} + \eta_2\int_{t-\Delta}^{t} e^{-\eta_1(t-s)}g(s)ds\Big) \\ &\quad \times \max_{s\in[-\Delta-\overline{D},0]} x(t+s) \\ &\leq \Big(e^{-\eta_1\Delta} + \eta_2\delta\Delta\Big)\max_{s\in[-\Delta-\overline{D},0]} x(t+s) \\ &\leq c\max_{s\in[-\Delta-\overline{D},0]} x(t+s)\end{aligned} \quad \text{(D.9)}$$

in which c is a positive constant such that $\lim_{\delta\to 0} c = e^{-\eta_1\Delta} < 1$.

Finally, we can always find a positive constant δ^* such that, if $\delta \leq \delta^*$, there exist $R > 0$ and $\gamma > 0$ such that x satisfies (D.3).