On observers, a meeting of many view points and applications

Laurent PRALY

CAS, Mines-ParisTech

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In most cases when we manipulate data coming from the real world, we implement in one way or another what I call an observer.

Depending on the field of applications, observers take different names – stochastic filters, soft sensors, state re-constructors, data assimilation, inverse problems, . . .

They are answers to the same problem: given measurements (= partial information), try to estimate internal variables of a dynamical system.

In this talk, I briefly go over the general (well known) design rules and give more details on one particular observer.

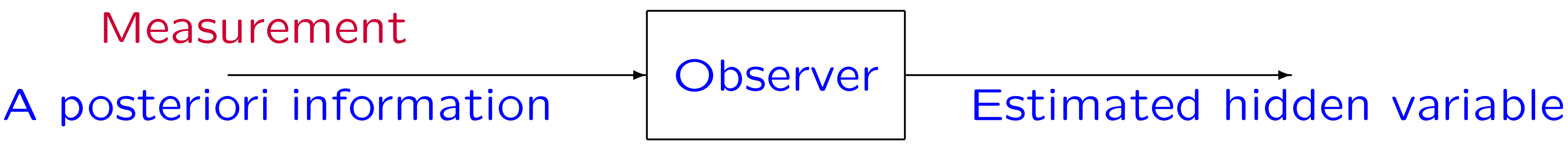
Contents

§ 1/8 Approaching the observation problem

Observation Problem:
Estimate *hidden variable*, denoted z, from observed/*measured variable*, denoted y.

The "object" solving this problem is called an *observer*.

Measurements make what is called the *a posteriori information*. It evolves with time as data accumulate.



(1st thought version of) observer structure

Galileo Galilei wrote in 1638 in a book entitled *Dialogues concerning two new sciences*,

I can easily know the length of a . . . string whose upper end is attached . . . . For, if I attach to the lower end of this string a rather heavy weight and give it a to-and-fro motion, and if a friend counts the number of its oscillations, while I, during the same time interval, count the number of oscillations of a pendulum which is exactly one cubit in length, then, knowing the number of oscillations which each pendulum makes in the given time interval, one can determine the string length. . . . by *taking the squares of these two numbers* of oscillations *. . . I shall divide* the larger square by the smaller one and *. . . the length of the* long *string* is the ratio in *cubits.*

Observation Problem:
Estimate a *string length* from the measurement of the *angular position* of a pendulum.

$\Rightarrow$ We need to know a “relation” between the hidden variable, the string length, and the measured variable, the angular position.

Rule 1 for observer design

We need a priori information,

= a "relation" between

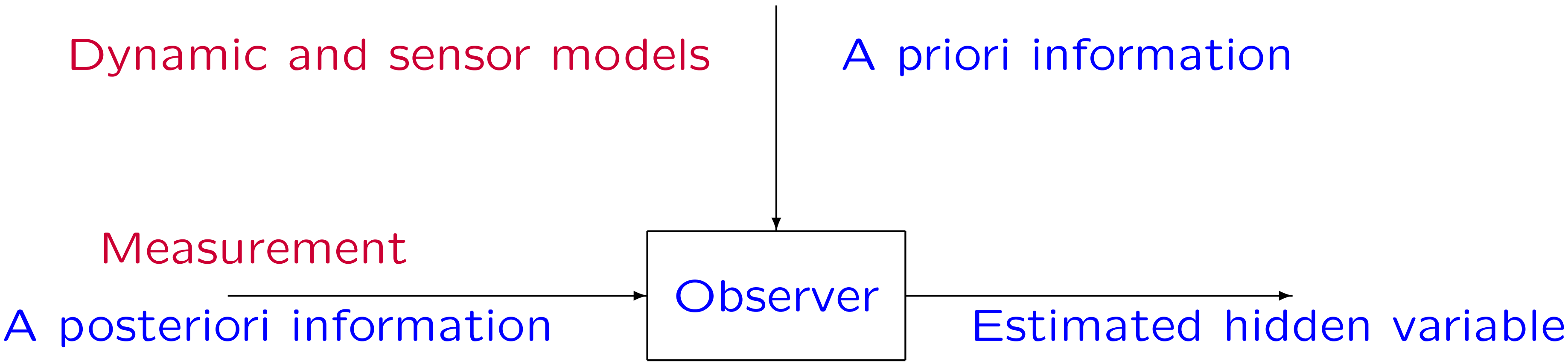
hidden and measured variables.

Usually this "relation" is not instantaneous and not explicit.

It can be given by dynamic and sensor models.

$$\begin{aligned}
&[\textit{time} \mapsto \textit{measured variable}\,(\textit{time})] \\
&\qquad = \textit{Function}\Big([\textit{time} \mapsto (\textit{hidden variable}\,(\textit{time}),\ \textit{others}\,(\textit{time})]\Big).
\end{aligned}$$

This makes what is called the a priori information.



(a more elaborate version of) observer structure

§1.6 Example — Length estimation following Galileo's comment

For Galileo, the "relation" is:

$$z = \ell = \frac{g}{4\pi^2} * \left(\frac{\textit{oscillation number}}{\textit{observation time}}\right)^2$$

It is sufficient to produce the oscillation number and the observation time from the measured pendulum angle.

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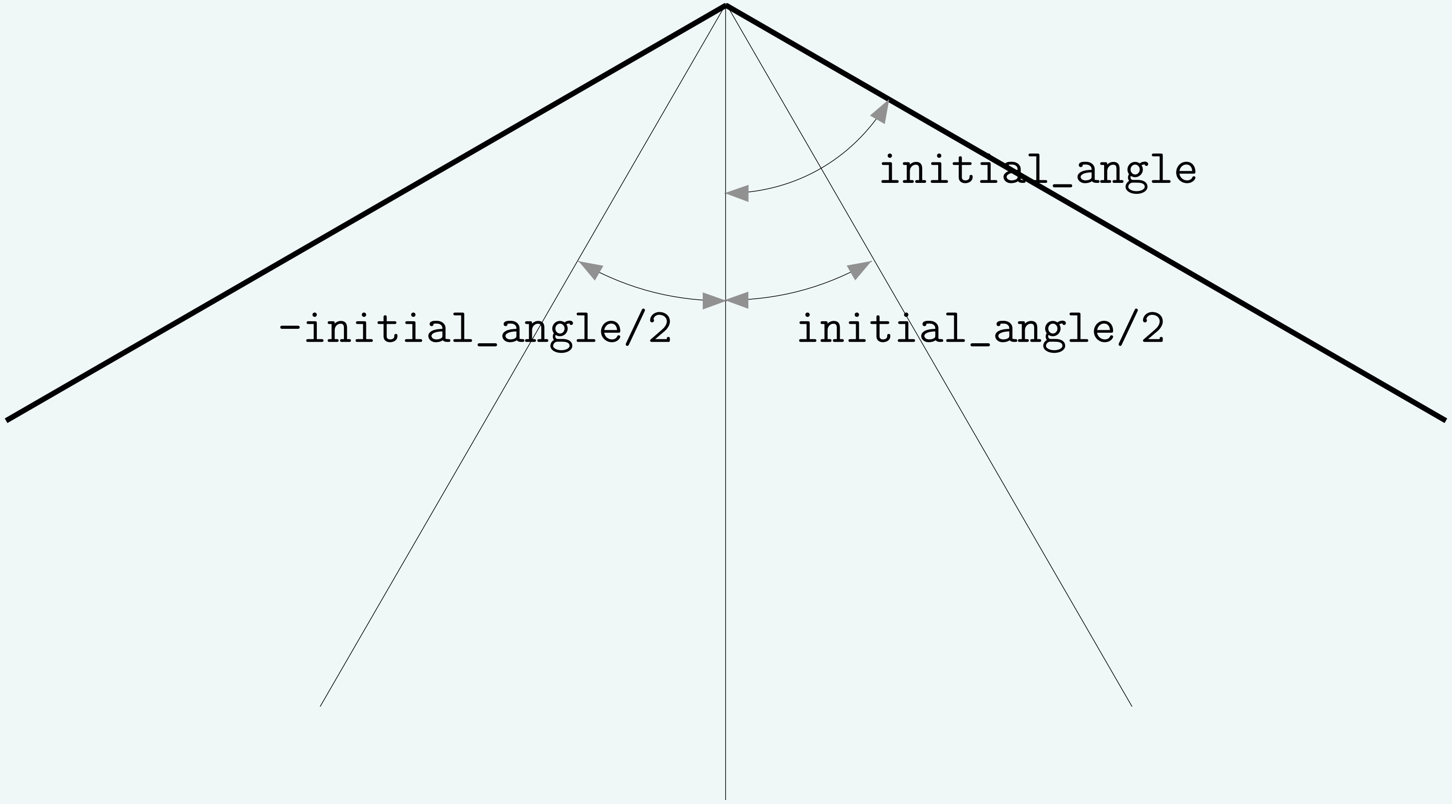

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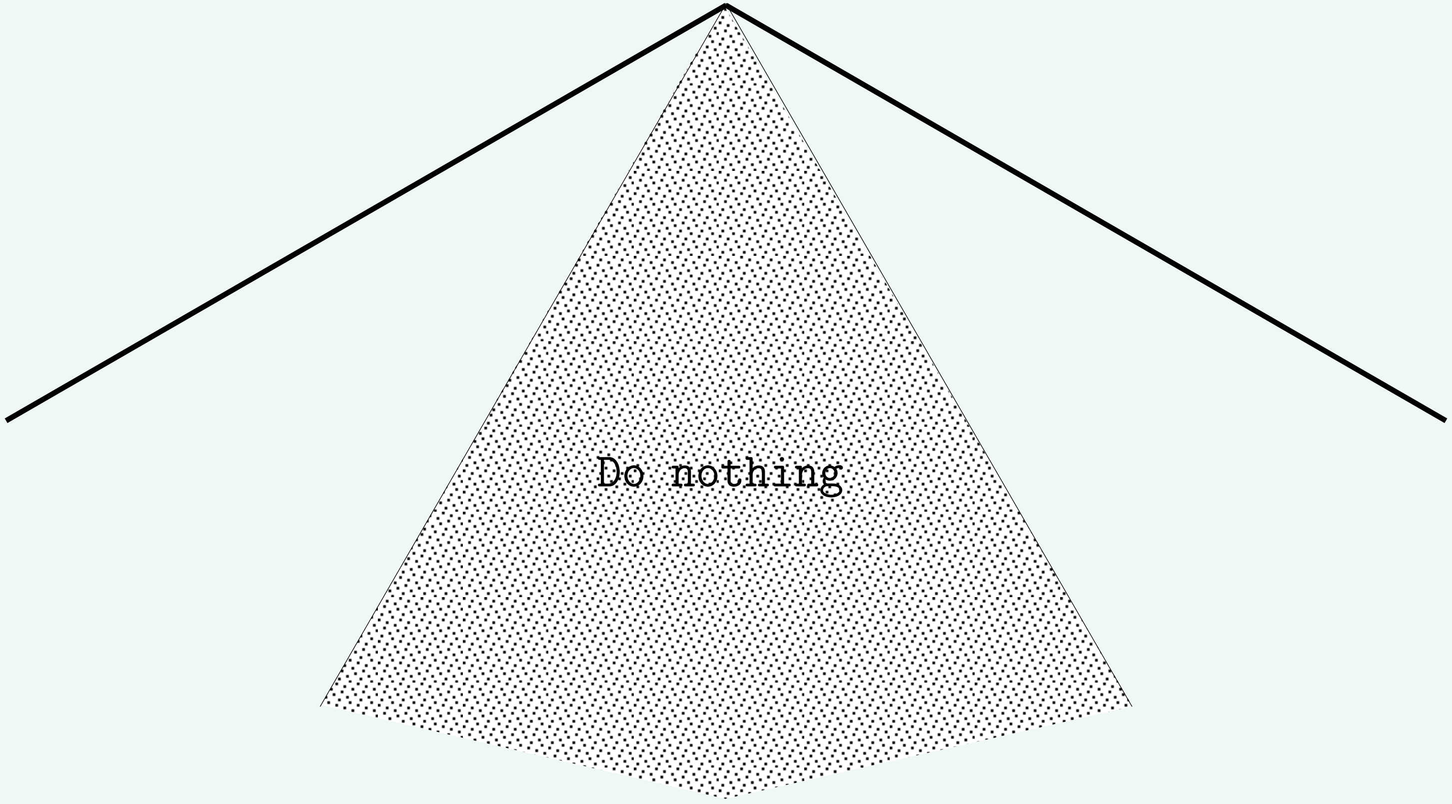

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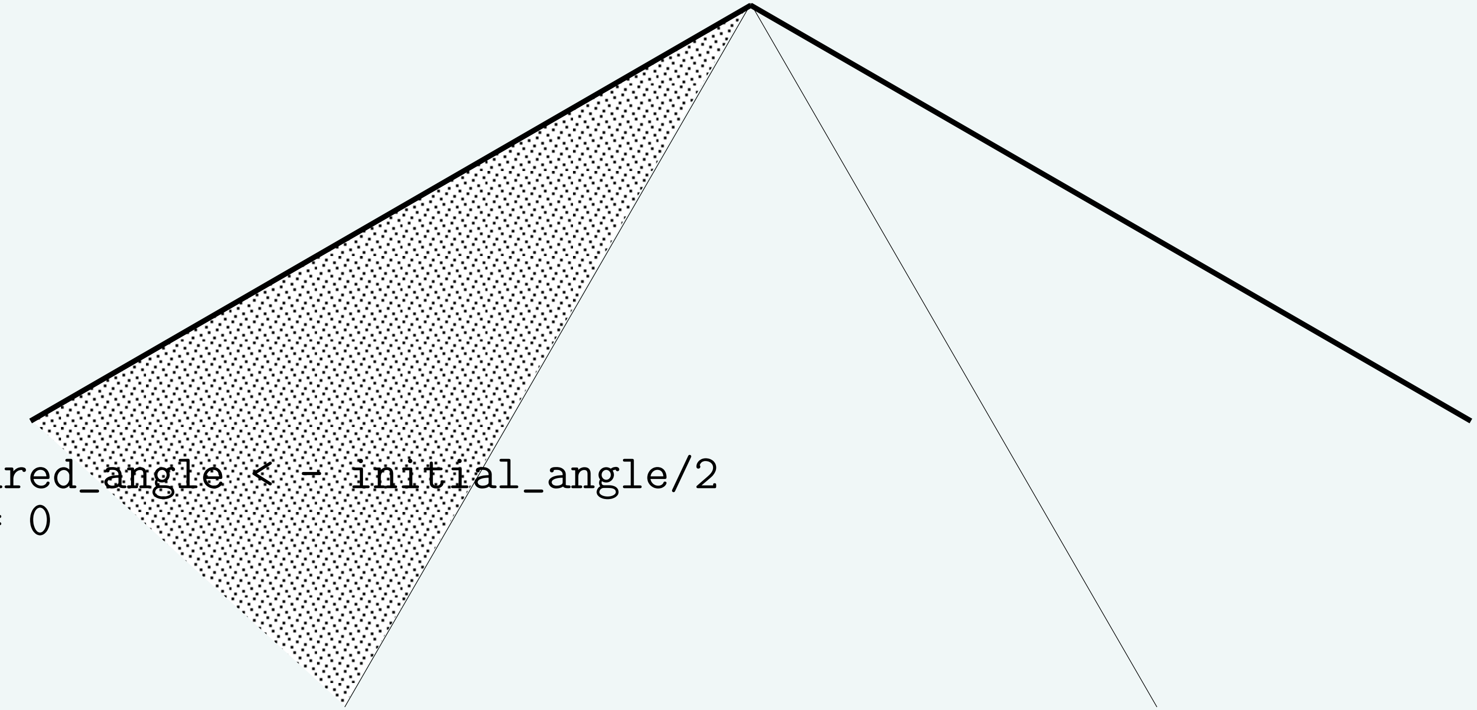

```
if measured_angle < - initial_angle/2
  pass = 0
end
```

Example Length estimation following Galileo's comment

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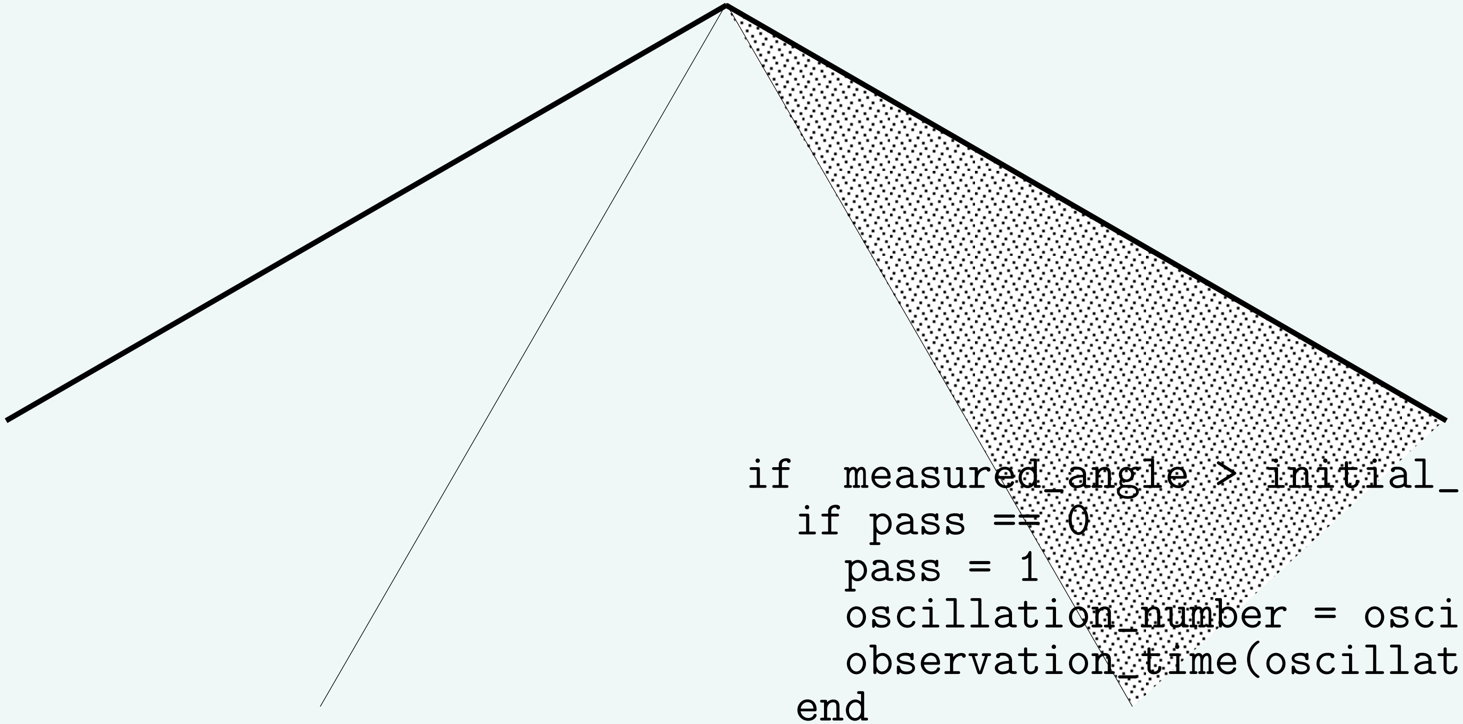
$$z = \ell = \frac{g}{4\pi^2} * \left(\frac{\textit{oscillation number}}{\textit{observation time}}\right)^2$$

It is sufficient to produce the oscillation number and the observation time from the measured pendulum angle.



```
if  measured_angle > initial_angle/2
  if pass == 0
    pass = 1
    oscillation_number = oscillation_number+1
    observation_time(oscillation_number) = t
  end
end
```

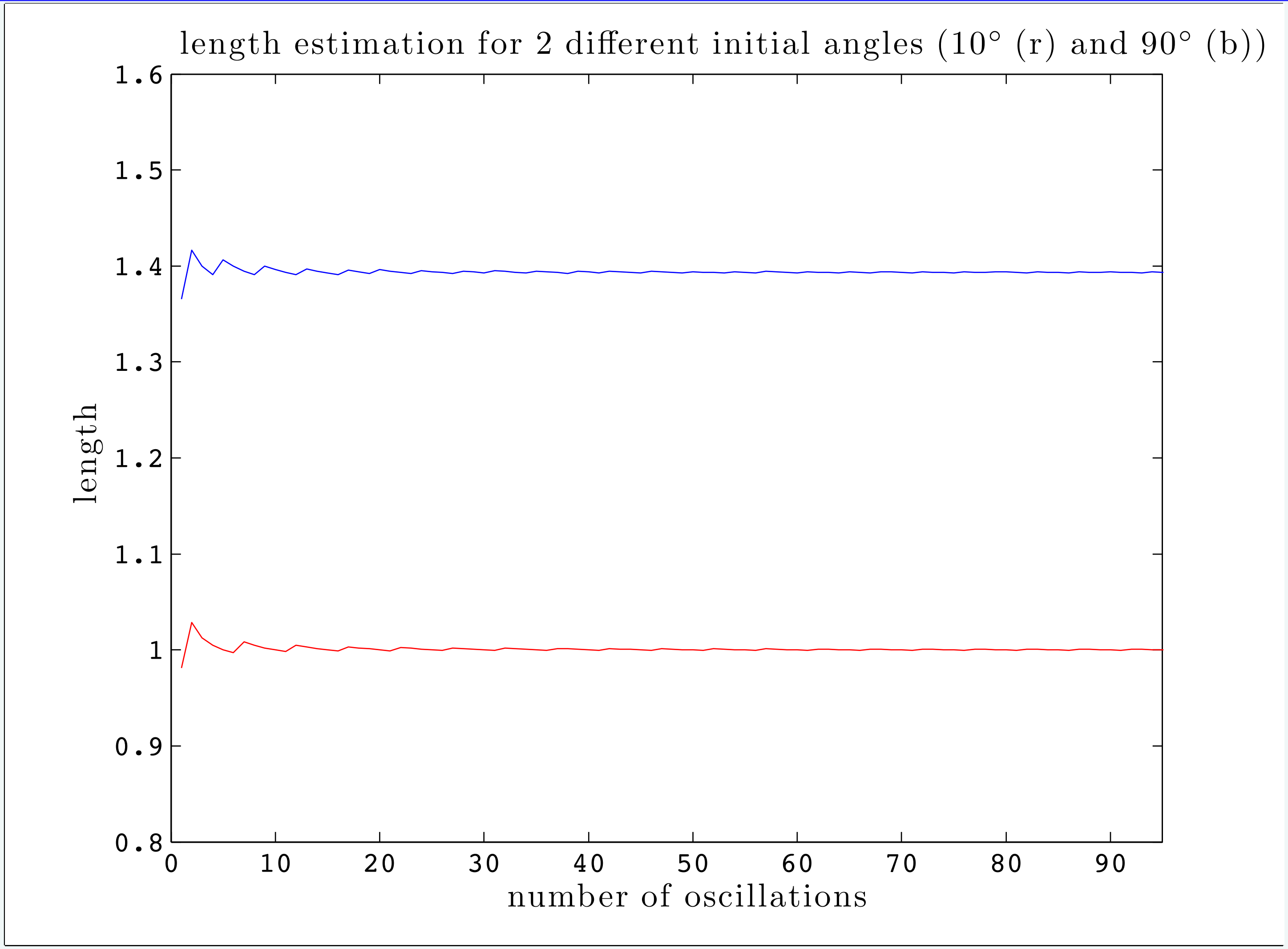
This leads to the observer :

```
dot t = 1
if  measured_angle > initial_angle/2
     if pass == 0
          pass = 1
          oscillation_number = oscillation_number+1
          observation_time(oscillation_number) = t
     end
elseif measured_angle < - initial_angle/2
     pass = 0
end
```
← Observer dynamic

```
z=(oscillation_number/observation time)^2 * g/4*pi^2
```
← Observer output

This observer is an hybrid dynamical system with continuous variable `t`, integer variable `oscillation_number`, and logical variable `pass` which make a state and with an output which is a function of this state.

Example   Length estimation following Galileo's comment



The estimation depends on the pendulum energy. **Bad!**

Problem: Galileo's "relation" is an approximation for small energy.
The correct relation involves Jacobi elliptic functions.

According to Lagrange, another "relation" is given by conservation of energy :

$$E \;=\; \frac{\ell}{2}\omega^2 \;+\; g\,(1 - \cos(\theta)) \;=\; \text{constant}$$

where g is the known local gravity acceleration, θ is the angular position and ω is the angular velocity.

When the function $s \in (t-T, t] \mapsto (\theta(s), \omega(s))$ is known, the two constant numbers E and ℓ can be obtained by solving the minimization problem:

$$\min_{E,\ell} J(E,\ell,t) \;=\; \min_{E,\ell} \int_{t-T}^{t} \left[E - \frac{\ell}{2}\omega(s)^2 - g\,(1 - \cos(\theta(s)))\right]^2 ds$$

The function $s \in (t-T,t] \mapsto \theta(s)$ is the a posteriori information, given by measurement.

$\Rightarrow$ sufficient to produce the function $s \in (t-T,t] \mapsto \omega(s) = \dot{\theta}(s)$
from the function $s \in (t-T,t] \mapsto \theta(s)$

But $\dot{\theta}$ is a mathematical concept involving a limit process.

Since only "standard" numbers can be used, we are led to use an approximation, e.g. :

$$\omega\left(s - \frac{\mathfrak{h}}{2}\right) \approx \widehat{\omega}\left(s - \frac{\mathfrak{h}}{2}\right) = \frac{\theta(s) - \theta(s-\mathfrak{h})}{\mathfrak{h}}$$

Naive observer:

$$\widehat{\omega}\left(t - \frac{\mathfrak{h}}{2}\right) = \frac{y(t) - y(t-\mathfrak{h})}{\mathfrak{h}}$$

$$\widehat{\ell}(t) = 2g \frac{\displaystyle\int_{t-T}^{t} \widehat{\omega}(s)^2 ds \int_{t-T}^{t} (1 - \cos(y(s))ds - T\int_{t-T}^{t} \widehat{\omega}(s)^2 (1 - \cos(y(s))ds}{\displaystyle T\int_{t-T}^{t} \widehat{\omega}^4(s) ds - \left(\int_{t-T}^{t} \omega(s)^2 ds\right)^2}$$

T and $\mathfrak{h}$ are two parameters to tune.

$$E = \tfrac{\ell}{2}\omega^2 + g\left(1 - \cos(\theta)\right)$$

A priori information

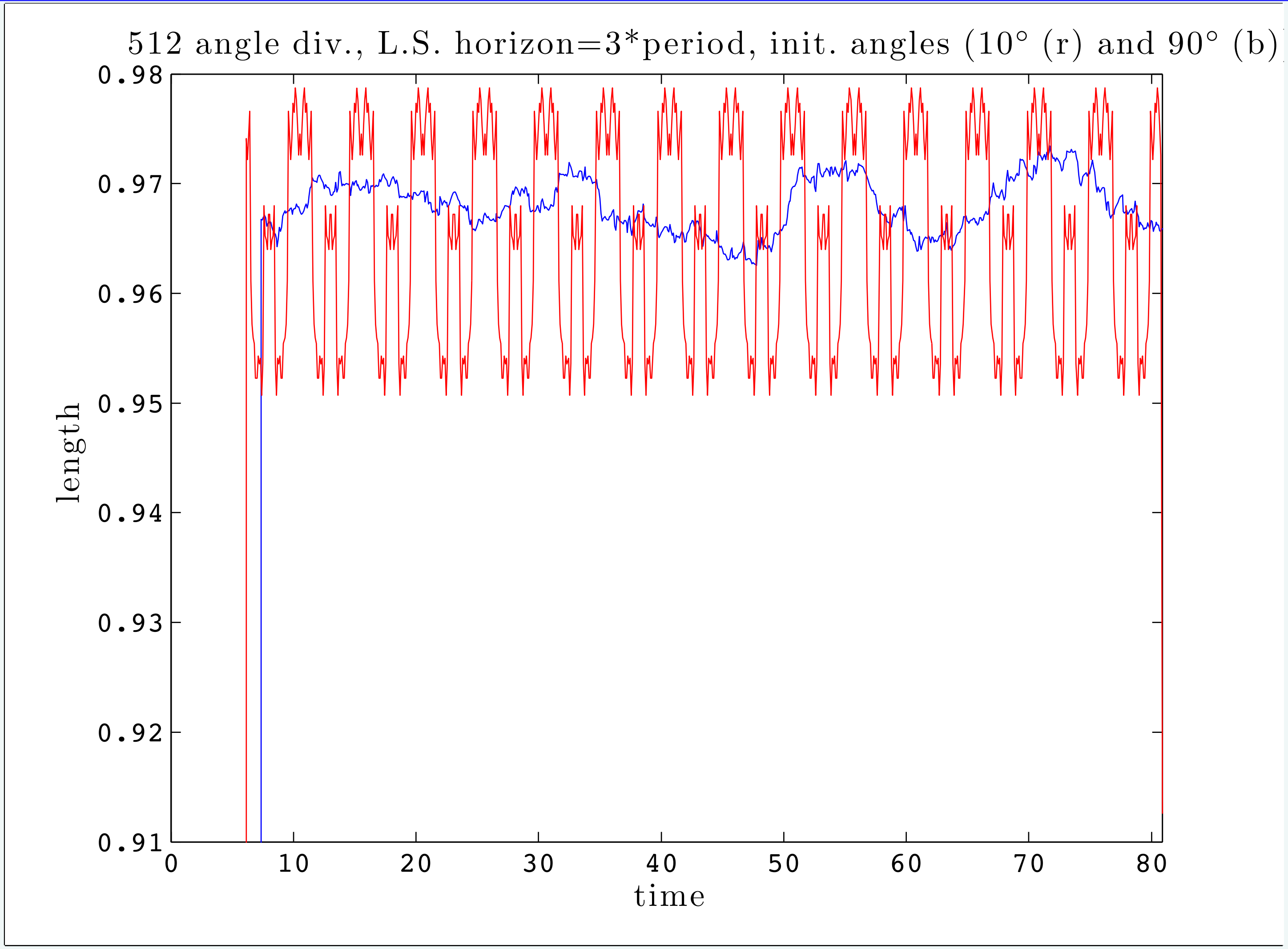
$s \in (t-T, t] \mapsto y(s)$

A posteriori information

Observer

$\widehat{\ell}(t)$

Estimated hidden variable



Compared with Galileo's approach, (the average is) less dependent on initial angle. But highly oscillatory estimated length though ℓ is constant. **Why ?**

Actually ...

The angle measurement is made with a disk divided in N sectors:

$$y = \texttt{round}\left(\frac{N\theta}{2\pi}\right)\frac{2\pi}{N} = \theta + q \qquad \leftarrow \text{sensor model}$$

$q \in [-\frac{\pi}{N}, \frac{\pi}{N}]$ = quantization error in the angle measurement.

This leads to an error in the angular velocity estimation which can be upperbounded as

$$|\omega(t) - \widehat{\omega}(t)| \leq \frac{1}{2}\frac{g}{l}|\sin(\theta(t))|\,\mathfrak{h} + \frac{\pi}{N}\frac{1}{\mathfrak{h}}$$

$\Rightarrow$ there exists an "optimal" step $\mathfrak{h}_{opt}$. It is decreasing with respect to solution energy but increasing with respect to the uncertainty N.

$\Rightarrow$ To tune the observer, we need to know N which models the uncertainty.

In all our simulations,

$N = 512$ (angle uncertainty $\approx 0.7^\circ$)

and

the sampling period t_{samp} is chosen equal to $\mathfrak{h}_{opt}$(N,solution energy).

Rule 2 for observer design

In the a priori information,

we need also a model of the uncertainties

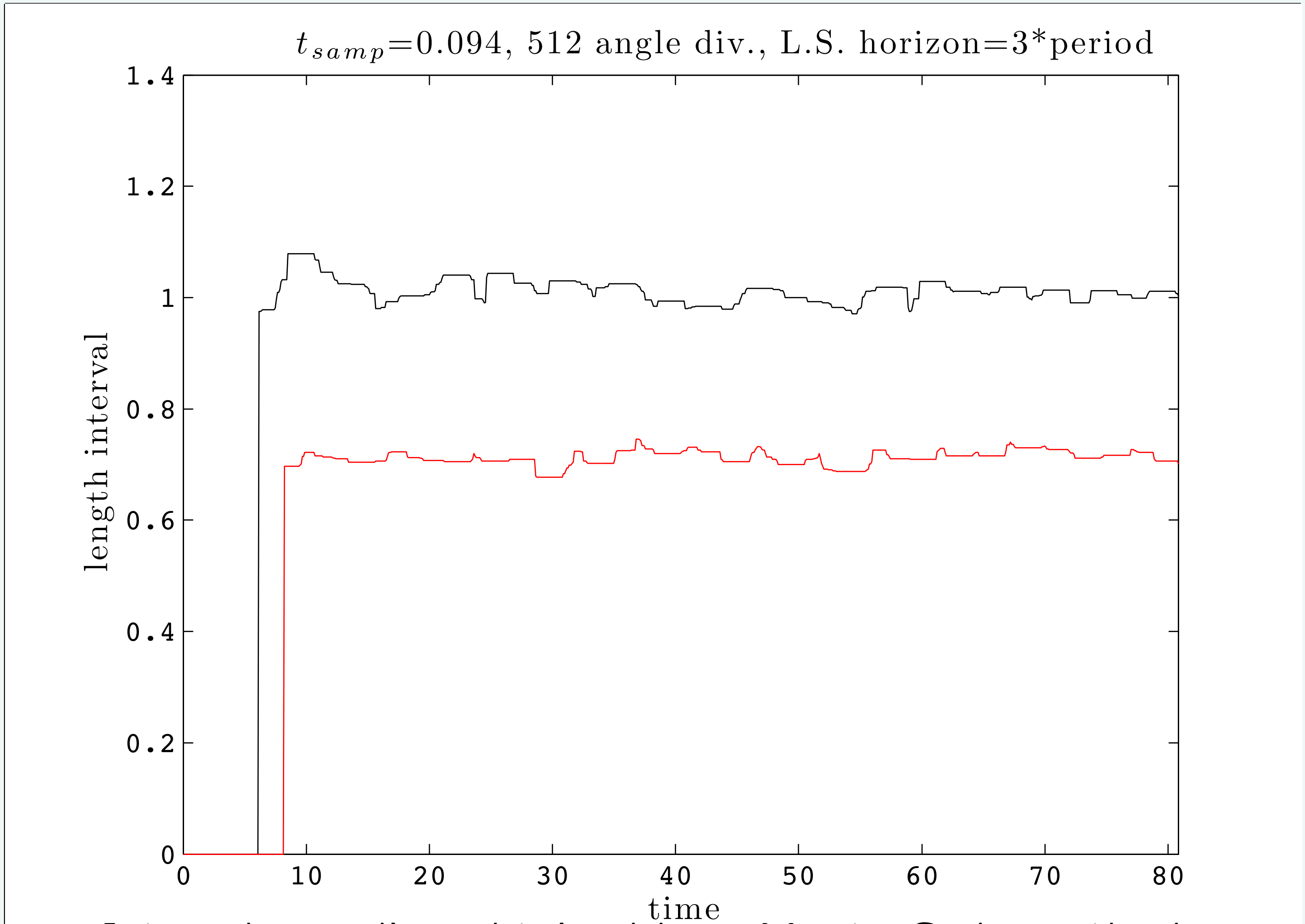
Another consequence:

the measurement gives only :

$$\theta \in \left[y - \frac{\pi}{N}, y + \frac{\pi}{N}\right] \qquad = \quad \text{The actual angle is in an interval.}$$

So we should use as observer input any angle in the interval $[y - \frac{\pi}{N}, y + \frac{\pi}{N}]$.

This leads to an interval for the estimated length



Interval sampling obtained by a Monte Carlo method

$\Rightarrow$ We can claim only :

At each time t, the length should be between the points given by the curves

Rule 3 for observer design

Due to uncertainties, the output of an observer
cannot be a point but only
a confidence set or a probability measure or . . . ,
(depending on the uncertainty model).

§ 2/8 The observation problem

Observation Problem:

A priori information: We are given a model :

or continuous time or hybrid version

$$x_{k+1} = f_k(x_k, u_k) \quad , \quad y_k = h_k(x_k, v_k) \quad , \quad z_k = g_k(x_k, w_k)$$
$$x_k \in X_k \quad , \quad y_k \in Y_k \quad , \quad z_k \in Z_k$$
$$u_k \in U_k \quad , \quad v_k \in V_k \quad , \quad w_k \in W_k$$

i.e. we are given the functions f, h, g and the sets X to W or the probability measure of these sets,

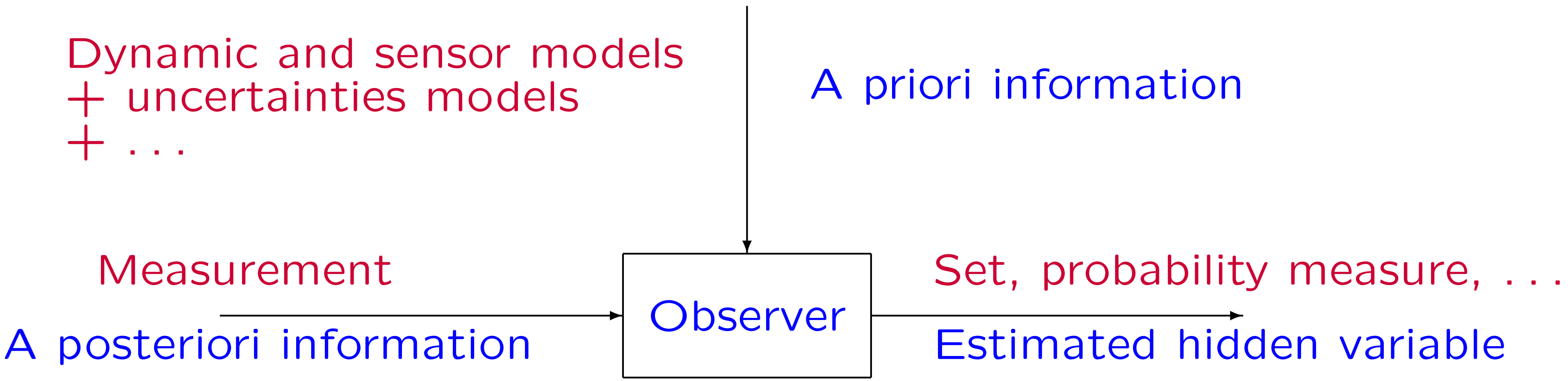
and,

A posteriori information: at time k, we have the observations

$$j \in (k - K, k] \mapsto y_j$$

Observation Problem:

Give a confidence set or probability measure for the *hidden variable*, z_k.



(a correct version of) observer structure

§2.3 Example   Length estimation following Newton's law

Following Newton, . . .

the a priori information for the length estimation problem can be :

$$\dot{\ell} = 0 \ , \ \dot{\theta} = \omega \ , \ \dot{\omega} = -\frac{g}{\ell}\,\mathsf{sin}(\theta) \ , \ \theta \in \left[y - \frac{\pi}{N} \, , \, y + \frac{\pi}{N}\right] \ , \ \ell > 0$$

with g and N given.

$$\dot{\ell} = 0\,,\ \dot{\theta} = \omega\,,\ \dot{\omega} = -\tfrac{g}{\ell}\,\mathsf{sin}(\theta)$$
$$\theta \ \in \ \left[y - \tfrac{\pi}{N}, y + \tfrac{\pi}{N}\right] \ , \ \ell > 0$$

A priori information

$$s \in (t-T, t] \ \mapsto \ y(s)$$

A posteriori information

Observer

Estimated hidden variable

§ 3/8 “Conceptual” solutions

At time k, the hidden variable z_k is in a set $\hat{Z}_k$ given by the following recursive algorithm, where $\hat{X}_k$ contains x_k,

initialization $$\hat{X}_0 = X_0$$

prediction $$\hat{X}_{k|k-1} = f_{k-1}(\hat{X}_{k-1}, U_{k-1})$$ flowing

restriction $$\hat{X}_k = \left\{x \in X_k \bigcap \hat{X}_{k|k-1} \, : \, y_k \in h_k(x, V_k) \bigcap Y_k\right\}$$ consistency

estimation $$\hat{Z}_k = g_k(\hat{X}_k, W_k) \bigcap Z_k$$

A priori information in blue

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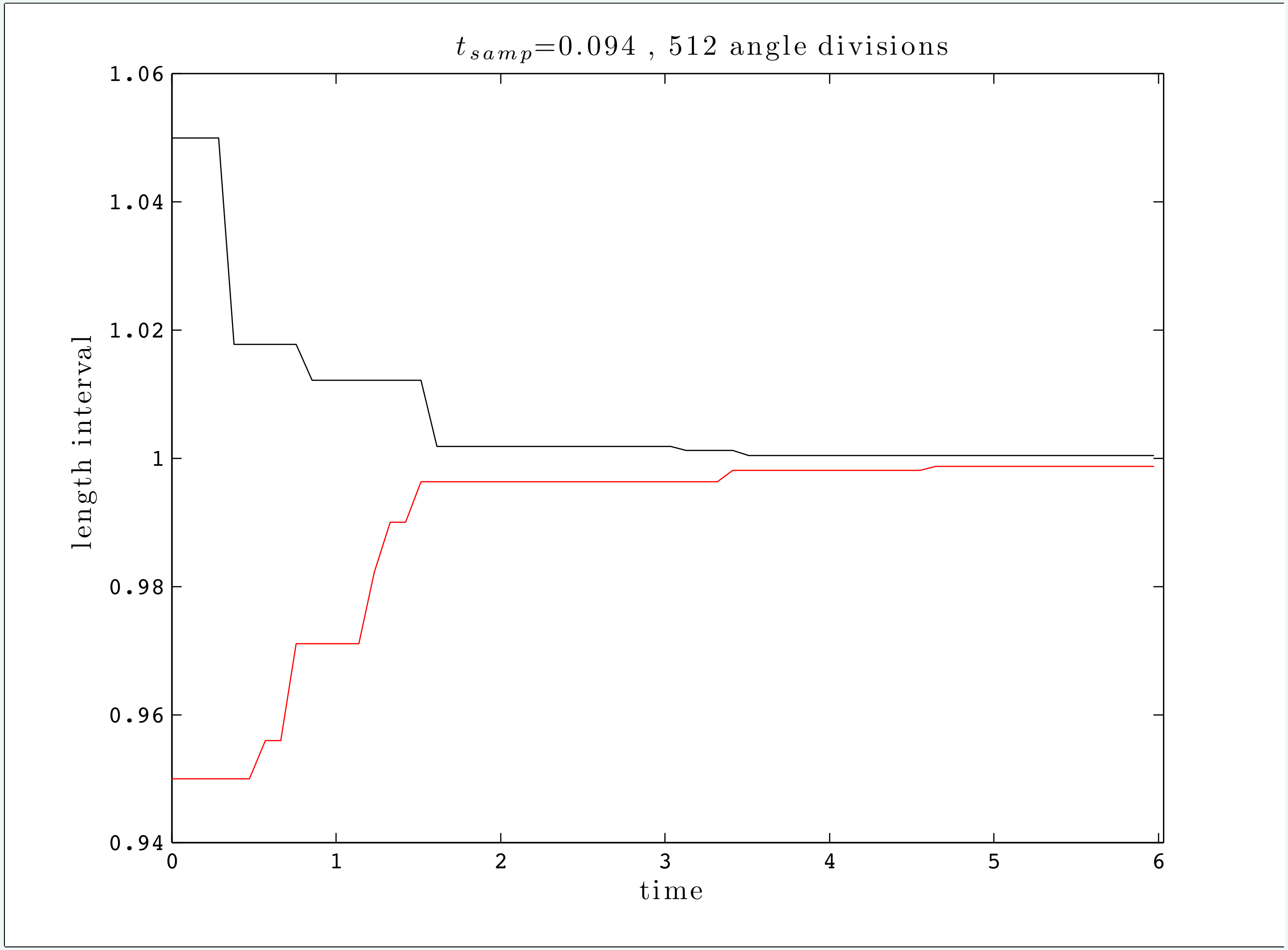
How to implement this algorithm in practice = set approximations

... ellipsoid, hypercubes, polyhedron, sampling ...

Length estimation using a set-valued estimator



The length is guaranteed to be between the curves

A priori information: x_0, u_k, v_k and w_k are independent random variables with known probability densities and we know

functional description	probabilistic description	notation
$x_0 = \mathcal{x}_0 \quad \rightarrow$	$p(x_0 = \mathcal{x}_0)$	$= X(\mathcal{x}_0)$
$x_{k+1} = f_k(x_k, u_k) \quad \rightarrow$	$p\left(x_{k+1} = \mathcal{x}_{k+1} \middle\vert x_k = \mathcal{x}_k,\ k\right)$	$= X_k(\mathcal{x}_{k+1}, \mathcal{x}_k)$
$y_k = h_k(x_k, v_k) \quad \rightarrow$	$p\left(y_k = \psi_k \middle\vert x_k = \mathcal{x}_k,\ k\right)$	$= Y_k(\psi_k, \mathcal{x}_k)$
$z_k = g_k(x_k, v_k) \quad \rightarrow$	$p\left(z_k = \zeta_k \middle\vert x_k = \mathcal{x}_k,\ k\right)$	$= Z_k(\zeta_k, \mathcal{x}_k)$

A priori information in blue

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A posteriori information = collected measurements

$(y_k = \psi_k,\ y_{k-1} = \psi_{k-1}, \ \ldots\ , y_1 = \psi_1)$ denoted $\mathcal{Y}_k = \Psi_k = (\psi_k, \Psi_{k-1})$

Observation problem: Compute at each time k the conditional probability

$$p\left(z_k = \zeta_k \middle\vert \mathcal{Y}_k = \Psi_k, k\right) \ = \ \widehat{Z}_k(\zeta_k, \Psi_k)$$

that the hidden variable z_k has value ζ_k knowing a priori and a posteriori information.

The conditional probability density $\widehat{Z}_k$ is given by the following recursive algorithm, where $\widehat{X}_k(x_k, \Psi_k)$ is the conditional probability that $x_k = x_k$ knowing that $\mathcal{Y}_k = \Psi_k$,

initialization $\widehat{X}_0(x_0) = X(x_0)$

prediction $\widehat{X}_{k|k-1}(x_k, \Psi_{k-1}) = \int_X X_{k-1}(x_k, x)\widehat{X}_{k-1}(x, \Psi_{k-1})dx$ flowing Chapman-Kolmogorov

restriction $\widehat{X}_k(x_k, \Psi_k) = \dfrac{Y_k(\psi_k, x_k)\widehat{X}_{k|k-1}(x_k, \Psi_{k-1})}{\int_X Y_k(\psi_k, x)\widehat{X}_{k|k-1}(x, \Psi_{k-1})dx}$ consistency Bayes rule

estimation $\widehat{Z}_k(\zeta_k, \Psi_k) = \int_X Z_k(\zeta_k, x)\widehat{X}_k(x, \Psi_k)dx$ Chapman-Kolmogorov

If all the probability distributions are gaussian this is the Kalman filter

In the general case, how to implement this algorithm in practice ?

⇒ approximate nonlinear filters:

- Extended Kalman filter,
- Unscented Kalman filter,
- Approximate grid based methods,
- Sequential Monte Carlo methods (particle filters),
- ...

These conceptual solutions give a confidence set, a probability measure, . . . , for the hidden variable.

How getting a single value ?

Rule 4 for observer design

To get a single value for the hidden variable from the confidence set, probability measure, . . .

we need to know what it is used for.

$\Rightarrow$ Quantify what the hidden variable is used for in a cost (acknowledging the equations of the a priori information cannot be satisfied "exactly").

For instance, at time k, a single-valued estimate $\widehat{z}_k$ for the hidden variable z_k is given by the following algorithm:

– Solve the minimization problem:

$$\min_{\sigma \in \Sigma_k} J_k\left((u_k, \ldots, u_{k-K}), (v_k, \ldots, v_{k-K}), (w_k, \ldots, w_{k-K})\right)$$

with

$$\sigma = ((x_k, \ldots, x_{k-K}), (\zeta_k, \ldots, \zeta_{k-K}))$$

$$\begin{aligned}\Sigma_k = \Big\{\sigma \ : \ x_{k-l} = f_{k-l-1}(x_{k-l-1}, u_{k-l-1}) \in X_{k-l} \ , \ u_{k-l-1} \in U_{k-l-1} \ , \\ \zeta_{k-l} = g_{k-l}(x_{k-l}, w_{k-l}) \in Z_{k-l} \ , \ w_{k-l} \in W_{k-l} \ , \\ h_{k-l}(x_{k-l}, v_{k-l}) = y_{k-l} \ , \ v_{k-l} \in V_{k-l}\Big\}\end{aligned}$$

– Select $\widehat{z}_k$ as the component ζ_k of σ in the set of minimizers of J_k.

How to implement this algorithm in practice $\Rightarrow$ approximation

Using a small angle approximation ($\sin(\theta) \approx \theta$) and the notations

$$\varrho = \sqrt{\frac{g}{l}} \quad , \qquad \mathcal{C} \;=\; \cos\left(\varrho t_{samp}\right) \quad , \qquad \mathcal{S} \;=\; \sin\left(\varrho t_{samp}\right) \;,$$

the a priori information is:

$$\theta_{k+1} \;=\; \mathcal{C}\,\theta_k \;+\; \tfrac{1}{\varrho}\,\mathcal{S}\,\omega_k \;, \quad \omega_{k+1} \;=\; -\varrho\,\mathcal{S}\,\theta_k \;+\; \mathcal{C}\,\omega_k \qquad \text{system dynamics}$$

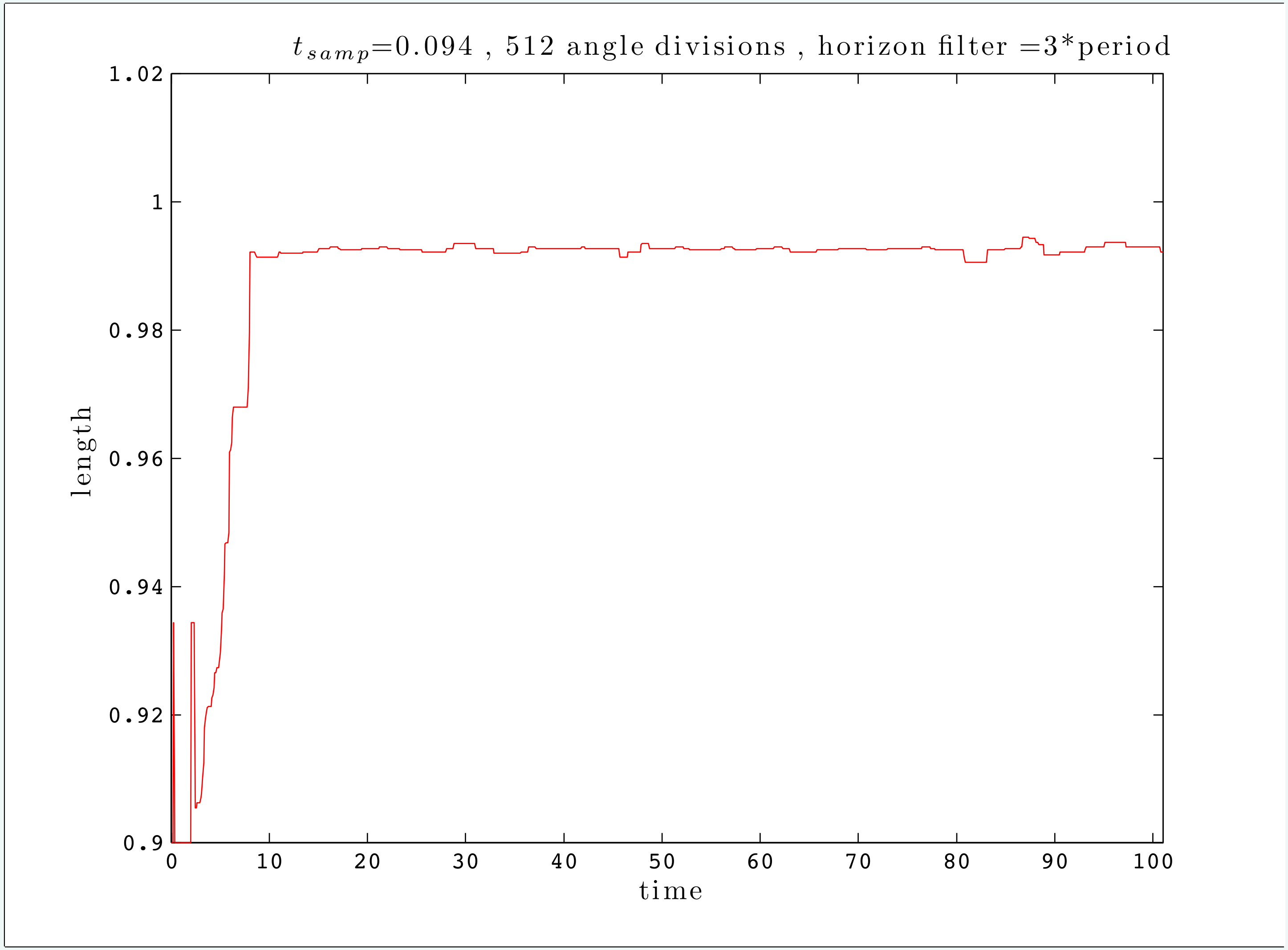
and the a posteriori information is :

$$\theta_k \;\in\; \left[y_k - \tfrac{\pi}{N}\,,\, y_k + \tfrac{\pi}{N}\right] \qquad \text{measurement with quantization}$$

We look for the length which is the most consistent, in a least square sense, with the a posteriori and a priori information, i.e. we look for $\bar{\varrho}$ minimizing

$$\begin{aligned} J(\bar{\varrho}, \bar{\theta}_{k-K}, \ldots, \bar{\theta}_k, \bar{\omega}_{k-K}, \ldots, \bar{\omega}_k) \;=\; & \\ r \sum_{l=k-K}^{k} \left(\min\left\{\bar{\theta}_l - y_l + \tfrac{\pi}{N}\,,\, 0\right\}^2 + \max\left\{0\,,\, \bar{\theta}_l - y_l - \tfrac{\pi}{N}\right\}^2\right) & \\ + q_\theta \sum_{l=k-K+1}^{k} \left(\bar{\theta}_l - \left[\mathcal{C}\,\bar{\theta}_{l-1} + \tfrac{1}{\bar{\varrho}}\,\mathcal{S}\,\bar{\omega}_{l-1}\right]\right)^2 & \\ + q_\omega \sum_{l=k-K+1}^{k} \left(\bar{\omega}_l + \left[\bar{\varrho}\,\mathcal{S}\,\bar{\theta}_{l-1} - \mathcal{C}\,\bar{\omega}_{l-1}\right]\right)^2 & \;. \end{aligned}$$

Length estimation following an optimization approach



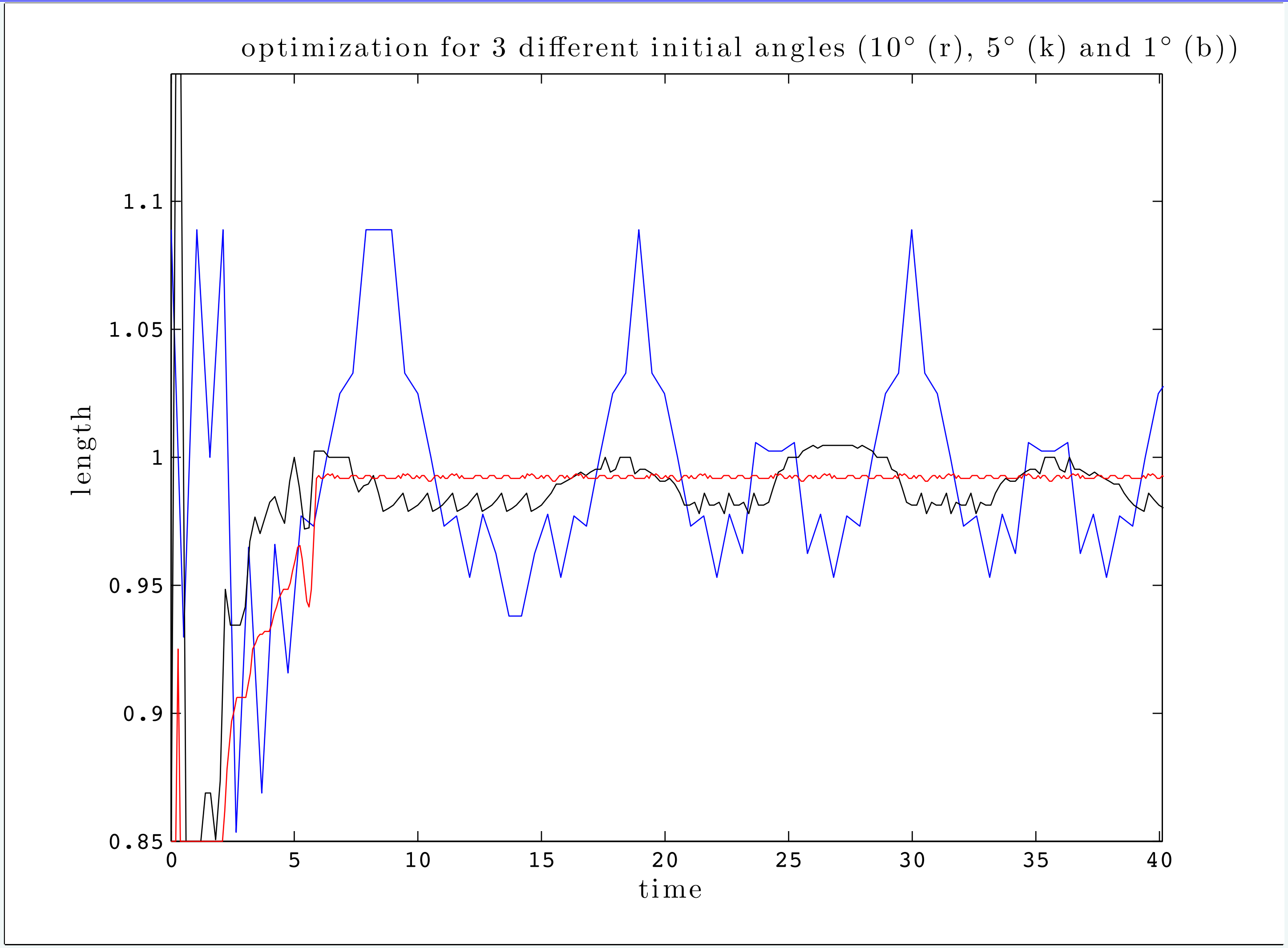
§ 4/8 Observability

Without any assumption on the models, the set-valued estimator and the conditional probability-valued estimator, in particular, are providing the best possible answer to the observation problem given the a priori and a posteriori information.

But this anwser is useful only when the a posteriori information is enriching the a priori one, i.e. when the a posteriori information allows us to make a better distinction among the hidden variable compatible with the a priori information.

Without this information enrichment, we may fear a too strong sensitivity in errors in the a priori information.

 Length estimation following an optimization approach



As the solution energy decreases,
($\Rightarrow$ the a posteriori information gets worse)
the estimation quality deteriorates

Let the system be $\dot{x} = f(x) \quad , \qquad y = h(x)$ with solutions denoted $\mathfrak{X}(x,t)$ and defined on the maximal time interval $(\sigma^-(x), \sigma^+(x))$.

If the system is time varying (may be due to known exogenous inputs), i.e.

$$\dot{x} = f(x,t) \quad , \qquad y = h(x,t) \; ,$$

we recover the above equations by replacing x by $x_e = (x,t)$, $(f(x,t),1)$ by $f_e(x_e)$, and $(h(x,t),t)$ by $h_e(x_e)$

...but then everything depends on the particular time function.

Differential observability with order k:

The system satisfies the differential observability property with order k if the function[1] $x \mapsto H_k(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{k-1} h(x) \end{pmatrix}$ is injective.

Proposition 1:

Differential observability with order k implies that, for all T strictly positive, the mapping $x \mapsto \{ h(\mathfrak{X}(x,t)),\, t \in [0\,, \min\{T, \sigma^+(x)\}\,)\,\}$ is injective.

$\Rightarrow$ if there is no uncertainty, the a posteriori information allows us to isolate **instantaneously** a single hidden variable.

The best for information enrichment

1

$$L_f^j h \;=\; L_f\left(L_f^{j-1} h\right) \quad , \qquad L_f h(x) \;=\; \lim_{t \to 0} \frac{h(\mathfrak{X}(x,t)) - h(x)}{t}$$

 Example Differential observability with order 4 for the pendulum

For the pendulum, we get : $$H_4(\ell,\theta,\omega) = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} \theta \\ \omega \\ -\frac{g}{\ell}\sin(\theta) \\ -\frac{g}{\ell}\omega\cos(\theta) \end{pmatrix}$$

We have $$\begin{pmatrix} \ell \\ \theta \\ \omega \end{pmatrix} = \begin{pmatrix} -\frac{y_3\sin(y_1)+y_4y_2\cos(y_1)}{y_3^2+y_4^2}\, g \\ y_1 \\ y_2 \end{pmatrix}$$ where :

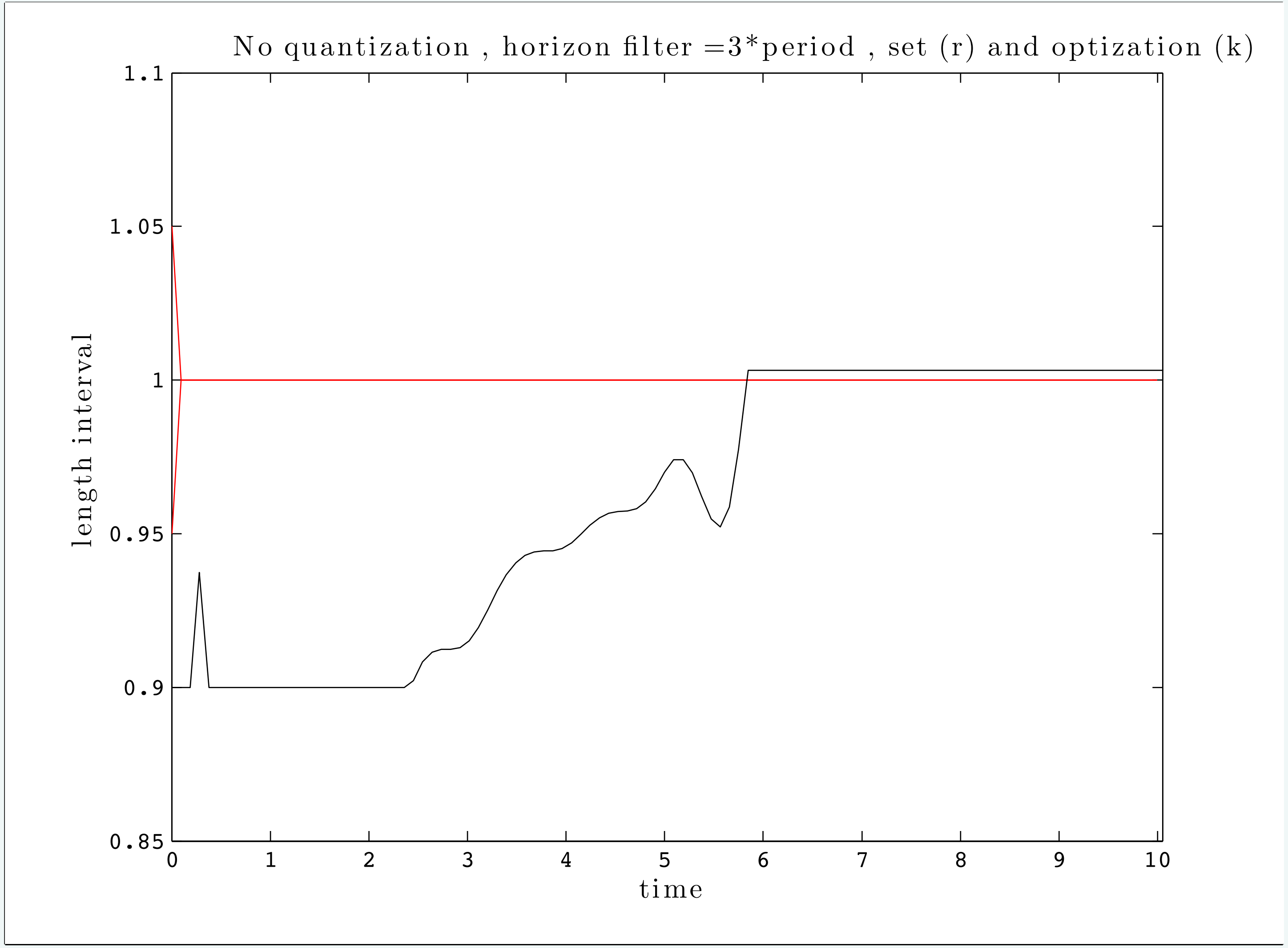
$$y_3^2+y_4^2 = \frac{g^2}{\ell^2}\sin(\theta)^2+\omega^2\cos(\theta)^2 = 0 \quad \Longleftrightarrow \quad \text{state energy} = 0$$

$\Rightarrow$ H_4 is left invertible when state energy $\neq 0$

$\Rightarrow$ H_4 is injective when state energy $\neq 0$

$\Rightarrow$ The pendulum satisfies the differential observability property with order 4 except for states with zero energy. But its state space dimension is 3.

The differential observability property implies that,
when there is no quantization error, the set-valued estimate, the conditional probability-valued estimate, and the optimization-based estimate converge.



§ 5/8 The convergence problem

Conceptual Observer

$$\{s \in (t-T, t] \mapsto y(s)\} \xrightarrow[\text{(= data compression . . .)}]{\text{time to "state" transformation}} \Sigma\{x, t, T\}$$

Measurement

$$\Sigma\{x, t, T\} \xrightarrow{\text{output}} \widehat{Z}(t)$$

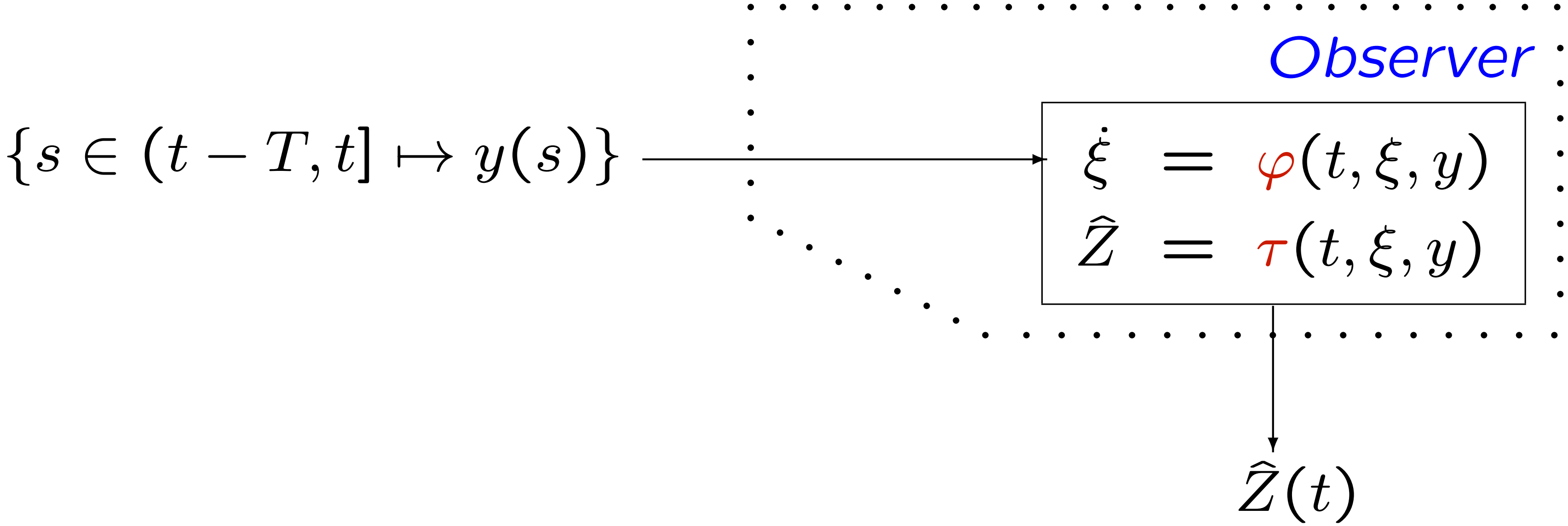
$\Sigma\{x, t, T\}$ is a "conceptual" observer "state". It is :

$\widehat{X}(t)$,	$\widehat{X}(x, \Psi(t), t)$,	$\frac{\partial J}{\partial \chi}(\chi, t, T)$,	$\left(y(t), \dot{y}(t), \ldots, \overset{\cdot\cdots\cdot}{y}(t)\right)$
set appr.	stochastic appr.	optimization appr.	differential obs.

$\widehat{Z}(t)$ is the hidden variable estimate. It is :

$\widehat{Z}(t)$,	$\widehat{Z}(\zeta, \Psi(t), t)$,	$\widehat{z}(t)$,	$\widehat{z}(t)$
set appr.	stochastic appr.	optimization appr.	differential obs.

Because of approximation – in the sets, in the probabilities, in the optimization, in the derivatives, in the observer output, . . . , a real observer is only an approximation of the "conceptual" observer.



The observer convergence problem : When there is no uncertainty, do we have $\hat{Z}(t)$ "converging to a single point" $z(t) = g(\mathfrak{X}(x,t))$ if the solution exists on $[0,+\infty)$ (respectively within the domain of existence of the solution $[0,\sigma^+(x))$) ?

§ 6/8 Necessary conditions for convergence

The technical context:

- Let d_x and d_ξ be given distances on $\mathbb{R}^n \times \mathbb{R}^n$ and $\mathbb{R}^m \times \mathbb{R}^m$ respectively;

- Let the system be defined by two functions f and h :

$$\dot{x} = f(x) \quad , \quad y = h(x) \quad , \quad z \underset{\uparrow}{=} x \tag{1}$$

hidden variable = system state

with x in an open set $\mathcal{O}$ of $\mathbb{R}^n$ and y in $\mathbb{R}^p$;

There is no uncertainty = perfect model

- Let the observer be defined by two functions φ and τ :

$$\dot{\xi} = \varphi(\xi, y) \quad , \quad \hat{x} = \tau(\xi, y) \tag{2}$$

with ξ in an open set Ω of $\mathbb{R}^m$ and $\hat{x}$ in $\mathcal{O}$;

Assumptions :

Reg : the functions $f : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}^p$, $\varphi : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^m$ and $\tau : \mathbb{R}^m \times \mathbb{R}^p \to \mathcal{O}$ are locally Lipschitz and all the solutions of (1),(2) with values in $\mathcal{O} \times \Omega$ can be right maximally defined on $[0, +\infty)$.

Conv : The zero error set $\mathcal{E} = \{(x, \xi) \in \mathcal{O} \times \Omega : x = \overbrace{\tau(\xi, h(x))}^{= \hat{x}} \}$ is asymptotically stable with domain of attraction containing $\mathcal{O} \times \Omega$.

Proposition 2: Under Assumptions *Reg* and *Conv*, we have :

Inv1 : there exists a subset $\mathcal{O}_0$ of $\mathcal{O}$ and a set-valued map $x \in \mathcal{O}_0 \mapsto \tau^*(x) \subset \Omega$ such that the zero error set $\mathcal{E}$ is the graph of τ^*, i.e.

$$\mathcal{E} = \{(x,\xi) \in \mathcal{O}_0 \times \Omega \, : \, \xi \in \tau^*(x)\} \ ,$$

$\Rightarrow$ the observer output function τ must be surjective given $h(x)$

Conv1 : We can "describe" the asymptotic stability of $\mathcal{E}$ by a Lyapunov function. This says there exist a C^∞ function $V : \mathcal{O} \times \Omega \to \mathbb{R}_+$, class $\mathcal{K}^\infty$ functions $\underline{\alpha}$ and $\overline{\alpha}$ and a continuous function $\varpi : \mathcal{O} \times \Omega \to \mathbb{R}_+$ such that :

$$L_{f,\varphi} V(x,\xi) \leq -V(x,\xi) \qquad \forall (x,\xi) \in \mathcal{O} \times \Omega \ ,$$

$$\underline{\alpha}\left(d_x(x, \tau(\xi, h(x)))\right) \leq V(x,\xi) \leq \overline{\alpha}\left(\varpi(x,\xi)\, d_x(x, \tau(\xi, h(x)))\right)$$

V is a function of two variables (x,ξ) not of $x - \hat{x} = x - \tau(\xi, h(x))$

Proposition 3: Under Assumption *Reg*, if there exists a C^1 function $V : \mathbb{R}^n \to \mathbb{R}_+$ satisfying

$$V(x - \tau(\xi, h(x))) > 0 \quad , \qquad L_{f,\varphi} V(x - \tau(\xi, h(x))) < 0$$
$$\forall (x, \xi) \in \mathcal{O} \times \Omega \,:\, x \neq \tau(\xi, h(x)))$$

then the function $x \in \mathcal{O}_0 \mapsto \begin{pmatrix} f(x) \\ h(x) \end{pmatrix}$ is injective.

Here V is a function of $x - \widehat{x}$ only. This is a coordinate dependent condition.

For the length estimation, the function

$$x = \begin{pmatrix} \ell \\ \theta \\ \omega \end{pmatrix} \mapsto \begin{pmatrix} f(x) \\ \\ \hline h(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \omega \\ -\sqrt{\frac{g}{\ell}}\,\sin(\theta) \\ \hline \theta \end{pmatrix}$$

is not injective on the set of strictly positive energy states.

(problem when $\sin(\theta) = 0$)

So there is no observer for which asymptotic stability of the zero error set $\mathcal{E}$ can be established with a strict Lyapunov function depending only on $(\ell - \widehat{\ell}, \theta - \widehat{\theta}, \omega - \widehat{\omega})$.

Proposition 4: Under Assumptions *Reg* and *Conv*, if the observer output function τ is injective given $h(x)$, specifically if there exists a class $\mathcal{K}^\infty$ function α_τ such that

$$d_\xi(\xi_a, \xi_b) \;\leq\; \alpha_\tau(d_x(x_a, \tau(\xi_b, h(x_a)))) \qquad \forall (x_a, \xi_b) \in \mathcal{O}_0 \times \Omega \;:\; \xi_a \in \tau^*(x_a)\,,$$

we have :

Inv2 : τ^* is singled valued and there exists a function $\ell : \Omega \times \mathbb{R}^p \to \mathbb{R}^n$ such that the dynamic observer function φ can be decomposed as

$$\varphi(\xi, y) \;=\; L_f \tau^*(\tau(\xi, y)) \;+\; \ell(\xi, y)$$

where

$$\ell(\xi, y) \;=\; 0 \qquad \forall (\xi, y) \in \Omega \times h(\mathcal{O}_0) \;:\; y = h(\tau(\xi, y))\,,\; \tau(\xi, y) \in \mathcal{O}_0$$

i.e. the observer dynamic is a copy of the image by τ^* of the system dynamic plus a correction term which is zero when actual measurement = estimated measurement.

Hence,

in some coordinates, the decomposition

"observer dynamic = copy of system dynamic + correction term"

is necessary when the observer output function τ is a bijection given $h(x)$, i.e. when there exists a function τ^* satisfying

surjective : $x \;=\; \tau(\tau^*(x), h(x)) \qquad \forall x \in \mathcal{O}_0$,

injective : $\xi \;=\; \tau^*(\tau(\xi, y))$
$$\forall (\xi, y) \in \Omega \times h(\mathcal{O}) \;:\; y = h(\tau(\xi, y)) \,,\; \tau(\xi, y) \in \mathcal{O}_0$$

Rule 5 for observer design

For an observer to have the convergence property with an output function τ which is a bijection given $h(x)$, the dynamic of its state ξ must be a copy of the system dynamic plus a correction term which is zero when $y = h(\tau(\xi, y))$ (actual measurement = estimated measurement).

There is no such rule if the output function τ is not injective given $h(x)$

§ 7/8 Observer with

τ possibly non injective given $h(x)$

The most frequently encountered observers have an output function τ which is a bijection given $h(x)$.

But we know at least 2 families of observers with τ possibly non injective given $h(x)$:

1. The high gain observer with immersion

2. The nonlinear Luenberger observer

The latter being less known let us focus our attention on it.

Proposition 5: Assume the set $\mathcal{O}$ is forward and backward invariant and there exist an integer m, a Hurwitz matrix A in $\mathbb{C}^{m\times m}$, continuous functions τ^* and B and a class $\mathcal{K}_\infty$ function ρ such that $L_f\tau^*$ exists and we have :

$$L_f\tau^*(x) = A\,\tau^*(x) + B(h(x))) \qquad \forall x \in \mathcal{O} \ , \qquad \text{(PDE)}$$

$$d_x(x_1,x_2) \leq \rho\left(d_\xi(\tau^*(x_1),\tau^*(x_2))\right) \qquad \forall x_1,\, x_2 \in \mathtt{cl}(\mathcal{O})$$

i.e. τ^* is uniformly injective

Then there exists a continuous function τ such that the observer

$$\dot{\xi} = \varphi(\xi,y) = A\xi + B(y) \quad , \qquad \hat{x} = \tau(\xi)$$

solves the observer convergence problem.

This observer is a nonlinear Luenberger observer. Its tunable parameters are the integer m (= dimension of ξ), the matrix A and the function B.

Comments :

a) The observer output function τ is not
a bijection given $h(x)$ in general.

b) The observer dynamic function φ is not
a copy of the image by τ^* of the system dynamic plus a correction term which is zero when y (actual measurement) $= h(\tau(\xi, y))$ (estimated measurement).

Existence of a solution to

$$L_f\tau^*(x) \;=\; A\,\tau^*(x) \;+\; B(h(x))) \qquad \forall x \in \mathcal{O}\ , \qquad \text{(PDE)}$$

Proposition 6: For each Hurwitz complex matrix A in $\mathbb{C}^{m\times m}$, we can find a C^1 injective function $B : \mathbb{R}^p \to \mathbb{C}^{m\times p}$ such that there exists a continuous function τ^* solution of (PDE).

With λ_i a complex number with negative real part, the nonlinear Luenberger observer dynamics are:

$$\dot{\xi}_i \;=\; \lambda_i\,(\xi_i - \sin(\theta)) \qquad i \in \{1,\ldots,m\}\ ,$$

with $\quad A = \mathtt{diag}(\lambda_1,\ldots,\lambda_m) \quad$ and $\quad B = -(\lambda_1\ \ldots\ \lambda_m)^T \sin(\theta)$.

The corresponding PDE with unknown τ_i^* is :

$$\frac{\partial \tau_i^*}{\partial \theta}(\ell,\theta,\omega,t)\,\omega \;-\; \frac{\partial \tau_i^*}{\partial \omega}(\ell,\theta,\omega,t)\,\frac{g}{\ell}\,\sin(\theta) \;+\; \frac{\partial \tau_i^*}{\partial t}(\ell,\theta,\omega,t)$$
$$=\; \lambda_i\,[\tau_i^*(\ell,\theta,\omega,t) - \sin(\theta)]\ .$$

There exists a solution in the form :

$$\tau_i^*(\ell,\theta,\omega,t) \;=\; c_{i\ell}(t)\,\frac{g}{\ell} \;+\; c_{i\omega}(t)\,\omega \;+\; \sin(\theta)$$

where

$$\dot{c}_{i\ell} \;=\; \lambda_i\,c_{i\ell} \;+\; c_{i\omega}\,\sin(\theta) \quad , \qquad \dot{c}_{i\omega} \;=\; \lambda_i\,c_{i\omega} \;-\; \cos(\theta)\ .$$

Uniform injectivity of τ^*

Proposition 7: Assume the system satisfies the differential observability property with some order on $\mathcal{O}$.
Then, if we select $m = n + 1$, there exist

– a positive real number μ

– and a subset S of $\mathbb{C}^m$ with zero Lebesgue measure,

such that,

with $A = \mathtt{diag}(\lambda_1, \ldots, \lambda_m)$ where the m λ_i are (arbitrary) in $\mathbb{C}^m \setminus S$ and with real part strictly smaller than μ,

we can find a function τ^* solution of (PDE) and injective.

Remark: On a compact set, continuity + injectivity $\Rightarrow$ uniform injectivity.

With $\dot{c}_{i\ell} = \lambda_i c_{i\ell} + c_{i\omega}\sin(\theta)$, $\dot{c}_{i\omega} = \lambda_i c_{i\omega} - \cos(\theta)$,

and
$$M_m(t) = \begin{pmatrix} c_{1\ell}(t) & c_{1\omega}(t) \\ \vdots & \vdots \\ c_{m\ell}(t) & c_{m\omega}(t) \end{pmatrix}$$

we have
$$\tau^*(\theta,\omega,\ell,t) = M(t)\begin{pmatrix} \frac{g}{\ell} \\ \omega \end{pmatrix} + \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}\sin(\theta)$$

It can be shown that it is sufficient to pick $m = 3$ (where complex counts for 2) λ_i to obtain the matrix M_3 with rank 2, at least for solutions with small but non zero energy.

Hence, for $m = 3$, given (θ, t), the function $(\omega, \ell) \mapsto \tau^*(\theta,\omega,\ell,t)$ is injective on the set

$$\left\{(\theta,\omega,\ell) \, : \, \ell > 0 \, , \, \frac{\ell}{2}\omega^2 + g(1-\cos(\theta)) > 0\right\}$$

Ultimately the nonlinear Luenberger observer is, for $i \in \{1, 2, 3\}$,

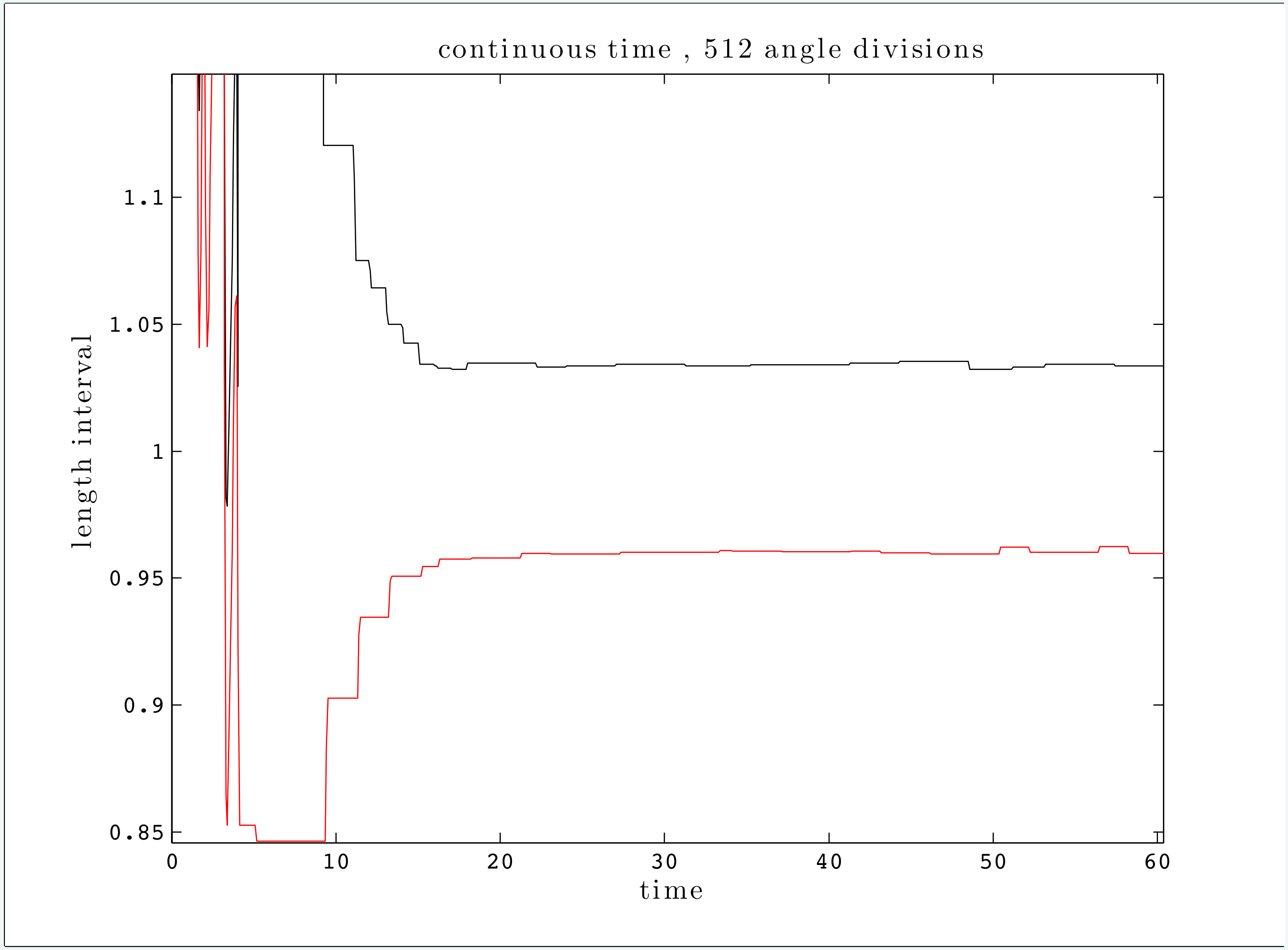
$$\dot{\xi}_i = \lambda_i \left(\xi_i - \sin(\theta)\right)$$

$$\dot{c}_{i\ell} = \lambda_i c_{i\ell} + c_{i\omega} \sin(\theta) \quad , \qquad \dot{c}_{i\omega} = \lambda_i c_{i\omega} - \cos(\theta)$$

$$M_3 = \begin{pmatrix} c_{1\ell} & c_{1\omega} \\ c_{2\ell} & c_{2\omega} \\ c_{3\ell} & c_{3\omega} \end{pmatrix}$$

$$\widehat{\ell} = \frac{g}{\begin{pmatrix} 1 & 0 \end{pmatrix} \left(M_3^T M_3\right)^{-1} M_3^T \left[\xi - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \sin(\theta)\right]}$$

Example The nonlinear Luenberger observer for the length estimation



The estimated length should be between the curves

Extra comments :

- An approximation theory about (PDE) solutions exists. In particular high gain observers can be seen as approximations of nonlinear Luenberger observers.

- No theory is available yet on how to select the tunable parameters m, A and B.

§ 8/8 Conclusions

The observation problem is to estimate a hidden variable from a measured variable.

I call observer the solution to this problem.

To design this observer, there are at least 5 rules.

Rule 1 : We need a priori information, as a "relation" between hidden and measured variables.

Rule 2 : In the a priori information, we need also a model of the uncertainties.

Rule 3 : Due to uncertainties, the output of an observer cannot be a point but only a confidence set or a probability measure or . . . , (depending on the uncertainty model).

Rule 4 : To get a single value for the hidden variable from the confidence set, probability measure, . . . , we need to know what it is used for.

Rule 5 : For an observer to have the convergence property with an output function which is a bijection given the measurement, its state dynamic must be a copy of the system dynamic plus a correction term which is zero when actual measurement = estimated measurement. There is no such rule if the observer output function is not injective given the measurement.

Observers whose output function is not injective given the measurement do exist, e.g. high gain observer with immersion or nonlinear Luenberger observer.

They exploit dynamic extension.

Their domain of application is broader than the one of usual observers (whose output function is a bijection given the measurement).

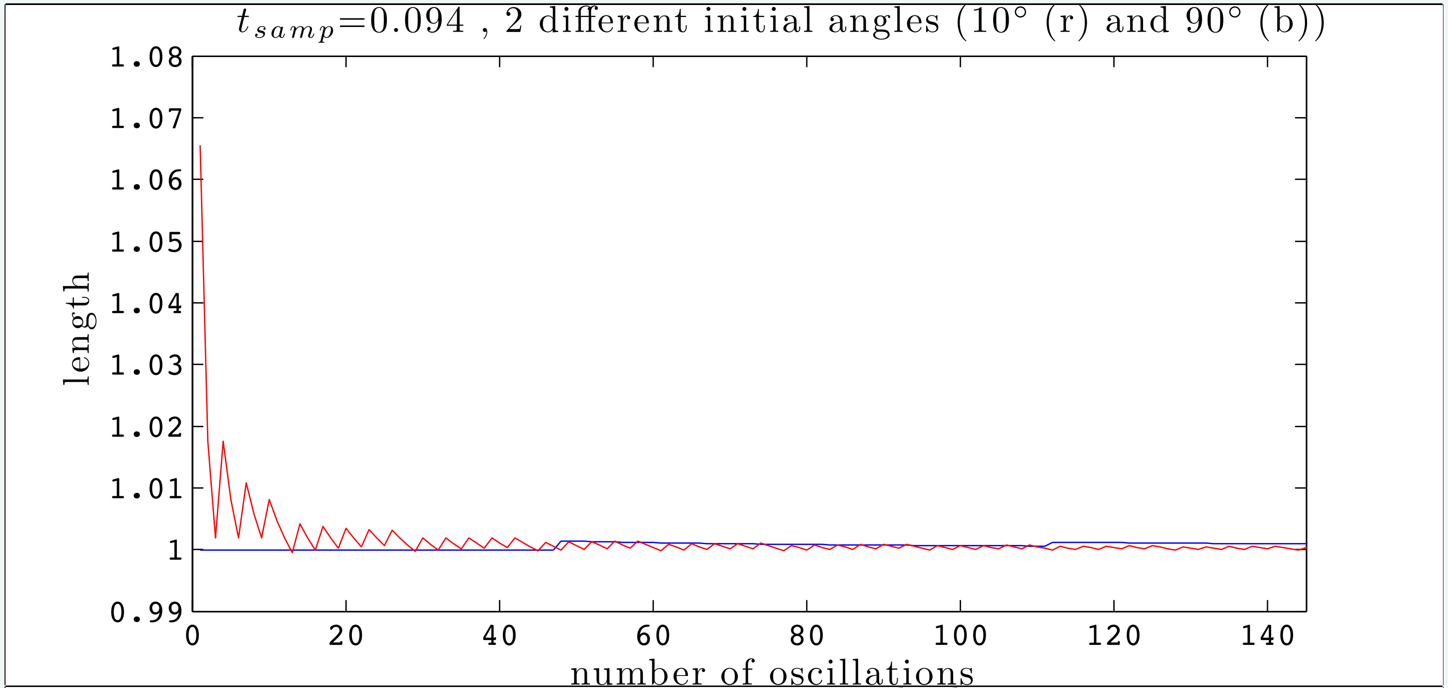
Future:

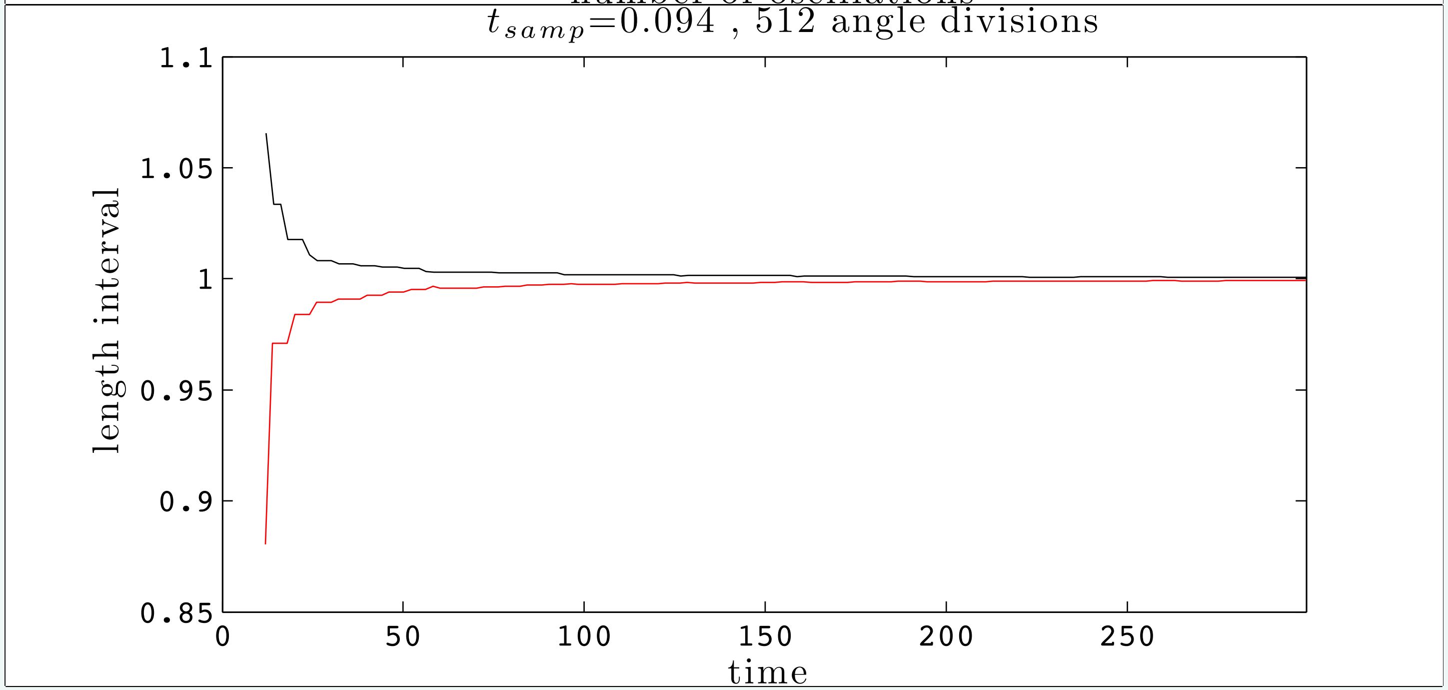
With progress in theory and technology, “conceptual” approaches such as set-valued estimators, conditional probability-valued estimators or optimization-based estimators will become very efficient tools. They can be used already for slow dynamic systems or with reduced number of states.

But ad hoc techniques will remain appropriate . . . By combining Galileo’s idea with Jacobi elliptic functions, we get a robust and very fast algorithm giving. . .

Example

Galileo's method with Jacobi elliptic functions





Bibliography

I apologize for not having given a single reference. This is a choice I had to make in view of the too many contributors I would have had to name.

Please contact me directly for any information.

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