COMMANDES D'ÉQUATIONS AUX DÉRIVÉES PARTIELLES : RÉSULTATS CLASSIQUES ET PROBLÈMES OUVERTS

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Outline of the lecture

1 - Introduction

2 - Controllability and observability

3 - Controllability of some equations

3.1 Wave equation
3.2 Plate equation, Schrödinger equation
3.3 Heat equation

4 - Stabilizability

5 - Nonlinear equations - Some open problems

1 - Introduction

Most physical phenomena are described by a Partial Differential Equation (PDE).

A natural (and commonly used) way to control a PDE is to perform a reduction to a finite dimensional system (e.g. by a Galerkin procedure), and next to control the ODE by classical tools.

This approach presents some drawbacks:

- the physics is not taken into account; actually, the location and the duration of the control play a great role in the control of PDE, and this fact is hidden in a finite dimension analysis.
- the convergence towards the PDE model is not guaranteed as the number of modes tends to infinity.

Methods to control a PDE will be exposed in this lecture.

Applications of the control of PDE:

- Noise reduction (wave equation);
- Vibrations reduction (plate equations);
- Turbulence reduction (Navier-Stokes equation);
- Laser control of chemical reactions (Schrödinger equation);
- $\cdots$

We focus here on controllability and stabilization issues.

Let $P(D)$ be a differential operator, with $P \in \mathbb{C}[\tau, \xi_1, ..., \xi_N]$, and $D = (-i\partial_t, -i\partial_{x_1}, ..., -i\partial_{x_N})$

Examples.

- $P = -\tau^2 + |\xi|^2$ gives the wave operator $P(D) = \partial_t^2 - \Delta$
- $P = i\tau + |\xi|^2$ gives the heat operator $P(D) = \partial_t - \Delta$
- $P = -\tau - |\xi|^2$ gives the Schrödinger operator $P(D) = i\partial_t + \Delta$

Let $\Omega \subset \mathbb{R}^N$ be a bounded (smooth) open set, whose boundary $\partial\Omega$ is denoted by Γ.

Two types of control problems are considered

- the internal control problem
- the boundary control problem

Internal Control Problem

Let $\omega \subset \Omega$ be some given open set. Consider the problem

$$\begin{aligned} P(D)z &= \chi_\omega u && \text{in } (0,T)\times\Omega; \\ B(D)z &= 0 && \text{in } (0,T)\times\Gamma; \\ z(0,.) &= z_0 && \text{in } \Omega. \end{aligned}$$

$u = u(t,x)$ is the internal control, $z = z(t,x)$ is the unknown function,

χ_ω is the characteristic function of ω ($\chi_\omega(x) = 1$ if $x \in \omega$, 0 otherwise)

Controllability issue

Given z_0 and z_1 in H, does it exist a control $u \in L^2(0,T;U)$ such that the solution z satisfies $z(T,x) = z_1(x)$?
(H and U are some given Hilbert spaces)

Boundary Control Problem

Let $\gamma \subset \Gamma = \partial\Omega$ be some given open set. Consider the problem

$$\begin{aligned} P(D)z &= 0 && \text{in } (0,T)\times\Omega, \\ B_1(D)z &= \chi_\gamma u && \text{in } (0,T)\times\Gamma, \\ B_2(D)z &= 0 && \text{in } (0,T)\times\Gamma, \\ z(0,.) &= z_0 && \text{in } \Omega. \end{aligned}$$

$u = u(t,x)$ is the boundary control, $z = z(t,x)$ is the unknown function

Controllability issue

Given z_0 and z_1 in H, does it exist a control $u \in L^2(0,T;U)$ such that the solution z satisfies $z(T,x) = z_1(x)$?
(H and U are some given Hilbert spaces)

We focus on the control theory for systems of the form

$$\Sigma_{A,B} \qquad \dot{z} = Az + Bu.$$

where

- $A : D(A) \subset H \rightarrow H$ is an unbounded operator generating a strongly continuous semigroup of operators on H, denoted by $(S(t))_{t\geq 0}$ (or $(e^{tA})_{t\geq 0}$).
- $B : U \rightarrow H$ is a bounded operator $(B \in L(U, H))$

Framework convenient for any internal control problem: typically, $U = H = L^2(\Omega)$ and $Bf = a(x)f$, with $a \in L^\infty(\Omega)$

Framework also convenient for boundary control problems, with this time $B \in L(U, D(A^*)')$, where $D(A^*)'$ is the dual of $D(A^*)$ with the pivot space H.

2 - Controllability and Observability

For given $z_0 \in H$, $u \in L^2(0,T;U)$, we consider the solution $z : [0,T] \to H$ of the Cauchy problem

$$\begin{cases} \dot{z} = Az + Bu, \\ z(0) = z_0. \end{cases} \tag{1}$$

If $z_0 \in D(A)$ and $u \in C^1([0,T];U)$, the Cauchy problem (CP) admits a unique classical solution $z \in C([0,T];D(A)) \cap C^1([0,T];H)$ given by Duhamel formula

$$z(t) = S(t)z_0 + \int_0^t S(t-s)Bu(s)\,ds \quad \forall t \in [0,T]$$

If $z_0 \in H$ and $u \in L^1(0,T;U)$, the above formula is still meaningful and define the mild solution of (CP).

The control system $\Sigma_{A,B}$ is said to be

- exactly controllable in time T if for any $z_0, z_T \in H$, there exists $u \in L^2(0,T;U)$ such that the solution z of (1) fulfills $z(T) = z_T$;
- null controllable in time T if for any $z_0 \in H$, there exists $u \in L^2(0,T;U)$ such that the solution z of (1) fulfills $z(T) = 0$;
- approximatively controllable in time T if for any $z_0, z_T \in H$ and any $\varepsilon > 0$, there exists $u \in L^2(0,T;U)$ such that the solution z of (1) fulfills $||z(T) - z_T||_H < \varepsilon$.

Let $L_T : L^2(0,T;U) \to H$ be defined by

$$L_T u = \int_0^T S(T-s)u(s)\,ds$$

Then

$$\begin{aligned}
\text{exact controllability in time } T &\iff \mathbf{Im}\ L_T = H \\
\text{null controllability in time } T &\iff S(T)H \subset \mathbf{Im}\ L_T \\
\text{approximate controllability in time } T &\iff \overline{\mathbf{Im}\ L_T} = H
\end{aligned}$$

In finite dimension (i.e., $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times M}$), the three concepts are equivalent, and equivalent to Kalman rank condition:

$$\text{rank } (B, AB, ..., A^{N-1}B) = N$$

$\Rightarrow$ T plays no role

The situation is more tricky for PDE:

- **There is no algebraic test for the (exact or null) controllability;**
- **The control time plays a role for hyperbolic PDE;**
- **The converses of**

$$\text{exact controllability} \Rightarrow \text{null controllability}$$
$$\text{exact controllability} \Rightarrow \text{approximate controllability}$$

are not true in general.

Adjoint operators

The adjoint of the bounded operator $B \in L(U, H)$ is the operator $B^* \in L(H, U)$, defined by $(B^*z, u)_U = (z, Bu)_H$ for all $z \in H$, $u \in U$.

Examples

1. If $Bu = \chi_\omega u$, with $\omega \subset \Omega$ and $U = L^2(\omega)$, $H = L^2(\Omega)$, then $B^*z = z_{|\omega}$.

2. If $Bu = au$, with $a \in L^\infty(\Omega)$ and $H = U = L^2(\Omega)$, then $B^* = B$.

The adjoint of the unbounded operator A is the unbounded operator A^* with domain

$$D(A^*) = \{z \in H |\ \exists C \in \mathbb{R}^+, \quad |(Ay, z)_H| \leq C||y||_H \quad \forall y \in D(A)\}$$

and defined by

$$(Ay, z)_H = (y, A^* z)_H \quad \forall y \in D(A),\ \forall z \in D(A^*).$$

A^* generates also a continuous semigroup $(e^{tA^*})_{t \geq 0}$ fulfilling $e^{tA^*} = (e^{tA})^* \quad \forall t \geq 0$.

If $A^* = A$ (resp. $A^* = -A$) the operator A is said to be self-adjoint (resp. skew-adjoint). Recall that a skew-adjoint operator generates a continuous group of isometries.

Exact Controllability

Theorem (Dolecki-Russell 1977)
The system $\Sigma_{A,B}$ is exactly controllable in time $T > 0$ if and only if there exists a constant $c > 0$ such that

$$\int_0^T ||B^* S^*(t) y_0||_U^2 \geq c||y_0||_H^2 \quad \forall y_0 \in H. \tag{2}$$

(2) is called an observability inequality.

It means that it is possible to recover a complete information about the initial state y_0 from a measurement of the output $B^*[S^*(t)y_0]$ on $[0, T]$ (observability property).

Example: internal control of the wave equation

$$\begin{cases} z_{tt} - \Delta z = \chi_\omega u & \textbf{in } (0,T)\times\Omega \\ z = 0 & \textbf{on } (0,T)\times\Gamma \\ z(0,.) = z_0,\ z_t(0,.) = z_1. \end{cases} \tag{3}$$

Here, $Z = (z, z_t) \in H = H_0^1(\Omega) \times L^2(\Omega)$, $U = L^2(\omega)$, $A = \begin{bmatrix} 0 & I \\ \Delta & 0 \end{bmatrix}$, $Bu = \begin{bmatrix} 0 \\ \chi_\omega u \end{bmatrix}$. **The exact controllability in** $H_0^1(\Omega) \times L^2(\Omega)$ **of** (3) **is equivalent to**

$$\int_0^T \int_\omega |y_t|^2 dx dt \geq c(||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)})$$

for any solution y **of**

$$\begin{cases} y_{tt} - \Delta y = 0 & \textbf{in } (0,T)\times\Omega \\ y = 0 & \textbf{on } (0,T)\times\Gamma \\ y(0,.) = y_0,\ y_t(0,.) = y_1. \end{cases}$$

Exact controllability via the Hilbert Uniqueness Method (HUM) of J.-L. Lions

We associate to the boundary-initial value problem

$$\Sigma \qquad \begin{cases} \dot{z} &= Az + Bu, \\ z(0) &= 0. \end{cases}$$

its adjoint problem, obtained by taking the distributional adjoint of the operator $\partial_t - A$, namely $-\partial_t - A^*$:

$$\Sigma^* \qquad \begin{cases} \dot{y} &= -A^* y \\ y(T) &= y_T. \end{cases}$$

Note that Σ^* is without control and backwards in time. For any $y_T \in H$, the solution y of Σ^* reads $y(t) = S^*(T-t)y_T$.

KEY IDENTITY. $(z(T), y_T)_H = \int_0^T (u, B^*y)_U \, dt.$

Proof. Integrate by part in $0 = \int_0^T (\dot{z} - Az - Bu, y)_H \, dt$

Assume that

$$\int_0^T ||B^*y||_U^2 \, dt \geq c||y_T||_H^2$$

For any $y_T \in H$**, set** $u(t) := B^*y(t)$ **(where** y **solves** Σ^***), and consider the solution** z **of** Σ **corresponding to that control** u**.**

This defines a bounded operator $\Gamma : y_T \in H \mapsto z(T) = L_T(B^*y(\cdot)) \in H$**. Since**

$$(\Gamma y_T, y_T)_H = \int_0^T ||B^*y(t)||_U^2 \, dt \geq c||y_T||_H^2,$$

it follows from Lax-Milgram theorem that Γ **is invertible.**

Remarks:

1. HUM provides a bounded operator $\Lambda : z_T \mapsto u$ giving the control.
2. In general we don't need to explicit B and B^*. The important ingredients in HUM are the key identity and the observability inequality.

Null Controllability

Theorem (Dolecki-Russell 1977)
The system $\Sigma_{A,B}$ is null controllable in time $T > 0$ if and only if there exists a constant $c > 0$ such that

$$\int_0^T ||B^* S^*(t) y_0||_U^2 \geq c||S^*(T) y_0||_H^2 \quad \forall y_0 \in H. \tag{4}$$

(4) **is a weak observability inequality: only $S^*(T)y_0$ may be recovered, not y_0**

Example: internal control of the heat equation

$$\begin{cases} z_t - \Delta z = \chi_\omega u & \textbf{in } (0,T)\times\Omega \\ z = 0 & \textbf{on } (0,T)\times\Gamma \\ z(0,.) = z_0. \end{cases} \tag{5}$$

Here, $z \in H = L^2(\Omega)$, $U = L^2(\omega)$, $A = \Delta$, $Bu = \chi_\omega u$. **The null controllability in** $L^2(\Omega)$ **of** (5) **is equivalent to**

$$\int_0^T \int_\omega |y|^2 dxdt \geq c||y(T)||^2_{L^2(\Omega)}$$

for any solution y **of**

$$\begin{cases} y_t - \Delta y = 0 & \textbf{in } (0,T)\times\Omega \\ y = 0 & \textbf{on } (0,T)\times\Gamma \\ y(0,.) = y_0. \end{cases}$$

Approximate Controllability

Theorem (Dolecki-Russell 1977)
The system $\Sigma_{A,B}$ is approximately controllable in time $T > 0$ if and only if the only $y_0 \in H$ for which

$$B^* S^*(t) y_0 = 0 \quad \forall t \in [0, T] \tag{6}$$

is $y_0 = 0$.

Example: $Bu = \chi_\omega u$ in $H = L^2(\Omega)$, $B^* y = y_{|\omega}$. (6) is a Unique Continuation Property (UCP), which is usually established with the aid of Holmgren theorem for an equation with analytic coefficients, or with some Carleman estimate for an equation with bounded coefficients.

Type of the PDE	Controllability	Control Region	Time Control
Hyperbolic (ex. wave equation)	Exact	Depends on Ω	$T \geq T_0$
Dispersive (ex. Schrödinger equation)	Exact	Depends on Ω	Any $T > 0$
Parabolic (ex. heat equation)	Null	Any ω	Any $T > 0$

Table 1: Controllability results for PDE. (Internal control)

3 - Controllability of some PDE

We first investigate the boundary controllability of the 1-D wave

$$
\begin{aligned}
z_{tt} - z_{xx} &= 0 \qquad 0 < t < T,\ 0 < x < \pi && (7)\\
z(t,0) &= 0 \qquad 0 < t < T, && (8)\\
z(t,\pi) &= h(t) \qquad 0 < t < T, && (9)\\
(z(0,.), z_t(0,.)) &= (z_0, z_1) && (10)
\end{aligned}
$$

where $h \in L^2(0,T)$ is the control input.

We shall prove the

Theorem

The system (7)-(10) is exactly controllable in $L^2(0,\pi) \times H^{-1}(0,\pi)$ in time T if and only if $T \geq 2\pi$.

The uncontrolled problem

$$\begin{aligned} z_{tt} - z_{xx} &= 0 \quad 0 < t < T,\ 0 < x < \pi, && (11)\\ z(t,0) = z(t,\pi) &= 0 \quad 0 < t < T, && (12)\\ (z(0,.), z_t(0,.)) &= (z_0, z_1) && (13) \end{aligned}$$

may be written $(z, z_t)_t = A(z, z_t)$ with $A(z_0, z_1) = (z_1, (z_0)_{xx})$ on the domain $D(A) = (H^2(0,\pi) \cap H_0^1(0,\pi)) \times H_0^1(0,\pi) \subset H := H_0^1(0,\pi) \times L^2(0,\pi)$.

It may be seen that A is skew-adjoint, so that it generates a group of isometries on H.

Actually, if $z_0 = \sum_{k\geq 1} a_k \sin(kx)$ and $z_1 = \sum_{k\geq 1} b_k \sin(kx)$, then the solution z of (11)-(13) reads

$$z(t,x) = \sum_{k\geq 1} [a_k \cos(kt) + \frac{b_k}{k} \sin(kt)] \sin(kx). \qquad (14)$$

To identify the adjoint problem and B^*, we multiply (7) by y, integrate over $(0,T)\times(0,\pi)$ and perform integrations by parts, assuming that the functions y and z are sufficiently regular.

We obtain

$$0 = \int_0^T\int_0^\pi (z_{tt} - z_{xx})y\,dxdt = \int_0^T\int_0^\pi (y_{tt} - y_{xx})z\,dxdt + [\int_0^\pi (z_t y - z y_t)\,dx]_0^T + \int_0^T [-z_x y + z y_x]_0^\pi dt.$$

If y is a solution of the adjoint problem (i.e. here the uncontrolled system), then we obtain

$$-\int_0^T h(t)y_x(t,\pi)\;dt = [\int_0^\pi (z_t y - z y_t)\,dx]_0^T. \tag{15}$$

Assume in addition that $(y(T,.), y_t(T,.)) = (y_{0,T}, y_{1,T}) \in H$**, with** $y_{0,T} = \sum_{k\geq 1} c_k \sin(kx)$, $y_{1,T} = \sum_{k\geq 1} d_k \sin(kx)$, **then**

$$y(t,x) = \sum_{k\geq 1} \left[c_k \cos(k(t-T)) + \frac{d_k}{k} \sin(k(t-T))\right] \sin(kx),$$

hence

$$y_x(t,\pi) = \sum_{k\geq 1} \left[kc_k \cos(k(t-T)) + d_k \sin(k(t-T))\right] (-1)^k. \qquad (16)$$

If $T = 2\pi$**, then since the functions** $\sin(kt)$ **and** $\cos(kt)$ **are orthogonal in** $L^2(0,2\pi)$**, we see that**

$$\int_0^{2\pi} |y_x(.,\pi)|^2 \, dt = \sum_{k\geq 1} (|kc_k|^2 + |d_k|^2) < \infty. \qquad (17)$$

It follows that $y_x(.,\pi) \in L^2_{loc}(\mathbb{R})$**, hence the integral term in (15) is meaningful.**

To define a solution of (7)-(10), we use the transposition method:

First, note that the dual of the space H is $H' = H^{-1}(0,\pi) \times L^2(0,\pi)$, and that the duality pairing $\langle .,. \rangle_{H',H}$ is defined by

$$\langle (z_1, z_0), (y_0, y_1) \rangle_{H',H} = \langle z_1, y_0 \rangle_{H^{-1}(0,\pi),H_0^1(0,\pi)} + \langle z_0, y_1 \rangle_{L^2(0,\pi),L^2(0,\pi)}.$$

Replacing T by any t, (15) may be rewritten as

$$\langle (-z_t(t), z(t)), (y(t), y_t(t)) \rangle_{H',H} = \int_0^t h(s) y_x(s,\pi)\, ds + \langle (-z_1, z_0), (y(0), y_t(0)) \rangle_{H',H}. \tag{18}$$

As the map $(y_{0,T}, y_{1,T}) \in H \mapsto (y(t), y_t(t)) \in H$ is an isomorphism of Hilbert spaces, (18) defines $(z_t(t), z(t))$ in H' in a unique way, and $(z_t, z) \in C([0,T]; H')$.

z is termed the solution by transposition of (7)-(10).

Assume now that $(z_0, z_1) = (0, 0)$. Then

$$\langle(-z_t(T), z(T)), (y_{0,T}, y_{1,T})\rangle_{H',H} = \int_0^T h(t) y_x(t, \pi)\, dt.$$

Then it is easily seen that (7)-(10) is exactly controllable in $H^{-1}(0, \pi) \times L^2(0, \pi)$ if and only if the following observability inequality

$$\int_0^T |y_x(t, \pi)|^2 dt \geq C||(y_{0,T}, y_{1,T})||_H^2. \tag{19}$$

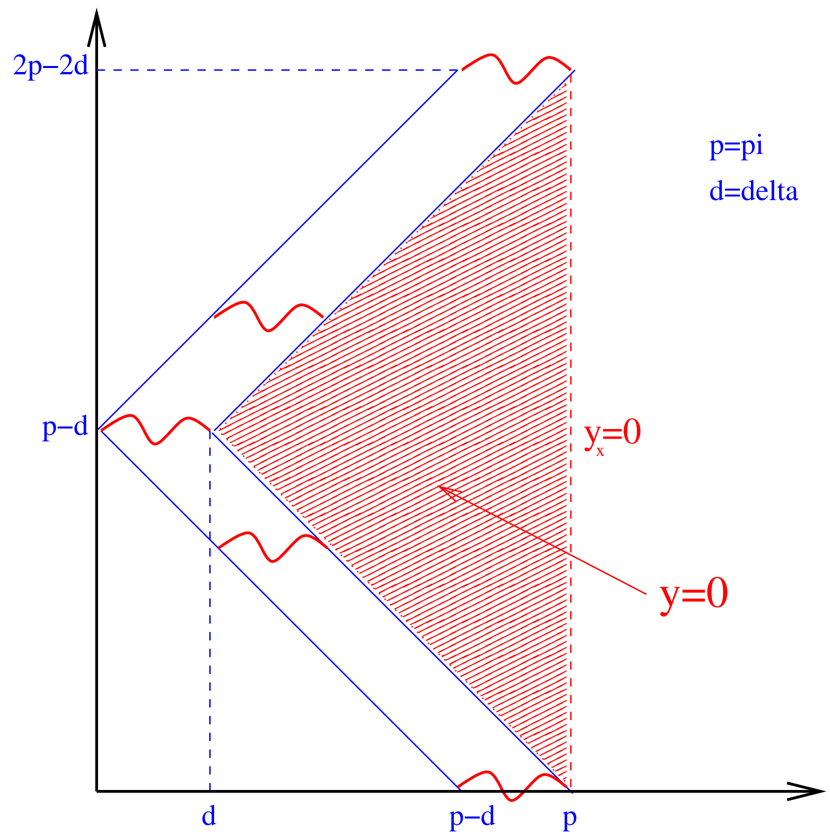
holds.

This is the case for $T = 2\pi$, since then

$$\int_0^{2\pi} |y_x(., \pi)|^2\, dt = \sum_{k \geq 1} (|kc_k|^2 + |d_k|^2) = ||(y_{0,T}, y_{1,T})||_H^2,$$

hence also for any $T \geq 2\pi$.

To see that (19) does not hold for $T = 2\pi - 2\delta < 2\pi$, it is sufficient to consider any solution y of (11)-(12) such that the functions $y(\frac{T}{2}, .)$ and $y_t(\frac{T}{2}, .)$ are supported in $(0, \delta)$ (not both null). Then $y_x(., \pi) \equiv 0$ on $(0, T)$, whereas $y \not\equiv 0$.



Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a boundary Γ of class C^∞. Let γ denote an open set in Γ, and consider the boundary control problem:

$$\begin{cases} z_{tt} - \Delta z = 0 & \textbf{in } (0,T)\times\Omega \\ z = h\chi_\gamma & \textbf{on } (0,T)\times\Gamma \\ z(0,.) = z_0,\ \ z_t(0,.) = z_1. \end{cases} \tag{20}$$

In (32), $z = z(t,x)$ is the state function, $h = h(t,x)$ is the control function, and χ_γ stands for the characteristic function of γ.

When $h \in L^2(0,T;L^2(\gamma))$ and $(z_0,z_1) \in L^2(\Omega)\times H^{-1}(\Omega)$, then a solution by transposition z of (32) may be defined in the class $C([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega))$.

An application of HUM shows that the exact controllability is equivalent to the following observability inequality

$$||(y_0,y_1)||^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq C \int_0^T \int_\gamma \left|\frac{\partial y}{\partial n}\right|^2 d\sigma dt \tag{21}$$

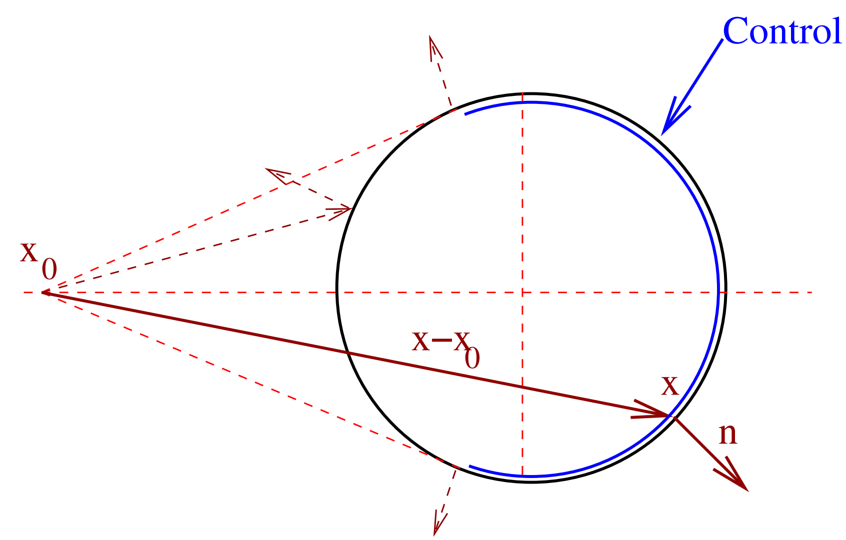
for any solution y of the uncontrolled wave equation with $(y_0,y_1) \in H_0^1(\Omega)\times L^2(\Omega)$ as initial data.

Multiplier method: Scaling in the wave equation by the Morawetz's multiplier $m(x) \cdot \nabla y$, where $m(x) = x - x^0$ ($x^0 \in \mathbb{R}^N$ given), it follows after some computations that the observability inequality holds **if γ takes the form**

$$\gamma = \Gamma(x^0) := \{x \in \Gamma | \; (x - x^0) \cdot n(x) > 0\} \tag{22}$$

and

$$T > T(x^0) := 2||x - x^0||_{L^\infty(\Omega)}$$



Exactly Controllable

Theorem (C. Bardos, G. Lebeau, and J. Rauch, 1992)
The observability inequality (21) holds if and only if the pair (γ, T) satisfies the so-called Geometric Control Condition (GCC): each ray which propagates in Ω and is reflected on Γ according to the laws of geometric optics has to meet γ in time less than T.

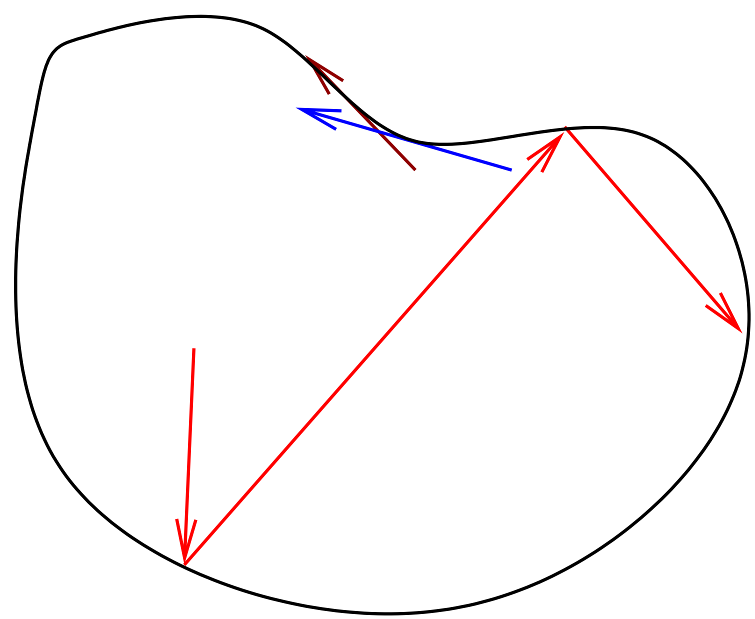


Figure 1: Laws of geometric optics

Examples

In 1D, for $\Omega = (0, L)$ the control time has to be

- **at least $2L$ with one control ($\gamma = \{L\}$),**
- **at least L with two controls ($\gamma = \{0, L\}$).**

In 2D, when Ω is a rectangle, the control region needs to contain a point of each line parallel to one side. Otherwise, there may exist trapped rays that support solutions that are not observable.

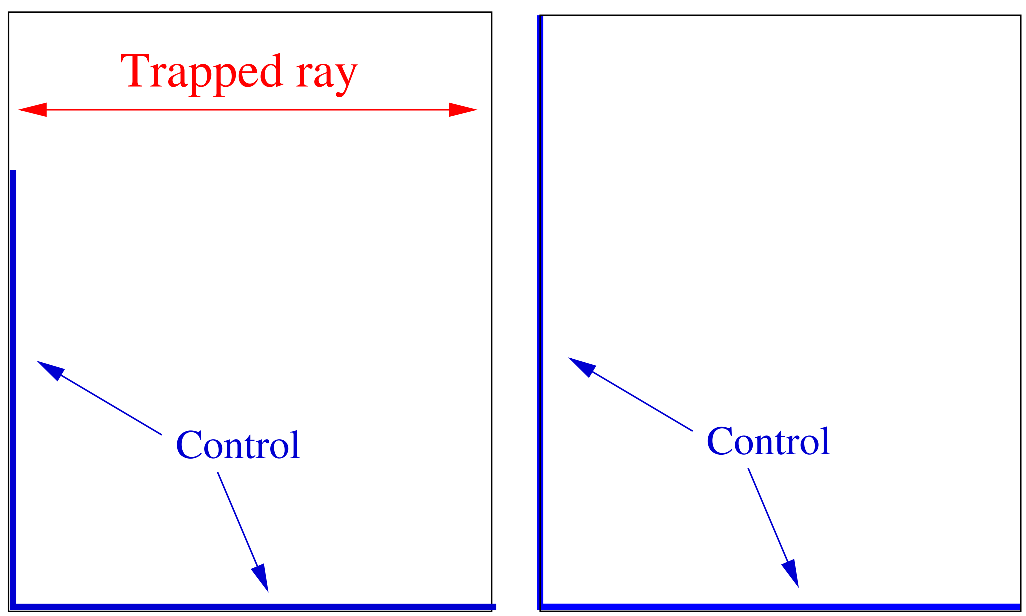
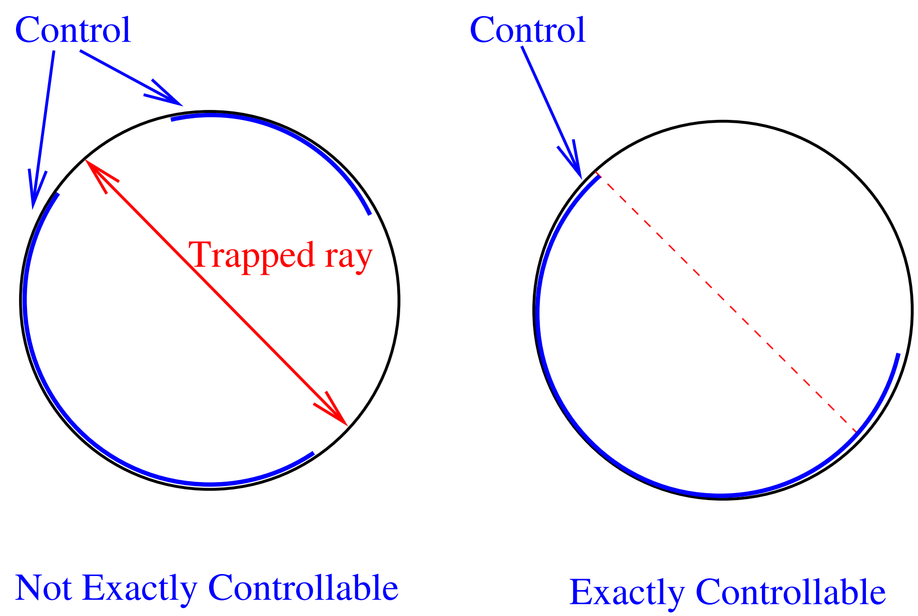


Figure 2: Not Exactly controllable - Exactly Controllable

In 2D, when Ω is a disk, the control region γ needs to contain a point of each diameter (otherwise there may exist trapped rays). The time control T has to be larger than $2d$, where d denotes the diameter of the ball.



- Bardos-Lebeau-Rauch theorem was proved by means of microlocal analysis.
- If we omit the problems due to the boundary, the main tool used is the theorem of propagation of singularities for the wave equation (also called generalized Huygens' principle):
  For y satisfying $\Box y := y_{tt} - \Delta y = 0$
  1. $WF(y) \subset \mathbf{car}(\Box) := \{(t, x, \tau, \xi);\ \tau^2 - |\xi|^2 = 0\}$
  2. If $\rho_0 = (t_0, x_0, \tau_0, \xi_0) \in \mathbf{car}(\Box)$ and $y \in H^s_{\rho_0}$, then $y \in H^s_\rho$ for all ρ in the bicharacteristic ray issuing from ρ_0.
- The proof has been simplified by N. Burq and P. Gérard in 1997 by using the microlocal defect measures introduced by L. Tartar and P. Gérard in 1991.

3.2 - Exact Controllability of the plate equation

We first investigate the boundary controllability of the beam equation (plate equation in dimension 1)

$$\begin{aligned} z_{tt} + z_{xxxx} &= 0 \qquad 0 < t < T,\ 0 < x < \pi && (23)\\ z(t,0) = z(t,\pi) = z_{xx}(t,0) &= 0 \qquad 0 < t < T, && (24)\\ z_{xx}(t,\pi) &= h(t) \quad 0 < t < T, && (25)\\ (z(0,.), z_t(0,.)) &= (z_0, z_1) && (26) \end{aligned}$$

where $h \in L^2(0,T)$ stands for the control input. Then the following result holds.

Theorem

The system (23)-(26) is exactly controllable in $H_0^1(0,\pi) \times H^{-1}(0,\pi)$ in any time $T > 0$.

- **If y is a solution of the uncontrolled problem, then we have the key identity**

$$\int_0^T h(t)y_x(t,\pi)\ dt = [\int_0^\pi (z_t y - z y_t)\, dx]_0^T. \qquad (27)$$

- **It may be seen that (23)-(26) is exactly controllable in $H_0^1(0,\pi) \times H^{-1}(0,\pi)$ in time T iff**

$$\int_0^T |y_x(t,\pi)|^2 dt \geq C||(y_{0,T}, y_{1,T})||_H^2. \qquad (28)$$

- **Writing $y_{0,T} = \sum_{k\geq 1} c_k \sin(kx)$, $y_{1,T} = \sum_{k\geq 1} d_k \sin(kx)$, we obtain**

$$y(t,x) = \sum_{k\geq 1} \left[c_k \cos(k^2(t-T)) + \frac{d_k}{k^2} \sin(k^2(t-T))\right] \sin(kx),$$

and (28) follows for $T = 2\pi$ from Parseval theorem.

Theorem (Ingham 1936, Ball-Slemrod 1979)
Let $(a_k)_{k\in\mathbb{Z}}$ be a sequence of complex numbers such that $\sum_{k\in\mathbb{Z}} |a_k|^2 < \infty$, and let $(\mu_k)_{k\in\mathbb{Z}}$ be a sequence of real numbers fulfilling the following properties

$$\mu_{k+1} - \mu_k \geq \gamma > 0 \qquad \text{for } |k| \geq K, \tag{29}$$

$$\mu_{k+1} - \mu_k \geq \rho > 0 \qquad \text{for } k \in \mathbb{Z} \tag{30}$$

where K is some integer. Then if $T > (2\pi)/\gamma$, one may find two positive constants $C_1, C_2 > 0$ such that

$$C_1 \sum_{k\in\mathbb{Z}} |a_k|^2 \leq \int_0^T \left| \sum_{k\in\mathbb{Z}} a_k e^{i\mu_k t} \right|^2 dt \leq C_2 \sum_{k\in\mathbb{Z}} |a_k|^2.$$

For the beam eq., $\mu_k = \mathrm{sgn}(k)|k|^2$.

Remarks

1. For $\mu_k = k$ and $T = 2\pi$ (harmonic setting) we have by Parseval theorem
$$\int_0^{2\pi} \left| \sum_{k\in\mathbb{Z}} a_k e^{ikt} \right|^2 dt = 2\pi \sum_{k\in\mathbb{Z}} |a_k|^2.$$
Ball-Slemrod's Theorem holds in a nonharmonic setting, that is, when the functions $e^{i\mu_k t}$ are NOT necessarily pairwise orthogonal in $L^2(0,T)$.

2. When $\rho = \gamma$, Ball-Slemrod's Theorem is just Ingham's lemma (1936). The infimum of the $\mu_{k+1} - \mu_k$ is called the spectral gap. The larger spectral gap, the smaller observability time T. With Ball-Slemrod's improvement of Ingham's lemma, only the asymptotic spectral gap γ has to be considered (provided that the frequencies are pairwise distinct). When $\lim_{|k|\to\infty} \mu_{k+1} - \mu_k = \infty$, then γ may be taken as large as wanted, and so T as small as wanted.

Boundary controllability problem

$$\begin{cases} z_{tt} + \Delta^2 z = 0 & \textbf{in } (0,T)\times\Omega \\ z = 0 & \textbf{on } (0,T)\times\Gamma \\ \Delta z = \chi_\gamma h & \textbf{on } (0,T)\times\Gamma \\ z(0,.) = z_0, \ \ z_t(0,.) = z_1. \end{cases} \tag{31}$$

Here, Ω is a bounded open set in $\mathbb{R}^N$, $\Gamma = \partial\Omega$, and γ denotes an open set in Γ.

For $h \in L^2(0,T;L^2(\gamma))$, a solution by transposition z exists with $(z,z_t) \in C([0,T];H_0^1(\Omega)\times H^{-1}(\Omega))$.

- G. Lebeau proved in 1992 that the EC holds when the Geometric Control Condition is fulfilled (i.e. the controllability of the wave equation implies the controllability of the Schrödinger equation and of the plate equation)
- K. Liu 1997 derived a similar result by using the multiplier method and a variant of Hautus lemma
- K.-D. Phung 2001 derived again Lebeau's result by using the transmutation method, based upon Kannai transform

For z_0 and $S >> 1$ given, let z solves

$$\begin{cases} z_{ss} - \Delta z = 0 & \textbf{in } (0,S)\times\Omega \\ z = h(s,x)\chi_\gamma & \textbf{on } (0,S)\times\Gamma \\ z(0,.) = z_0,\ \ z_s(0,.) = 0,\ \ z(S,.) = 0,\ \ z_s(S,.) = 0. & \end{cases}$$

Pick any $T > 0$ and a "null control" $u \in C^\infty([0,T])$ for the fundamental solution of the Schrödinger equation in the space domain $(-S,S)$, i.e. the solution ρ of

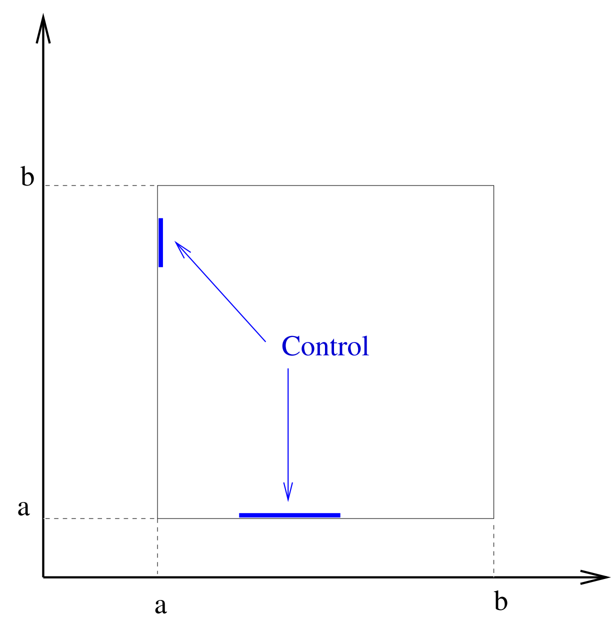
$$\begin{cases} i\rho_t + \rho_{ss} = 0 & \textbf{in } (0,T)\times(-S,S) \\ (\rho(t,-S),\rho(t,S)) = (0,u(t)) & \textbf{on } (0,T) \\ \rho(0,.) = \delta_{|s=0} & \end{cases} \tag{32}$$

satisfies $\rho(T,.) \equiv 0$. Let $\tilde{f}(s,x)$ denote the even extension of f w.r.t. s. Then $y(t,x) := \int_{-S}^{S} \tilde{z}(s,x)\rho(t,s)ds$ (Kannai transform) solves

$$\begin{cases} iy_t + \Delta y = 0 & \textbf{in } (0,T)\times\Omega \\ y = k\ \chi_\gamma & \textbf{on } (0,T)\times\Gamma \\ y(0,.) = z_0,\ \ y(T,.) = 0 & \end{cases}$$

with the control $k(t,x) := \int_{-S}^{S} \tilde{h}(s,x)\rho(t,s)ds$.

K. Ramdani et al. proved in 2005 that for the domain $\Omega = (0,\pi)^2$, the EC holds (in any time $T > 0$) iff the control region γ contains both a horizontal and a vertical segment of nonzero length.



Extension to $\Omega = (0,\pi)^N$ with $N > 2$ challenging. The EC holds if γ is a neighborhood of a vertex (LR and B.-Y. Zhang 2010).

- Assume that A is skew-adjoint, and consider again the control system

$$(\Sigma_{A,B}) \qquad \dot{z} = Az + Bu. \tag{33}$$

- Hautus criterion (Liu 1997, Burq-Zworski 2004, Miller 2006) $\Sigma_{A,B}$ is EC in some time $T > 0$ iff $\exists \delta > 0$

$$||(A^* - \lambda I)z||^2 + ||B^* z||^2 \geq \delta ||z||^2 \qquad \forall z \in D(A),\ \forall \lambda \in \mathbb{C}.$$

- Assume in addition that there is a Hilbert basis $(e_n)_{n \geq 0}$ of eigenvectors of A, $Ae_n = i\mu_n e_n$.
  Thm (Chen et al. 1991, Ramdani et al. 2005) $\Sigma_{A,B}$ is EC in some time $T > 0$ iff there exist $\varepsilon > 0$, $\delta > 0$ s.t. for all $\mu \in \mathbb{R}$, for all

$$z = \sum_{n;\ |\mu_n - \mu| < \varepsilon} c_n e_n$$

  we have $||B^* z|| \geq \delta ||z||$.

We consider the following internal control problem

$$z_t - z_{xx} + d(x)z = \chi_{(a,b)}f, \tag{34}$$
$$z(t,0) = z(t,L) = 0, \tag{35}$$
$$z(0,x) = z_0(x) \tag{36}$$

in which $z = z(t,x)$ is the state, $f = f(t,x)$ is the control input, and $d = d(x)$ is some potential. We assume that $d \in L^\infty(0,L)$, that $f \in L^2(0,T;L^2(a,b))$, and that $z_0 \in L^2(0,L)$. Here, a and b are given numbers with $0 \leq a < b \leq L$.

By standard regularity results for parabolic equations

$$z \in C([0,T]; L^2(0,L)) \cap C((0,T], H_0^1(0,L))$$

Since $z(T) \in H_0^1(0,L)$, the system (34)-(36) is NOT Exactly Controllable in $L^2(0,L)$.

Actually, if $d \equiv 0$, then $z \in C^\infty((0,T) \times (0,a))$ by hypoellipticity of the heat operator $\partial_t - \partial_x^2$, hence $z(T)$ has to be smooth on $(0,a)$.

The following result may however be proved.

Theorem
The system (34)-(36) is null controllable in $L^2(0,L)$ for any $T > 0$.

The system (34)-(36) may be put in the form $\dot{z} = Az + Bf$, with $Az := z_{xx} - d(x)z$ on the domain $D(A) = H^2(0,L) \cap H_0^1(0,L)$ and $B : f \in L^2(a,b) \mapsto \chi_{(a,b)} f \in L^2(0,L)$ (i.e., Bf is the extension of f by 0). Clearly $A^* = A$, so $S^*(t) = S(t)$.

According to the Null Controllability test, we have to check that there exists some constant $C > 0$ such that for any $z_0 \in L^2(0,L)$

$$\int_0^T \int_a^b |z(t,x)|^2 \, dx dt \geq C \int_0^L |z(T,x)|^2 \, dx \qquad (37)$$

where $z(t,x)$ denotes the solution of

$$\begin{aligned} z_t - z_{xx} + d(x)z &= 0, \\ z(t,0) = z(t,L) &= 0, \\ z(0,x) &= z_0(x). \end{aligned}$$

By a density argument, we may as well assume that $z_0 \in D(A)$.

The key ingredient is the following Carleman estimate.

Lemma

There exist two positive numbers s_0, C_0, and a function $\psi \in C^2([0,L])$ with $\psi(x) > 0 \quad \forall x \in [0,L]$ such that for any $z \in L^2(0,T;H^2(0,L) \cap H_0^1(0,L)) \cap H^1(0,T;L^2(0,L))$ and for any $s \geq s_0$ we have

$$\int_0^T \int_0^L e^{-2s\frac{\psi(x)}{t(T-t)}} \left(\frac{t(T-t)}{s}(|z_t|^2 + |z_{xx}|^2) + \frac{s}{t(T-t)}|z_x|^2 + \frac{s^3}{t^3(T-t)^3}|z|^2 \right) dxdt$$
$$\leq C_0 \left(\int_0^T \int_0^L e^{-2s\frac{\psi(x)}{t(T-t)}} |z_t - z_{xx}|^2 \, dxdt + \int_0^T \int_a^b e^{-2s\frac{\psi(x)}{t(T-t)}} \frac{s^3}{t^3(T-t)^3} |z|^2 \, dxdt \right).$$

Remark

Such estimate gives at once the Unique Continuation Property for the heat equation, with $\omega = (a,b) \subset (0,L) = \Omega$.

3.3 - NC of the heat eq. (completed)

G. Lebeau and L. Robbiano, and independently O. Imanuvilov and A. Fursikov, proved in 1995 the null controllability of the heat equation in any dimension by using Carleman estimates.

Since then, Carleman estimates have been used to prove the zero controllability of semilinear heat equations and of the Navier-Stokes equations. For these nonlinear equations, the Carleman estimates are combined with a fixed-point theorem (Schauder theorem) or an appropriate version of the inverse mapping theorem.

4 - Stabilizability

We investigate the stabilizability of a control system

$$\Sigma_{A,B} \qquad \dot{z} = Az + Bu, \tag{38}$$

where

- A generates a continuous semigroup of operators $(S(t))_{t\geq 0}$ on some Hilbert space H
- B is a linear, bounded operator acting from a Hilbert space U to H.

Let $A \in \mathbb{C}^{N\times N}$ **and** $B \in \mathbb{C}^{N\times M}$. $\sigma(A) := \{\lambda \in \mathbb{C} | \ \det(\lambda I - A) = 0\}$ **is the spectrum of** A.

- **The origin is asymptotically stable (** $\Longleftrightarrow$ **exponentially stable) for** $\dot{x} = Ax$ **if and only if** $\sigma(A) \subset \mathbb{C}_- := \{z \in \mathbb{C} | \ \mathbf{Re}\ z < 0\}$.

  Furthermore

  $$\sup\{\mathbf{Re}\ \lambda | \ \lambda \in \sigma(A)\} = \inf\{\mu \in \mathbb{R} | \ \exists C > 0, \ ||e^{tA}|| \leq Ce^{\mu t} \ \forall t \geq 0\}.$$

- **Wonham's Theorem**
  $\Sigma_{A,B}$ **is controllable if and only if it is exponentially stabilizable with an arbitrary exponential decay rate:**

  $$\forall \mu \in (-\infty, 0), \ \exists K \in \mathbb{C}^{M,N}, \ \exists C > 0 \quad ||e^{t(A+BK)}|| \leq Ce^{\mu t} \quad \forall t \geq 0.$$

Definition

The resolvent set of A, denoted by $\rho(A)$, is the set of complex numbers λ for which the operator $\lambda I - A$ is boundedly invertible (i.e., its inverse $(\lambda I - A)^{-1} : H \to D(A)$ exists and is bounded from H to H).

As A is closed, an application of the closed graph theorem shows at once that $\lambda \in \rho(A)$ if and only if $\lambda I - A : D(A) \to H$ is onto and one-to-one.

The map $\lambda \in \rho(A) \mapsto (\lambda I - A)^{-1} \in L(H, H)$ is called the resolvent of A. The spectrum of A, denoted by $\sigma(A)$, is the complement of the resolvent set: $\sigma(A) = \mathbb{C} \setminus \rho(A)$

λ is in the spectrum of A if and only if

- $\lambda I - A$ is not one-to-one (i.e., λ is an eigenvalue of A)

OR

- $\lambda I - A$ is not onto.

$\sigma(A)$ is NOT reduced to the set of the eigenvalues of A in general!!

Let us consider the following properties:

- **(i)** **For some constants** $C > 0$ $\mu > 0$ **and all** $t \geq 0$ $||S(t)|| \leq Ce^{-\mu t}$;
- **(ii)** **For any** $z_0 \in H$ $z(t) \to 0$ **exponentially as** $t \to +\infty$;
- **(iii)** **For any** $z_0 \in H$, $\int_0^{+\infty} ||z(t)||_H^2 \, dt < \infty$;
- **(iv)** **For any** $z_0 \in H$ $z(t) \to 0$ **as** $t \to +\infty$;
- **(v)** $\sup\{\text{Re } \lambda | \ \lambda \in \sigma(A)\} < 0$.

In finite dimension, there are all equivalent!!

In infinite dimension, we ONLY have

Theorem (Datko)
$(i) \iff (ii) \iff (iii)$ **and**
$(ii) \Rightarrow (iv)$, $(ii) \Rightarrow (v)$.

In infinite dimension, we still have that

$$\omega \leq \omega_0$$

where

$\omega := \sup\{\text{Re } \lambda |\ \lambda \in \sigma(A)\}$ **is the spectral abscissa, and**
$\omega_0 := \inf\{\mu \in \mathbb{R} |\ \exists C > 0,\ ||S(t)|| \leq Ce^{\mu t}\ \forall t \geq 0\}$ **is the best decay rate.**

Proof. If $\mu > \omega_0$**, then** $||S(t)|| \leq Ce^{\mu t}\ \forall t \geq 0$**. Then for all** $\lambda \in \mathbb{C}$ **with Re** $\lambda > \mu$**, we have that** $\lambda \in \rho(A)$ **and** $(\lambda I - A)^{-1} = \int_0^{+\infty} e^{-\lambda t} S(t)\, dt$**. Thus** $\sigma(A) \subset \{z \in \mathbb{C} |\ \text{Re } z \leq \mu\}$**, and** $\omega \leq \mu$**.**

A **gap** between the spectral abscissa and the best decay rate may exist:

Lemma

Let $(\lambda_k)_{k\geq 1}$ be an arbitrary sequence of real numbers such that $|\lambda_k| \to \infty$. Then there exists a semigroup $(S(t))_{t\geq 0}$ on $H = L^2_{\mathbb{C}}(\mathbb{N})$ such that

$$||S(t)|| = e^t \quad \forall t \geq 0 \quad \textbf{and} \quad \sigma(A) = \{i\lambda_k;\ k = 1, 2, ...\}$$

where A denotes the generator of $(S(t))_{t\geq 0}$.

In particular, $\omega = 0 < \omega_0 = 1$.

Let $(S(t))_{t\geq 0}$ be a continuous semigroup on H.

Definition

The semigroup $(S(t))_{t\geq 0}$ is said to be

- **exponentially stable if the condition (i) holds:**

$$\exists C > 0,\ \mu > 0 \qquad ||S(t)|| \leq Ce^{-\mu t} \quad \forall t \geq 0$$

- **strongly stable if the condition (iv) holds:**

$$\forall z_0 \in H, \qquad S(t)z_0 \longrightarrow 0 \quad \text{as } t \longrightarrow +\infty$$

We now review classical stability results based on frequency-domain considerations.

Strong Stability

Arendt-Batty Theorem 1988.
If $(S(t))_{t\geq 0}$ is bounded and $i\mathbb{R} \subset \rho(A)$, then $(S(t))_{t\geq 0}$ is strongly stable

Remarks.
1. $(S(t))_{t\geq 0}$ bounded means $||S(t)|| \leq C$ for some constant C and all t
2. Only the location of the spectrum plays a role
3. The condition $i\mathbb{R} \subset \rho(A)$ is ***not necessary*** for the strong stability to hold.

Exponential Stability

When we look at the decay rate, the behavior of the resolvent $(\lambda I - A)^{-1}$ as $\lambda \to \infty$ has also to be considered:

Huang-Prüss Theorem 1984

Assume that $(S(t))_{t\geq 0}$ is bounded. Then $(S(t))_{t\geq 0}$ is exponentially stable if and only if $i\mathbb{R} \subset \rho(A)$ and

$$\sup_{\beta\in\mathbb{R}} ||(i\beta I - A)^{-1}|| < \infty.$$

Polynomial Decay Rate

When the resolvent is not bounded, but has a polynomial growth on the imaginary axis, a "polynomial" decay rate holds:

Theorem (Liu-Rao, 2005)
Assume that $(S(t))_{t\geq 0}$ **is bounded, and that**

$$\sup_{|\beta|\geq 1} \frac{1}{|\beta|^s} ||(i\beta I - A)^{-1}|| < \infty.$$

Then, for any $k \in \mathbb{N}^*$ **there exists a constant** $C_k > 0$ **such that**

$$||S(t)z||_H \leq C_k \left(\frac{\ln t}{t}\right)^{\frac{k}{s}} (\ln t)||z||_{D(A^k)} \quad \forall z \in D(A^k)$$

For any bounded operator $K \in L(H, U)$, we let A_K denote the operator $A_K z = Az + BKz$ with domain $D(A_K) = D(A)$, and by $(S_K(t))_{t \geq 0}$ the semigroup generated by A_K.

Definition

The control system $\Sigma_{A,B}$ is said to be

- **exponentially stabilizable** if there exists a feedback $K \in L(H, U)$ such that the operator A_K is exponentially stable; i.e., for some constants $C > 0$, $\mu > 0$,

$$||S_K(t)|| \leq Ce^{-\mu t} \quad \forall t \geq 0.$$

- **completely stabilizable** if it is exponentially stabilizable with an **arbitrary exponential decay rate**; i.e., for any $\mu \in \mathbb{R}$, there exists a feedback $K \in L(H, U)$ and a constant $C > 0$ such that

$$||S_K(t)|| \leq Ce^{-\mu t} \quad \forall t \geq 0.$$

Theorem (Datko 1972)
If the system $(\Sigma_{A,B})$ is null controllable, then it is exponentially stabilizable.

Application: heat equation with potential.

Theorem (Slemrod 1974, Megan 1975, Russell 1978, Liu 1997)
Assume that A is skew-adjoint. Then the following properties are equivalent.

1. $(\Sigma_{A,B})$ is exactly controllable in some time $T > 0$;
2. $(\Sigma_{A,B})$ is null controllable in some time $T > 0$;
3. $(\Sigma_{A,B})$ is completely stabilizable;
4. $(\Sigma_{A,B})$ is exponentially stabilizable;
5. For every positive definite self-adjoint operator $S \in L(U)$, the operator $A - BSB^*$ generates an exponentially stable semigroup on H.

5 - Nonlinear equations - Some open problems

- The usual way to show the controllability/stabilizability of a nonlinear equation is to establish the controllability/stabilizability of the linearized equation, and next to apply a perturbation argument to incorporate the nonlinear term (fixed-point argument). A good WellPosedness theory is needed.
- For the exact controllability (wave eq., Schrödinger eq.), we use the control operator given by HUM and apply the contraction mapping theorem (Banach-Picard).
- For the null controllability (heat eq., Navier-Stokes eq.), we use Carleman estimate and Schauder (or Tichonov) fixed point theorem.
- For the Korteweg-de Vries eq. $z_t + z_{xxx} + z_x + zz_x = 0$, both approaches are needed (LR 1997, 2004): if we control from the right endpoint, we have exact controllability (hyperbolic), while if we control from the left endpoint we have null controllability (parabolic). Reason: the dispersion relation $\omega = k^3 - k$ indicates that high frequencies propagate to the right.

- The perturbation approach described above give local results. To get global results, we need to combine a local controllability result with a global stabilization result.
- The choice of the space is crucial. In spaces of high regularity, functions are bounded and the nonlinear term is easy to deal with. (Sobolev embedding $H^s(\Omega) \subset L^\infty(\Omega)$ if $s > N/2$). However, the natural energy space is usually $L^2(\Omega)$, where a WP theory may require a lot of analysis (e.g. Bourgain theory for KdV, NLS). It turns out that a simple feedback law providing a global stabilization is often found in the energy space.

Consider the NonLinear Schrödinger (NLS) equation on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ ($a \in C^\infty(\mathbb{T})$ given)

$$iz_t + z_{xx} + \lambda |z|^2 z = ia(x)u, \qquad x \in \mathbb{T}$$

- R. Illner et al. 2003 (loc. EC in $H^1(\mathbb{T})$)
- LR -B.-Y. Zhang 2009 (loc. EC in $L^2(\mathbb{T})$, loc. stab. in $L^2(\mathbb{T})$ with the feedback $u = -a(x)z$)
- C. Laurent 2010 (glob. stab. in $L^2(\mathbb{T})$, same feedback). Method: compactness/uniqueness argument (Zuazua) + propagation of compactness/regularity.
- Open question: do we have global controllability in small time ??

Nonlinear Methods ??

- Some popular methods in finite dimension (controllability via Lie brackets tests, CLF, nonsmooth control) do not seem to have (useful) generalizations for PDE
- Lyapunov theory (direct and converse results) is classical for reaction-diffusion equations (only ??)
- The return method introduced by J.-M. Coron has proved to be useful for PDE (Euler eq., Saint-Venant eq., conservation laws)
- The power series expansion method has been applied to KdV in situations when the linearized eq. fails to be EC (Crépeau-Coron 2004, Cerpa 2006, Cerpa-Crepeau 2009)
- Time-varying feedback laws were used to achieve a global stabilization with arbitrary large decay rate for KdV (Laurent-LR-Zhang, 2010)
- Homogeneity is used by Coron-Guerrero-LR for a system of PDE

Fluid-structure interaction systems

- Class of systems modeling the motion of a rigid body (or an elastic body) in a fluid. Combines a PDE (Euler or Navier-Stokes for the fluid) to some ODE (Newton laws for the solid)
- Example: control of the motion of a boat (Glass-LR, preprint), of a submarine (Lecaros-LR, in progress), swimming of a fish (Chambrion-Munnier, preprint)
- To be done: tracking, optimal control, role played by the vortex generated at the end of a fish

Control of system of PDE

- **A Kalman type test has been discovered by Ammar Khodja et al 2005 for systems of heat eq. with constant coefficients**

$$\begin{cases} z_{1,t} = \Delta z_1 + a_{11} z_1 + a_{12} z_2 + bu \\ z_{2,t} = \Delta z_2 + a_{21} z_1 + a_{22} z_2 \end{cases}$$

- **The case of a system with variable coefficients is quite open (only a positive result by L. de Teresa-LR 2011 in dim. 1)**

$$\begin{cases} z_{1,t} = \Delta z_1 + b(x) u \\ z_{2,t} = \Delta z_2 + a(x) z_1 \end{cases}$$

- **System of wave eq. much harder to deal with!**
- **Nonlinear coupling terms (Coron-Guerrero-LR 2010)**

$$\begin{cases} z_{1,t} = \Delta z_1 + F(z_1, z_2) + bu \\ z_{2,t} = \Delta z_2 + [z_1]^{2k+1} \end{cases}$$

Thank You !