This is the simplest problem I have found to address Knut’s work from a theoretical point of view.

1 Original problem

The dynamics are
\[
\frac{dx}{dt} = \nu
\]  
subject to an control constraint
\[
\nu \in [\nu-, \nu+]
\]  
and a state constraint
\[
x \in [x-, x+]
\]  
We wish to minimize
\[
J(u) = \int_a^b L(x, \nu) dt
\]  
where \(L\) is continuous. We shall try to avoid strict convexity since we wish to address minimum time problems.

2 Saturation functions and new problem statements

There exist infinitely differentiable, strictly monotone functions \(\xi_1\) and \(\xi_2\) which map the intervals \((x-, x+)\) and \((u-, u+)\) respectively, into \((-\infty, +\infty)\). If we set
\[
x = \psi_1(\xi) \, , \, \nu = \psi_2(u)
\]
then for any \((x, v)\) which satisfy (1) and \(v \in (v-, v+)\), \(x \in (x-, x+)\) then there exist unique functions of time \(\psi\) and \(u\) defined by (5) which satisfy

\[
\frac{d\xi}{dt} = \left[ \frac{d\psi_1}{d\xi}(\xi) \right]^{-1} \psi_2(u) \tag{6}
\]

Indeed, we have

\[
\frac{dx}{dt} = \frac{d}{dt}(\psi_1(\xi)) \tag{7}
\]

\[
= \frac{d\psi_1}{d\xi} \frac{d\xi}{dt} \tag{8}
\]

\[
= v \tag{9}
\]

\[
= \psi_2(u) \tag{10}
\]

The differential equation (6) is well defined since \(\frac{d\psi_1}{d\xi}\) never vanishes. Its right handside is, however, not lipschitz if \(\xi\) is not bounded, since \(\frac{d\psi_1}{d\xi}\) vanishes at the infinity. This means that we may have a finite time blowup of \(\xi\), e.g. saturation of the state constraints in the original coordinates and in a finite time, if consider general solutions of (6), and not only the transforms of (1) with \(x \in (x-, x+)\) and \(v \in (v-, v+)\). As an example, a typical saturation function \(\psi_1(\xi)\) behaves like \(e^{-\xi}\) when \(\xi\) is close to the \(+\infty\); since \(\psi_2\) is bounded, (6) behaves roughly like

\[
\frac{d\xi}{dt} = a e^{\xi}. \tag{11}
\]

Setting classically \(z = e^{\xi}\), we end up with an equation of
the kind
\[ \frac{dz}{dt} = az^2 \]  \hspace{1cm} (12)
whose solution \( z \) behaves (for \( a < 0 \)) like \( \frac{1}{t-t_i} \), i.e. we have
\[ \xi(t) \sim \log(1/(t-t_i)) = -\log(t-t_i) \]  \hspace{1cm} (13)
which blows up when \( t \) goes to \( t_i \).

Note: the same problem exists with the dynamics (25b) in the IFAC Goddard paper by Graichen & Petit.

Conversely, if \((\xi, v)\) is a maximal solution of (6), then \((x, v)\) defined by (5) satisfy (1) over an interval which may be bounded and with \( v \in (v-, v+) \), \( x \in (x-, x+) \).

We consider now the family of optimisation problems
\[
\min J_\varepsilon(u) = \int_a^b \left[ L(\psi_1(\xi), \psi_2(u)) + \varepsilon(\xi^2 + u^2) \right] dt \hspace{1cm} (14)
\]
where \( \xi \) and \( u \) are unconstrained and (6) has a solution on \([a,b]\) for the given initial condition.

The cost \( J_\varepsilon \) has an infimum because \( \xi^2 + u^2 \) is non negative and \( L(\psi_1(\xi), \psi_2(u)) \) is lower bounded. We show now that \( \xi \) and \( u \) are bounded in \( L^2 \).

**Lemma 1** If \( u_0 \) is an admissible control and \( \mu = J_\varepsilon(u_0) \) then, if \( u \) is admissible with \( J_\varepsilon(u) \leq \mu \), we have
\[
\int_{-\infty}^{+\infty} (\xi^2 + u^2) dt \leq \frac{1}{\varepsilon} (\mu - (b-a) \inf L) \hspace{1cm} (15)
\]
where \( \inf L \) is the lower bound of \( L(x, v) \) over \((x-, x+) \times (v-, v+)\).
**Question:** can trajectories with $\xi$ and $u$ in $L^2$ blow up in finite time?

A partial answer lies here: in the neighbourhood of 0, $\log t$ is square integrable. Indeed, we have

\[
\int (\log t)^2 dt = t(\log t)^2 - \int \frac{2 t}{t} \log t dt = t(\log t)^2 - 2 t \log t + \int t dt
\]

hence we may have finite a time explosion of $\xi$ while $\xi$ is $L^2$.

Any minimizing sequence of $J_\varepsilon$ will have, after some time, bounded $\xi$ and $u$. At the very least, we will have weak convergence of some subsequence. Observe that, if $u$ converges weakly, $\xi$ converges strongly in $L^{21}$. Also, on the numerical side, putting a lower bound on the mesh refinement prevents high frequency oscillation, i.e. weak convergence.

Another approach is to assume

1. existence of an optimal control for the the cost $J_\varepsilon$ and the new variables (something we cannot reasonably assume for $\varepsilon = 0$).

2. existence and local regularity with respect to param-

\footnote{for the nonlinear case averaging may help}
eters of the control solutions of $\frac{\partial H_\varepsilon}{\partial u} = 0$ where

$$H(\xi, u, \lambda, \varepsilon) = L(\psi_1(\xi), \psi_2(u)) + \varepsilon(\xi^2 + u^2) + \lambda \left[ \frac{d\psi_1}{d\xi}(\xi) \right]^{-1} \psi_2(\xi)$$  \hspace{1cm} (19)

Observe that $H$ is continuous and lower bounded if $\xi$ and $\lambda$ are bounded.

Since the original optimal control may lead to constraint saturations, one must keep in mind that, for $\varepsilon = 0$,

$$\arg\min_u H(\xi, u, \lambda, 0) = \pm \infty \hspace{1cm} (20)$$

along the optimal primal/dual trajectories. On the contrary, for $\varepsilon > 0$, the control $u$ which minimizes $H(\xi, u, \lambda, \varepsilon)$ is necessarily finite. Indeed, we have, if $u^\#$ minimizes $H(\xi, u, \lambda, \varepsilon)$:

$$(u^\#)^2 \leq -\xi^2 + \frac{1}{\varepsilon} \min_H - \min_{x \in [x-, x+]} L - \lambda \left[ \frac{d\psi_1}{d\xi}(\xi) \right]^{-1} \min_{v \in [v-, v+]}$$

which is bounded if $\lambda$ and $\xi$ are bounded, and $\varepsilon > 0$.

To compute an optimal control (which is assumed to exist), we seek to solve the boundary value problem

$$\frac{d\xi}{dt} = \left[ \frac{d\psi_1}{d\xi}(\xi) \right]^{-1} \psi_2(u^\#(\xi, \lambda, \varepsilon)) \hspace{1cm} (22)$$

$$\frac{d\lambda}{dt} = -\frac{\partial H}{\partial \xi}(\xi, u^\#(\xi, \lambda, \varepsilon), \lambda, \varepsilon) \hspace{1cm} (23)$$

with $u^\#(\xi, \lambda, \varepsilon) = \arg\min_u H(\xi, u, \lambda, \varepsilon) \hspace{1cm} (24)$
with suitable initial and final conditions.

Observe that taking the min in (24) is not equivalent to \( \frac{\partial H}{\partial u} = 0 \) since, especially for small \( \epsilon \), \( H \) may not be convex with respect to \( u \). While it is reasonable to assume that a unique global optimum exists, we cannot assume in the general case that multiple extrema do not occur.

The question is: for a nonnegative \( \epsilon \), does this BVP has a solution over \([a, b]\)?