This is the simplest problem I have found to address Knut’s work from a theoretical point of view.

1 Original problem

The dynamics are

\[
\frac{dx}{dt} = v
\]  

(1)

subject to an control constraint

\[
v \in [v-, v+]
\]  

(2)

and a state constraint

\[
x \in [x-, x+]
\]  

(3)

We wish to minimize

\[
J(u) = \int_{a}^{b} L(x, v) dt
\]  

(4)

where \( L \) is continuous. We shall try to avoid strict convexity since we wish to address minimum time problems.

2 Saturation functions and new problem statements

There exist infinitely differentiable, strictly monotone functions \( \xi_1 \) and \( \xi_2 \) which map the intervals \( (x-, x+) \) and \( (u-, u+) \) respectively, into \( (-\infty, +\infty) \). If we set

\[
x = \psi_1(\xi), \quad v = \psi_2(u)
\]  

(5)
then for any \((x, v)\) which satisfy (1) and \(v \in (v-, v+), x \in (x-, x+)\) then there exist unique functions of time \(\psi\) and \(u\) defined by (5) which satisfy

\[
\frac{d\xi}{dt} = \left[ \frac{d\psi_1}{d\xi}(\xi) \right]^{-1} \psi_2(u) \quad (6)
\]

Indeed, we have

\[
\frac{dx}{dt} = \frac{d}{dt}(\psi_1(\xi)) = \frac{d\psi_1}{d\xi} \frac{d\xi}{dt} = v \quad (7)
\]

\[
= \psi_2(u) \quad (8)
\]

The differential equation (6) is well defined since \(\frac{d\psi_1}{d\xi}\) never vanishes. Its right handside is, however, not lipschitz if \(\xi\) is not bounded, since \(\frac{d\psi_1}{d\xi}\) vanishes at the infinity. This means that we may have a finite time blowup of \(\xi\), e.g. saturation of the state constraints in the original coordinates and in a finite time, if consider general solutions of (6), and not only the transforms of (1) with \(x \in (x-, x+)\) and \(v \in (v-, v+)\). As an example, a typical saturation function \(\psi_1(\xi)\) behaves like \(e^{-\xi}\) when \(\xi\) is close to the \(+\infty\); since \(\psi_2\) is bounded, (6) behaves roughly like

\[
\frac{d\xi}{dt} = ae^{\xi} \quad (11)
\]

Setting classically \(z = e^\xi\), we end up with an equation of the kind

\[
\frac{dz}{dt} = az^2 \quad (12)
\]

whose solution \(z\) behaves (for \(a < 0\)) like \(\frac{1}{t-t_i}\), i.e. we have

\[
\xi(t) \sim \log(1/(t-t_i)) = -\log(t-t_i) \quad (13)
\]
which blows up when t goes to \( t_i \).

Note: the same problem exists with the dynamics (25b) in the IFAC Goddard paper by Graichen & Petit.

Conversely, if \((\xi, v)\) is a maximal solution of (6), then \((x, v)\) defined by (5) satisfy (1) over an interval which may be bounded and with \( v \in (v-, v+) \), \( x \in (x-, x+) \).

We consider now the family of optimisation problems

\[
\min J_\epsilon(u) = \int_a^b L(\psi_1(\xi), \psi_2(u)) + \epsilon(\xi^2 + u^2) \, dt
\]

where \( \xi \) and \( u \) are unconstrained and (6) has a solution on \([a,b]\) for the given initial condition.

The cost \( J_\epsilon \) has an infimum because \( \xi^2 + u^2 \) is non negative and \( L(\psi_1(\xi), \psi_2(u)) \) is lower bounded. We show now that \( \xi \) and \( u \) are bounded in \( L^2 \).

**Lemma 1** If \( u_0 \) is an admissible control and \( \mu = J_\epsilon(u_0) \) then, if \( u \) is admissible with \( J_\epsilon(u) \leq \mu \), we have

\[
\int_{-\infty}^{+\infty} (\xi^2 + u^2) \, dt \leq \frac{1}{\epsilon} (\mu - (b - a) \inf L)
\]

where \( \inf L \) is the lower bound of \( L(x, v) \) over \((x-, x+) \times (v-, v+)\).

**Question:** can trajectories with \( \xi \) and \( u \) in \( L^2 \) blow up in finite time?

A partial answer lies here: in the neighbourhood of 0, \( \log t \) is square integrable. Indeed, we have

\[
\int (\log t)^2 \, dt = t(\log t)^2 - \int \frac{2}{t} \log t \, dt 
\]

(16)

\[
= t(\log t)^2 - 2t \log t + \int \frac{t}{t} \, dt
\]

(17)

(18)
hence we may have finite a time explosion of $\xi$ while $\xi$ is $L^2$.

Any minimizing sequence of $J_\epsilon$ will have, after some time, bounded $\xi$ and $u$. At the very least, we will have weak convergence of some subsequence. Observe that, if $u$ converges weakly, $\xi$ converges strongly in $L^2$1. Also, on the numerical side, putting a lower bound on the mesh refinement prevents high frequency oscillation, i.e. weak convergence.

Another approach is to assume

- existence of an optimal control for the the cost $J_\epsilon$ and the new variables (something we cannot reasonably assume for $\epsilon = 0$).
- existence and local regularity with respect to parameters of the control solutions of $\frac{\partial H_\epsilon}{\partial u} = 0$ where

$$H(\xi, u, \lambda, \epsilon) = L(\psi_1(\xi), \psi_2(u)) + \epsilon(\xi^2 + u^2) + \lambda \left[ \frac{d\psi_1}{d\xi}(\xi) \right]^{-1} \psi_2(u)$$

(19)

Observe that $H$ is continuous and lower bounded \textbf{if} $\xi$ and $\lambda$ are bounded.

Since the original optimal control may lead to constraint saturations, one must keep in mind that, for $\epsilon = 0$,

$$\arg\min_u H(\xi, u, \lambda, 0) = \pm\infty$$

(20)

along the optimal primal/dual trajectories. On the contrary, for $\epsilon > 0$, the control $u$ which minimizes $H(\xi, u, \lambda, \epsilon)$ is necessarily finite. Indeed, we have, if $u^\#$ minimizes $H(\xi, u, \lambda, \epsilon)$:

$$(u^\#)^2 \leq -\xi^2 + \frac{1}{\epsilon} \left[ \min_x H - \min_{\psi_2 (\xi)} L - \lambda \left[ \frac{d\psi_1}{d\xi}(\xi) \right]^{-1} \min_{\nu (\xi)} \nu \right]$$

(21)

\footnote{for the nonlinear case averaging may help}
which is bounded if \( \lambda \) and \( \xi \) are bounded, and \( \epsilon > 0 \).

To compute an optimal control (which is assumed to exist), we seek to solve the boundary value problem

\[
\frac{d\xi}{dt} = \left[ \frac{d\psi_1}{d\xi}(\xi) \right]^{-1} \psi_2(u^\#(\xi, \lambda, \epsilon)) \tag{22}
\]

\[
\frac{d\lambda}{dt} = -\frac{\partial H}{\partial \xi}(\xi, u^\#(\xi, \lambda, \epsilon), \lambda, \epsilon) \tag{23}
\]

with \( u^\#(\xi, \lambda, \epsilon) = \operatorname{argmin}_u H(\xi, u, \lambda, \epsilon) \tag{24} \)

with suitable initial and final conditions.

Observe that taking the min in (24) is not equivalent to \( \frac{\partial H}{\partial u} = 0 \) since, especially for small \( \epsilon \), \( H \) may not be convex with respect to \( u \). While it is reasonable to assume that a unique global optimum exists, we cannot assume in the general case that multiple extrema do not occur.

The question is: for a nonnegative \( \epsilon \), does this BVP has a solution over \([a, b]\)?